Graduate Texts in Physics

Florian Scheck

Electroweak and Strong Interactions

Phenomenology, Concepts, Models

Third Edition

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With 59 Figures

Springer

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To the memory of Res Jost, who was an outstanding scientist and a truly exceptional personality

Aber sage nur niemand, daß uns das Schicksal trenne! Wir sinds, wir! Wir haben unsere Lust daran, uns in die Nacht des Unbekannten, in die kalte Fremde irgend einer andern Welt zu stürzen, und wär' es möglich, wir verließen der Sonne Gebiet und stürmten über des Irrsterns Grenzen hinaus.

(Friedrich Hölderlin, Hyperion, 1. Band, 1. Buch)

But never let it be said that Fate puts us asunder. We do, we ourselves, we delight in flinging ourselves into the dark unknown, into the cold abroad of some other world, we'd leave the zone of the sun altogether if we could and career beyond the frontiers of the wandering star.

(Hyperion by F. Hölderlin, Vol. 1, Part 1, 1797) translation taken from D. Constantine, "Hölderlin", Clarendon Press, Oxford, 1988, p. 348

Preface to the Third Edition

This book has its roots in a book on *Leptons, Hadrons and Nuclei* which I published, under that title in 1983, and, naturally enough, in the lectures and courses on elementary particle physics that I have given over the years, first at the Eidgenössische Technische Hochschule in Zurich, and later at the Johannes Gutenberg University in Mainz. Since 1983 – the year of the discovery of the W and Z bosons – experimental tests of what is now called the standard model of electroweak and strong interactions have made dramatic progress and, in fact, have reached a qualitatively new level. It now appears that the standard model, in its minimal version, is very well confirmed and, as yet, there is little hint of physics at higher mass scales, going beyond the standard model. This consolidation of the standard model concerns not only its building blocks and its general pattern but also the radiative corrections it predicts. For the student and for the beginner in the field of particle physics it is important to learn about the way to the standard model, the experimental basis on which it rests, its predictive power and its limitations. In particular, the novice in theoretical particle physics who sets out either to find a better foundation for the model, or else to recklessly dethrone it, hence to revolutionize our field, should know where he or she is sailing and what he or she is searching for. This is the reason why I decided to concentrate on the foundations and the phenomenology of electroweak and strong interactions rather than giving yet another account of the intricacies of quantized gauge theory.

There are many excellent textbooks and monographs on quantized field theory (for example, [ITZ80], [ZIJ94], [COL84], [DWS86]) and, more specifically, on quantized gauge field theory ([CHL84], [HUA92], [OKU82], [BEB94] and many more), but only few books covering in depth the phenomenology or the contact to nuclear physics (noteworthy exceptions are [PER87], [NAC94], [POR95]). This book is at the level of what might be termed *advanced quantum mechanics;* that is, I assume that the reader is familiar with nonrelativistic quantum mechanics and with the foundations of special relativity. In writing it I made every effort to define, to explain and to illustrate the basic notions and to explicitly show the path from

them to the final physical results. Although the level is not elementary, with a little effort and perseverance, the reader, whether experimentally or theoretically oriented, should be able to follow the complete argument or derivation in every subject that this book addresses, without having to resort to other sources. I do hope, of course, that this aspect will contribute to the fun and the satisfaction in learning the topics dealt with in this book. Sections marked with an asterisk contain more detailed material that may be skipped in a first reading.

Being largely self-contained, the book can be read as an independent text by anyone eager to learn this physics or to refresh his or her knowledge. As it originated in graduate-level lectures it may also serve as an accompanying textbook for a one- or two-semester course, perhaps with some cuts. Every chapter is followed by a set of exercises, some of which are simple whereas others require a little more time and effort. Solutions to selected exercises are given at the end of the book. All exercises will help the reader to test his or her understanding, and some serve the purpose of further illustrating the content of the corresponding chapter.

As compared to the second edition of 1996 various new developments and experimental results in electroweak physics are brought to date. The sections on deep inelastic scattering and QCD in Chaps. 2 and 3, as well as the discussion of neutrino oscillations in Chap. 4 are revised and extended. In turn, the fields of hadron scattering on nuclei and of hadronic atoms no longer are in focus of present-day experimental and theoretical research. Therefore the two chapters dealing with these topics were dropped here. If the need arises they may be consulted in the earlier edition of 1996.

In a field as vast and rapidly expanding as particle physics, the bibliography is bound to be incomplete, biased and to some extent unbalanced. I have adopted the following compromise: Within each chapter and at its end I give a selection of references, mostly on experimental results, which have direct bearing on the content of the chapter. In addition, towards the end of the book, there is a list of handbooks, textbooks and monographs to which I refer throughout all chapters using the notation [ABCxy](author(s) and year). Anyone who wishes to delve deeper into some topic dealt with in this book is advised to turn first to the review articles quoted here which will be helpful in retracing the complete literature.

Theoretical physics is a synthesis of lonely work and lively interaction with others. My colleagues and friends, my collaborators and students from whom I learnt a great deal and who directly or indirectly contributed to the genesis of this book, are too numerous to list here. I am very grateful to all of them for much stimulation, fruitful criticism, lively discussions or simply for the pleasure of collaborating with them.

Let me, quite presumptiously, adapt for my purpose the beautiful dedication that Johann Sebastian Bach chose for his Well-Tempered Clavier in 1722,

> "Zum Nutzen und Gebrauch der Lehr-begierigen *Physicali*schen Jugend, als auch derer in diesem *studio* schon *habil* seyenden besonderem Zeit Vertreib auffgesetzet und verfertiget..."

which means “Written and composed both for the benefit and use of young physicists desirous of instruction and for the particular diversion of those already advanced in this study…”, (translation taken from British Library Music Facsimiles I, The British Library, 1980).

Mainz *Florian Scheck*
August 2011

Contents

Notation and Conventions

To a large extent, the notation is explained in the text. Nevertheless it may be useful to first look through this short section on notations and conventions, or to return to it when one is not absolutely sure about a symbol or definition that is used in the text.

(i) *Units*. We use natural units

$$\hbar = 1, c = 1$$

throughout this book. Taking $c = 1$ means that coordinates and time have the same dimension, $[q_{\text{nat}}] = [t_{\text{nat}}]$, likewise, momenta and energy have the same dimension $[p_{\text{nat}}] = [E_{\text{nat}}]$. As in a physical system of units (SI, Gauss, or other) $\hbar$ has dimension energy $\times$ time and as this is also the dimension of an *action* such as $p \cdot q$, taking this constant equal to 1 means that

$$[p_{\text{nat}}] = [E_{\text{nat}}] = [q_{\text{nat}}]^{-1} = [t_{\text{nat}}]^{-1}.$$

This convention is not sufficient to fix the units completely. What is needed, in addition, is a unit of energy (or mass, or time). Following standard practice, we use multiples of the electron Volt,

$$1\ \text{meV} = 10^{-3}\,\text{eV}\quad 1\ \text{keV} = 10^{3}\ \text{eV}$$
$$1\ \text{MeV} = 10^{6}\,\text{e}V\quad 1\ \text{GeV} = 10^{9}\ \text{eV}\quad 1\ \text{TeV} = 10^{12}\,\text{eV},$$

i.e., milli, kilo, mega, giga, tera eV, respectively.

It is easy to translate length l, cross section σ, and time t from natural units back to conventional units. Let $\mathring{l}, \mathring{\sigma}, \mathring{t}$ be such quantities expressed in natural units, l, σ, t the same quantities in standard units. Then

$$l[\text{fm}] \stackrel{\Delta}{=} \hbar c \cdot \mathring{l}\,[\text{MeV}^{-1}],$$

$$t[\text{s}] \stackrel{\Delta}{=} \frac{\hbar c}{c} \cdot \mathring{t}[\text{MeV}^{-1}],$$

$$\sigma[\text{fm}^2] \stackrel{\Delta}{=} (\hbar c)^2 \cdot \mathring{\sigma}[\text{MeV}^{-2}],$$

with $\hbar c = 197.3270\,\text{MeV} \cdot \text{fm}$, $1\ \text{fm} = 10^{-13}\ \text{cm}$, $c = 2.99792458 \times 10^{10}\ \text{cm/s} = 2.99792458 \times 10^{23}\ \text{fm/s}$. Cross sections are usually expressed in units of $1\ \text{b} = 10^{-24}\ \text{cm}^2 = 10^2\ \text{fm}^2$. Thus, if $\mathring{\sigma}$ is found in GeV^{-2}, for example, then it follows that

$$1\ \text{GeV}^{-2} \stackrel{\Delta}{=} 0.3894\,\text{mb}.$$

Another example is the relationship between the width Γ and the lifetime τ of an unstable state,

$$\Gamma = \left(\frac{\hbar c}{c}\right)\frac{1}{\tau} = 6.58212 \times 10^{-22}\text{MeVs}\frac{1}{\tau}.$$

Momenta p are expressed in energy units if h and c are set equal to one, and are given in (energy unit)/c when conventional units are used.

(ii) *Experimental results and errors* are generally quoted with the error of the last digits in parentheses. For example,

$$0.7773(13)\ \text{means}\quad 0.7773 \pm 0.0013,$$

$$0.51126(5)\ \text{means}\ 0.51126 \pm 0.00005.$$

The abbreviation ppm stands for "parts per million". For example, a measurement giving the result 0.510 999 06(15) MeV for the electron mass is a "0.3 ppm measurement".

(iii) *Metric and normalization*. The metric is explained in more detail in Chap. 1. We use the form $g^{00} = +1$, $g^{ii} = -1$ $(i = 1, 2, 3)$ for the diagonal metric tensor. A contravariant vector is denoted by $x^\mu = (x^0, \boldsymbol{x})$ so that $x_\mu = (x^0, -\boldsymbol{x})$. One-particle states appear with the covariant normalization

$$< \boldsymbol{p}'|\boldsymbol{p} >= 2E_p\delta(\boldsymbol{p} - \boldsymbol{p}')$$

for both bosons and fermions. In the case of particles with spin there is an additional Kronecker δ-symbol for the spin indices.

(iv) *Some symbols. T* denotes the scattering matrix and is defined in App. B.

$\overleftrightarrow{\nabla}$ and $\overleftrightarrow{\partial}_\mu$ are short-hand notations for antisymmetric derivatives

$$f(\boldsymbol{x})\overleftrightarrow{\nabla}g(\boldsymbol{x}) = f(\boldsymbol{x})(\nabla g(\boldsymbol{x})) - (\nabla f(\boldsymbol{x}))g(x),$$

$$f(x)\overleftrightarrow{\partial}_\mu g(x) = f(x)(\partial_\mu g(x)) - (\partial_\mu f(x))g(x).$$

A list of symbols is found below.
(v) *Rotation matrices*. For the representation coefficients of the rotation group we use the definitions of e.g. [FAR59], that is

$$D^{(j)}_{KM}(\psi,\theta,\phi) = \mathrm{e}^{iK\psi} d^{(j)}_{KM}(\theta) \mathrm{e}^{iM\phi}$$

with Euler angles as defined in Fig. 1.1 (p.12). As explained in more detail in Sect. 1.2, these are the matrices which transform the expansion coefficients. Basis functions then transform according to D^*.

(vi) *Natural units for Maxwell's equations*. With $\boldsymbol{E}$ and $\boldsymbol{H}$ denoting the electric and magnetic fields, respectively, and $\boldsymbol{D}$ and $\boldsymbol{B}$ the electric displacement and magnetic induction, respectively, Maxwell's equations read in any system of units

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{E} + f_1 \tfrac{\partial \boldsymbol{B}}{\partial t} = 0$$
$$\nabla \cdot \boldsymbol{D} = f_2 \rho, \quad \nabla \times \boldsymbol{H} - f_3 \tfrac{\partial \boldsymbol{D}}{\partial t} = f_4 \boldsymbol{j}\,.$$

With q denoting the electric charge, the Lorentz force is

$$\boldsymbol{F} = q(\boldsymbol{E} + f_1 \boldsymbol{v} \times \boldsymbol{B}).$$

As is well known from electrodynamics, the fundamental fields are $\boldsymbol{E}$ and $\boldsymbol{B}$. In vacuum the derived fields $\boldsymbol{H}$ and $\boldsymbol{D}$ are related to the former by

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}, \quad \boldsymbol{B} = \mu_0 \boldsymbol{H}\,.$$

By convention the constants ε_0 and μ_0 are chosen in such a way that $f_1 = f_3$. Furthermore, the continuity equation which follows from the two inhomogeneous Maxwell equations, requires the relation $f_4 = f_1 f_2$, thus leaving three constants to be fixed by a suitable choice of physical units: f_1, f_2, and ε_0. The reader will easily verify that the SI system is based on the choice $f_1 = f_2 = 1$ and $\varepsilon_0 = 10^7/(4\pi c^2)$, while the Gauss system is characterized by the choice $f_1 = 1/c, f_2 = 4\pi$, $\varepsilon_0 = 1$.

The *system of natural units* that is used in elementary particle physics takes the velocity of light c to be unity, (as well as Planck's constant divided by 2π, $\hbar = 1$), cf. (i) above, and is also designed such that the factors 4π disappear from Maxwell's equations. This means setting

$$f_1 = f_2 = 1, \quad \varepsilon_0 = \mu_0 = 1$$

and absorbing square roots of 4π into the fields and charges as follows,

$$E|_{\mathrm{nat}} = \frac{1}{\sqrt{4\pi}} E|_{\mathrm{Gauss}}, \; B|_{\mathrm{nat}} = \frac{1}{\sqrt{4\pi}} B|_{\mathrm{Gauss}},$$
$$\rho|_{\mathrm{nat}} = \sqrt{4\pi}\,\rho|_{\mathrm{Gauss}}, \quad j|_{\mathrm{nat}} = \sqrt{4\pi}\, j|_{\mathrm{Gauss}},$$

Obviously, this is a convenient framework for doing calculations. When returning to customary units such as Gauss' system, and evaluating the results of a calculation, all that remains to be done at the end is to multiply external fields by $\sqrt{4\pi}$ and to replace the squared elementary charge e^2 by $4\pi\alpha$, where α is Sommerfeld's fine-structure constant. Indeed, from what we said above, we have

$$\alpha = \frac{e^2|_{\text{Gauss}}}{\hbar c} = \frac{e^2|_{\text{nat}}}{4\pi} \approx \frac{1}{137.036}$$

(vii) All equations, figures, tables and sections are numbered beginning with the number of the chapter in which they occur. They can thus easily be located when they are mentioned elsewhere.

List of Symbols

$A_\alpha(x) = \mathrm{i}e\sum_k^N T_k A_\alpha^{(k)}(x)$:	gauge potential taking its values in the Lie algebra spanned by generators T_k
$a_\alpha(x)$:	weak axial vector current
a_B:	Bohr radius of atomic orbit
$\boldsymbol{C}$:	charge conjugation operator
$\mathbb{C}$:	field of complex numbers
$D_\alpha(A) = \mathbb{1}\partial_\alpha + A_\alpha$:	covariant derivative
∂_μ:	partial derivative with respect to x^μ, with x a point in Minkowski space-time
∂^μ:	partial derivative with respect to x_μ, with x a point in Minkowski space-time
ε_{ijk}:	totally antisymmetric tensor in 3 real dimensions
$\varepsilon_{\mu\nu\sigma\tau}$:	totally antisymmetric tensor in 4 real dimensions (convention $\varepsilon_{0123} = +1$)
$f(x) \equiv \psi^{(f)}(x)$:	shorthand for Dirac field describing fermion f
$F_{\alpha\beta}(x) = \mathrm{i}e\sum_k^N T_k F_{\alpha\beta}^{(k)}(x)$:	field strength tensor of Abelian ($N = 1$) or non-Abelian gauge theory
$f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$:	kinetic part of field strength tensor $F_{\alpha\beta}$
${}_1F_1\,(a; b; x)$:	confluent hypergeometric function
$F_i(q^2)$:	form factors in hadronic matrix elements of electromagnetic and weak vector currents
$F_\mathrm{A}(q^2)$, $F_\mathrm{P}(q^2)$:	axial and pseudoscalar form factors for weak axial current
f_π:	decay constant of charged pion
$\phi_a(x)$, $a = 1, 2$:	spinor field of first kind
$\chi^A(x)$, $A = \mathrm{I}, \mathrm{II}$:	spinor field of second kind
$G/\sqrt{2} = g^2/(8m_\mathrm{w}^2)$:	Fermi's constant (G) of weak interactions

g, g' or e:	(dimensionless) coupling constant in gauge theories
$g_{\mu\nu}$, $g^{\mu\nu}$:	metric tensor in Minkowski space–time
$G_{\mathrm{E}}(q^2)$, $G_{\mathrm{M}}(q^2)$:	electric and magnetic form factors of nucleon
$\mathscr{H}$:	Hamiltonian density
$h(\nu_f)$:	helicity of neutrino in lepton family $f = e$, μ or τ
$\mathbb{1}$:	unit matrix or, more generally, identity operation
$\mathscr{L}$:	lagrangian density
L_f:	lepton family number for $f = e$, μ or τ
Λ:	Lorentz transformation
$L(\boldsymbol{v})$:	special Lorentz transformation or boost
$\overset{\circ}{M}$:	diagonal form of a hermitean or symmetric matrix
$\boldsymbol{P}$:	space-reflection or parity operation
ψ_L, ψ_R:	left- and right-handed components of Dirac field
$\overline{\psi(x)} = \psi^\dagger(x)\gamma_0$:	conjugate Dirac field
$\mathscr{R}$:	rotation in four dimensions, $\mathscr{R} = \mathrm{diag}(1, \boldsymbol{R})$, where $\boldsymbol{R} \in \mathrm{SO}(3)$ is a rotation matrix
$\mathbb{R}$:	field of real numbers
$S = \mathbb{1} + R$:	S-matrix, decomposed into the identity ("no scattering") and the reaction matrix R
$\mathrm{SU}(N)_{\mathrm{f}}$:	unitary group describing N flavours
$\mathrm{SU}(3)_{\mathrm{c}}$:	colour group of quarks and gluons
$\Theta = \boldsymbol{PCT}$:	combined operation of space reflection, charge conjugation and time reversal
θ_{W}:	Weinberg, or weak interaction, angle. Note that only $\sin^2\theta_{\mathrm{w}}$ is physical
$\mathbf{T}$:	time-reversal operation
T:	scattering matrix which enters cross-section formulae.
$T_k, k = 1, \ldots N$:	abstract generators of Lie group G, with N the dimension of the Lie algebra $Lie(G)$
$U(g)$:	unitary representation of group element $g \in G$ in a given representation space
$U(T_k)$:	representation of the generator T_k in a given representation space
V_{CKM}:	mixing matrix of *down*-type quarks
$\upsilon_\alpha(x)$:	weak vector current

Chapter 1
Fermion Fields and Their Properties

The fundamental building blocks of matter, i.e. quarks and leptons, carry spin 1/2. There are two formally different but in essence equivalent methods of describing particles with spin: The representation theory of the Poincaré group, in the framework of Wigner's classification hypothesis of particles (see e.g. [QP07], Chap. 6), and the Van der Waerden spinor calculus based on SL(2, $\mathbb{C}$). In this chapter we derive the Dirac equation in its natural framework: the spinor representations of SL(2, $\mathbb{C}$). The properties of this equation and of its solutions are discussed in detail. The quantization of Dirac fields is developed in the light of covariance and causality, and all important consequences of quantization are worked out.

Sections 1.1–7 contain the fundamentals while the remaining sections 8 to 10 deal with questions of practical importance in embedding fermions in a more comprehensive theory of elementary particles: the description of masses in the Lagrange density; the description of spin in the language of (covariant) density matrices; and the coupling of a charged Dirac field to the electromagnetic field. The case of the coupling to a non-Abelian local gauge field is treated in Chap. 3, in the context of weak interactions.

1.1 Lorentz Group and SL(2, $\mathbb{C}$)

The natural basis for the construction of a Lorentz invariant theory of free fermions (i.e. particles with spin 1/2) is provided by the isomorphism that links the group of proper, orthochronous Lorentz transformations to SL(2, $\mathbb{C}$), the special linear group in two complex dimensions. This connection is completely analogous to the relation between the rotation group SO(3) and the special, unimodular group SU(2) in two dimensions. It is not difficult to construct spinor representations of SL(2, $\mathbb{C}$). One then discovers, in fact, that SL(2, $\mathbb{C}$) – in contrast to SU(2) – admits *two* inequivalent spinor representations both of which are relevant for the Lorentz covariant description of fermions. This connection is essential for the understanding of the properties of Dirac theory, such as the space–time structure of Dirac fields, the

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particle–antiparticle symmetry, and the relation between spin and statistics. These matters are no more difficult to understand than the traditional, historical approach to Dirac theory but from a logical point of view this approach is much more satisfactory. The reader should not be frightened by the few group theoretical terms that we will use and which are necessary for the understanding of these matters. The text is sufficiently self-contained so that it should be understandable even with only a rudimentary knowledge of group theory.

We begin by recalling the definition and some properties of the group of Lorentz transformations, (for a more detailed, elementary, introduction see e.g. Chap. 4 of [SCH10]). We then work out the precise relationship between the proper orthochronous Lorentz group, denoted $\mathrm{L}_+^\uparrow$, and the special linear group in two complex dimensions, denoted SL(2, $\mathbb{C}$).

1.1.1 Lorentz Transformations

Physical theories which obey the postulates of special relativity are formulated over Minkowski space–time M^4. This is a 4-dimensional Euclidean space $\mathbb{R}^4$ which is endowed with the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) \tag{1.1}$$

Here, the indices μ and ν take the value 0 (referring to the time axis), and 1,2,3 (referring to the spatial axes), rows and columns in (1.1) are numbered in the order 0,1,2,3 for time and space[1]. When written in coordinates, a point x of M^4 reads

$$x^\mu = (x^0, \boldsymbol{x}) \quad \text{with} \quad x^0 = ct,$$

with x^0 a point on the real axis $\mathbb{R}$, $\mathbf{x}$ a point in $\mathbb{R}^3$. Note that by multiplying time t by the velocity of light, x^0 acquires physical dimension length.

Likewise, four-vectors a are called *contravariant* vectors if their coordinate representation is $a^\mu = (a^0, \boldsymbol{a})$, the position of the indices indicating the behaviour under Lorentz transformations. The *covariant* vector that corresponds to a^μ, that is the vector which is contragredient to a^μ with respect to Lorentz transformations, is defined by

[1]We have inserted a vertical and a horizontal line in order to emphasize the space–space components in the lower right 3×3 block as opposed to the time–time, time–space, and space–time components. We shall keep these auxiliary lines occasionally for the sake of clarity, but we shall drop them later in the text.

$$a_\mu = g_{\mu\nu} a^\nu = (a^0, -\boldsymbol{a}),$$

(summation over pairs of identical upper and lower indices is implied).

Let $z{=}x - y$ be the vector joining the points x and y in M^4, i.e. $z^\mu{=}x^\mu{-}y^\mu$ with respect to an arbitrary inertial frame of reference, and let z' be the same vector, expressed with respect to any other inertial frame. The group of Lorentz transformations is defined to be the group of all linear transformations Λ

$$z \mapsto z' = \Lambda z, \quad \text{i.e.} \quad z'^\mu = \Lambda^\mu{}_\nu z^\nu$$

which leave invariant the square of the generalized space–time distance

$$z^2 = z^\mu g_{\mu\nu} z^\nu = z^\mu z_\mu = (z^0)^2 - z^2.$$

The requirement $z'^2 = z^2$ yields the condition on Λ

$$\Lambda^T g \Lambda = g \text{ or } g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}, \tag{1.2}$$

from which one deduces the following properties of the 4×4 matrices Λ:

(i) By taking the determinant of (1.2), noting that Λ is real and that det $g = -1$, one finds $(\det \Lambda)^2 = 1$, or,

$$\det \Lambda = \pm 1. \tag{1.3}$$

(ii) Equation (1.2) with $\alpha = 0$ and $\beta = 0$ yields the equation $\left(\Lambda^0{}_0\right)^2 - \sum_{i=1}^3 \left(\Lambda^i{}_0\right)^2 = 1$, and hence leads to the conclusion

$$\Lambda^0{}_0 \geq +1 \text{ or } \Lambda^0{}_0 \leq -1. \tag{1.4}$$

Lorentz transformations which have determinant $+1$ are called *proper* transformations and are denoted by a suffix $+$. Transformations which have the property $\Lambda^0{}_0 \geq 0$ are called *orthochronous* (they map the time coordinate "forward") and are identified by an arrow pointing upward. The matrices Λ which have both properties form a subgroup called the proper orthochronous Lorentz group, viz.

$$\mathrm{L}_+^\uparrow = \{\Lambda | \text{real } 4 \times 4 \text{ matrices}, \Lambda^T g \Lambda = g, \det \Lambda = +1, \Lambda^0{}_0 \geq 1\}. \tag{1.5}$$

Analogously, Lorentz transformations with det $\Lambda = -1$ are denoted by a suffix $-$, those with $\Lambda^0{}_0 \leq -1$ by an arrow pointing downward, all combinations of $\pm$ and $\uparrow$ or $\downarrow$ being possible. Thus, the Lorentz group has four connected components,

(a) $\mathrm{L}_+^\uparrow$ as defined in (1.5). This component contains the identy $\mathbb{1} = \mathrm{diag}(1, 1, 1, 1)$. Each of its elements can be deformed continuously into the unit element, i.e. can be taken into $\mathbb{1}$ by continuous tuning of the parameters on which it depends.

(b) $\mathrm{L}_-^\uparrow$: This component contains $P = \mathrm{diag}(1, -1, -1, -1)$, the operation of space reflection. Its elements cannot be deformed continuously into the unit element

because they all have determinant equal to -1 while $\mathbb{1}$ has determinant equal to $+1$. Similarly, the remaining two components are disconnected from $\mathbb{1}$, either for the same reason, or because their time–time component is smaller than or equal to -1, in contrast to $\mathbb{1}$. They are

(c) $\mathrm{L}_-^\downarrow$ which contains the operation of time reversal $T = \mathrm{diag}(-1,\ 1,\ 1,\ 1)$, and

(d) $\mathrm{L}_+^\downarrow$ which contains the product $PT = \mathrm{diag}(-1,\ -1,\ -1,\ -1)$.

While $\mathrm{L}_+^\uparrow$ is a group by itself and, hence, a subgroup of the Lorentz group, the remaining three components clearly cannot be subgroups. They do not contain the unit element, the product of any two elements of one branch lies in another, different, branch. Note, however, that the elements of $\mathrm{L}_-^\uparrow$ are obtained from those of $\mathrm{L}_+^\uparrow$ by multiplication with P. Similarly, $\mathrm{L}_-^\downarrow$ is obtained from $\mathrm{L}_+^\uparrow$ by multiplication with T, and $\mathrm{L}_+^\uparrow$ is obtained from $\mathrm{L}_+^\uparrow$ by multiplication with PT. Thus, the structure of the entire Lorentz group is obtained from a knowledge of the proper, orthochronous, Lorentz group $\mathrm{L}_+^\uparrow$ and of the discrete transformations P, T, and (PT).

For the sake of reference, we summarize some important properties of $\mathrm{L}_+^\uparrow$, the proper orthochronous Lorentz group, but refer to [SCH10] or one of the more specialized treatises on special relativity for more details.

There are two types of transformation in $\mathrm{L}_+^\uparrow$ which are basic in the following sense. Any $\Lambda \in \mathrm{L}_+^\uparrow$ can be written as a product of a rotation in space and a special Lorentz transformation (also called boost). Indeed, rotations in three-dimensional space have the form

$$\mathscr{R} = \left(\begin{array}{c|c} \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R} \end{array}\right), \text{ with } \mathbf{R} \in \mathrm{SO}(3). \tag{1.6}$$

They fulfill condition (1.2) which reduces here to $\mathbf{R}^T\mathbf{R} = \mathbb{1}$. As $\mathscr{R}_0^0 = +1$, and $\det \mathscr{R} = \det \mathbf{R} = +1$, they belong to $\mathrm{L}_+^\uparrow$.

A special Lorentz transformation taken along the direction $\hat{\upsilon} = \upsilon/|\upsilon|$ and characterized by the absolute value $\upsilon = |\boldsymbol{\upsilon}|$ of the velocity $\boldsymbol{\upsilon}$, is given by

$$L(\boldsymbol{\upsilon}) = \left(\begin{array}{c|c} \gamma & \gamma\upsilon^k \\ \hline \gamma\upsilon^i & \delta^{ik} + \frac{\gamma^2}{1+\gamma}\upsilon^i\upsilon^k \end{array}\right), \tag{1.7}$$

where $\gamma = 1/\sqrt{1-\upsilon^2}$. Like in (1.6) the first row gives the time–time and the time–space components, respectively, the first column contains the time–time and the space-time components, while the 3×3 matrix in the right lower square contains the space–space components. That $L(\boldsymbol{\upsilon})$ belongs to $\mathrm{L}_+^\uparrow$ is verified as follows.

Obviously, $L_0^0 \geq +1$. As a preparation to verifying condition (1.2), we note that (1.7) can be written in an appealing notation in terms of "ket" $|\rangle$ and "bra" $\langle|$ which makes obvious the action of $L(\boldsymbol{\upsilon})$ on a four-vector, i.e.

$$L(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma\langle \boldsymbol{v}| \\ \gamma|\boldsymbol{v}\rangle & \mathbb{1}_{3\times 3} + \frac{\gamma^2}{1+\gamma}|\boldsymbol{v}\rangle\langle \boldsymbol{v}| \end{pmatrix}.$$

Indeed, with $\langle \boldsymbol{v}|\boldsymbol{w}\rangle = \boldsymbol{v}\cdot\boldsymbol{w}$ being the ordinary scalar product in space $\mathbb{R}^3$, $L(\boldsymbol{v})$ applied to a column vector with time component a^0 and space part $|\boldsymbol{a}\rangle$, again gives such a vector.

First we verify that $L(\boldsymbol{v})$ fulfills relation (1.2). The time–time component of the product $L^T(\boldsymbol{v})g\ L(\boldsymbol{v})$ is $\gamma^2(1-\boldsymbol{v}^2) = 1$. The time–space components and the space–time components contains the factor $\gamma^2 - \gamma - \gamma^3 u^2/(1+\gamma)$ which is zero by the relation $\boldsymbol{v}^2 = (\gamma^2-1)/\gamma^2$. The space–space components are $-\mathbb{1}_{3\times 3}$ plus a term in $|\boldsymbol{v}\rangle\ \langle \boldsymbol{v}|$ whose factor again vanishes. Thus $L^T gL = g$. It remains to verify that $\det L(\boldsymbol{v}) = +1$. For this purpose we calculate the product $\mathscr{R}L\mathscr{R}^T$, with $\mathscr{R}$ as given in (1.6),

$$\mathscr{R}L\mathscr{R}^T = \begin{pmatrix} \gamma & \gamma\langle \boldsymbol{v}|\boldsymbol{R}^T \\ \gamma\mathbf{R}|\boldsymbol{v}\rangle & \mathbb{1} + \frac{\gamma^2}{1+\gamma}\mathbf{R}|\boldsymbol{v}\rangle\langle \boldsymbol{v}|\mathbf{R}^T \end{pmatrix}.$$

The determinant of this product is the same as the determinant of $L(\boldsymbol{v})$. As $\mathbf{R}\in \mathrm{SO}(3)$ can always be chosen such that $\boldsymbol{v}$ points along one of the coordinate axes, say the 1-axis, we find

$$\det L(\boldsymbol{v}) = \det\ (\mathscr{R}L(\boldsymbol{v})\mathscr{R}^T) = \gamma\left(1 + \frac{\gamma^2}{1+\gamma}\boldsymbol{v}^2\right) - \gamma^2\boldsymbol{v}^2 = 1,$$

where we have made use of the relation $\boldsymbol{v}^2 = (\gamma^2-1)/\gamma^2$. This proves the assertion.

The structure of the proper, orthochronous, Lorentz group $\mathrm{L}_+^\uparrow$ is clarified by the decomposition theorem[2] for Lorentz transformations:

Theorem (I): Every element $\Lambda \in \mathrm{L}_+^\uparrow$ can be written uniquely as the product of a rotation followed by a special Lorentz transformation,

$$\Lambda = L(\boldsymbol{v})\mathscr{R}, \tag{1.8}$$

with $\mathscr{R}$ as in (1.6), the parameters of the two transformations being given by

$$\boldsymbol{v}^i = \frac{\Lambda^i{}_0}{\Lambda^0_0}, \quad \mathbf{R}^{ik} = \Lambda^i{}_k - \frac{1}{1+\Lambda^0_0}\Lambda^i{}_0\Lambda^0{}_k. \tag{1.9}$$

[2]Note that the rotations (1.6) are orthogonal matrices, i.e. $\mathscr{R}^{-1} = \mathscr{R}^T$, while the boosts (1.7) are symmetric matrices. The theorem writes an arbitrary $\Lambda \in \mathrm{L}_+^\uparrow$ as the product of a symmetric and an orthogonal matrix. This is analogous to the decomposition of a complex number in its (real) modulus and a (unitary) phase. This analogy will become even more striking for SL(2, $\mathbb{C}$) representations below.

For a proof see e.g. [SCH10].

We close this subsection with a few remarks:

(i) Of course, Λ may also be decomposed with a different order of the factors,

$$\Lambda = \mathscr{R} L(\boldsymbol{w}). \tag{1.10}$$

One can easily show that the rotation is the same as with the order of factors chosen in the decomposition (1.8), and that the velocities $\boldsymbol{v}$ and $\boldsymbol{w}$ are related by $\boldsymbol{v} = \mathbf{R}\boldsymbol{w}$ where $\mathbf{R}$ is the space part of $\mathscr{R}$.

(ii) Another way of parameterizing the special transformations is the following. As v, the absolute value of $\boldsymbol{v}$, may take on any value between 0 (state at rest) and 1 (velocity of light, in natural units), the parameter $\gamma = 1/\sqrt{1-v^2}$ varies between 1 and infinity. Therefore, γ and u may be parametrized by hyperbolic functions of a real parameter λ as follows

$$\gamma = \cosh\lambda, \quad \gamma v = \sinh\lambda. \tag{1.11}$$

The parameter λ lies in the interval $\lambda \in [0, \infty]$ and is called *rapidity parameter*. For $\lambda = 0$ we have $\gamma = 1$, $v = 0$, while for $\lambda \to \infty$ we have $\gamma \to \infty$ and $v \to 1$. When written in terms of the rapidity parameter, the boost (1.7) becomes

$$L(\boldsymbol{v}) = \begin{pmatrix} \cosh\lambda & \hat{v}^i \sinh\lambda \\ \hat{v}^k \sinh\lambda & \hat{v}^i \hat{v}^k \cosh\lambda + (\delta^{ik} - \hat{v}^i \hat{v}^k) \end{pmatrix}. \tag{1.12}$$

(iii) Suppose the velocity $\boldsymbol{v}$ is the instantaneous velocity of an elementary particle of mass m with respect to an inertial frame of reference. Its energy E_p and its spatial momentum $\boldsymbol{p}$ are related by $E_{\boldsymbol{p}}^2 = \boldsymbol{p}^2 + m^2$ and are expressed in terms of its mass, its velocity $\boldsymbol{v}$, and the corresponding parameter γ by $E_{\boldsymbol{p}} = \gamma m$ and $\boldsymbol{p} = \gamma m \boldsymbol{v}$. Thus, the transformation (1.7) can be expressed in terms of E_p and $\boldsymbol{p}$ as follows

$$L(p) = \begin{pmatrix} \dfrac{E_p}{m} & \dfrac{p^k}{m} \\ \dfrac{p^i}{m} & \delta^{ik} + \dfrac{p^i p^k}{m(m+E_p)} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} E_p & \langle \boldsymbol{p}| \\ |\boldsymbol{p}\rangle & m\mathbb{1} + \dfrac{1}{m+E_p} |\boldsymbol{p}\rangle \langle \boldsymbol{p}| \end{pmatrix}, \tag{1.13}$$

the relation to the previous parameterization (1.11) being

$$\cosh\lambda = \frac{E_p}{m}, \quad \sinh\lambda = \frac{|\boldsymbol{p}|}{m}.$$

The form (1.13) is particularly well adapted for an interpretation. For instance, $L(\boldsymbol{p})$, when applied to the 4-vector $(m, \mathbf{0})$, gives the 4-vector $p = (E_p, \boldsymbol{p})$,

$$L(\boldsymbol{p})(m, \mathbf{0}) = (E_p, \boldsymbol{p}). \tag{1.14}$$

This shows that $L(\boldsymbol{p})$ indeed "boosts" the particle from a state of rest (with respect to an inertial frame of reference) to its instantaneous 4-momentum $p = (E_p, \boldsymbol{p})$. Conversely, the inverse transformation is $L(-\boldsymbol{p})$ and takes the particle from a state with 4-momentum $p = (E_p, \boldsymbol{p})$ back to the state of rest $(m, \mathbf{0})$. (This is the *active* interpretation of the transformation. In a *passive* interpretation, i.e. one where the particle's state of motion remains unchanged, $L(\boldsymbol{p})$ and $L(-\boldsymbol{p})$ relate the instantaneous (co-moving) rest frame of the particle and the inertial frame.) Of course, $L(\boldsymbol{p})$ may also be used to boost any other Lorentz 4-vector that is needed in the description of an elementary particle. For instance, if the particle is massive (i.e. $m \neq 0$), and carries spin whose expectation value is assigned the 4-vector $(0, \boldsymbol{s})$, in the particle's rest frame, that same vector, when expressed with respect to the inertial frame, is

$$L(\boldsymbol{p})(0, \boldsymbol{s}) = \left(\frac{1}{m} \boldsymbol{p} \cdot \boldsymbol{s}, \boldsymbol{s} + \frac{(\boldsymbol{p} \cdot \boldsymbol{s})}{m(E_p + m)} \boldsymbol{p} \right). \tag{1.15}$$

Denoting the 4-vector that describes the expectation value of the spin by s, with $s = (0, \boldsymbol{s})$ in the rest frame and $s = L(\boldsymbol{p})(0, \boldsymbol{s})$ as calculated in (1.15), it is easy to verify the following, Lorentz invariant, properties

$$s^2 = -1, \quad s \cdot p = 0. \tag{1.16}$$

This follows from the fact that the scalar products s^2 and $s \cdot p$ are Lorentz invariant, and, indeed, have the values $-\mathbf{s}^2 = -1$ and 0, respectively, in the rest system. Alternatively, one may wish to confirm by direct calculation from (1.15), using $\mathbf{p}^2 = E_p^2 - m^2$,

$$\begin{aligned} s^2 &= \frac{1}{m^2} (\mathbf{p} \cdot \mathbf{s})^2 - \left\{ \mathbf{s} + \frac{(\mathbf{p} \cdot \mathbf{s})}{m(E_p + m)} \mathbf{p} \right\}^2 \\ &= (\mathbf{p} \cdot \mathbf{s})^2 \left\{ \frac{1}{m^2} - 2 \frac{1}{m(E_p + m)} - \frac{\mathbf{p}^2}{m^2 (E_p + m)^2} \right\} - \mathbf{s}^2 = -\mathbf{s}^2, \\ s \cdot p &= (\mathbf{p} \cdot \mathbf{s}) \left\{ \frac{E_p}{m} - 1 - \frac{\mathbf{p}^2}{m(E_p + m)} \right\} = 0. \end{aligned}$$

Note that if the spin expectation value $\boldsymbol{s}$ is meant to describe a particle which is only partially polarized then $\boldsymbol{s}^2 < 1$, or $(-s^2) < 1$.

(iv) Theorem (I) shows that every proper, orthochronous, Lorentz transformation depends on six parameters: the rotation is characterized by three angles which may be taken to be the angles of rotation about the three Cartesian coordinate axes, or, alternatively, to be Euler angles (see below); the special Lorentz transformation is determined by the three components of the velocity $\boldsymbol{v}$, or,

alternatively, by the direction $\hat{\boldsymbol{v}}$ into which the boost acts (fixed by its two polar angles) and the rapidity parameter λ. Accordingly, the Lie algebra of $\mathrm{L}_+^\uparrow$ has six generators, three for rotations, and three for special Lorentz transformations.

(v) Partial derivatives with respect to the components of the contravariant variable $x^\mu = (x^0, \boldsymbol{x})$ are written as follows

$$\partial_\mu := \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right), \tag{1.17}$$

with $\partial_0 = \partial/\partial x^0$, $\partial_i = \partial/\partial x^i$. Thus, ∂_μ is a generalization to the four dimensions of Minkowski space of the well-known gradient operator, its spatial part being the ordinary nabla operator. The notation ∂_μ with a *lower* index becomes clear (and also easier to remember) if one considers the derivative of a Lorentz *invariant* such as $(x \cdot p)$:

$$\partial_\mu (x \cdot p) = \frac{\partial}{\partial x^\mu}(x^\nu p_\nu) = p_\mu .$$

The result is indeed a covariant vector.

The 4-divergence of a vector field $A^\mu(x) = (A^0, \boldsymbol{A}(x))$ is

$$\partial_\mu A^\mu(x) = \frac{\partial}{\partial x^0} A^0(x) + \nabla \cdot \boldsymbol{A}(x). \tag{1.18}$$

Note the relative plus sign in this expression. Likewise, the partial derivative with repect to the covariant variable x_μ gives a contravariant result, so that it should be written with an *upper* index, viz.

$$\partial^\mu := \frac{\partial}{\partial x_\mu} = g^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial x^0}, -\nabla\right), \tag{1.19}$$

with a minus sign in its spatial part. Thus, if ∂_μ and ∂^μ are contracted, we obtain a Lorentz invariant operator, the four-dimensional analogue of the Laplace operator,

$$\Box := \partial_\mu \partial^\mu = \frac{\partial^2}{(\partial x^0)^2} - \Delta, \tag{1.20}$$

that appears, e.g., in the wave equation

$$\Box \psi(x) = \frac{\partial^2 \psi(x)}{(\partial x^0)^2} - \Delta \psi(x) = 0 \tag{1.21}$$

and in the Klein–Gordon equation

$$(\Box + m^2)\phi(x) = 0. \tag{1.22}$$

1.1.2 The Relation Between $\mathrm{L}_+^\uparrow$ and the Special Linear Group in Two Complex Dimensions

We begin by working out the precise relationship between $\mathrm{L}_+^\uparrow$, the proper orthochronous Lorentz group, and SL(2, $\mathbb{C}$). This we do by establishing the following correspondence between the four-vectors x^μ in the space–time continuum and the hermitean matrices in two complex dimensions.

To any four-vector $x = \{x^0, \boldsymbol{x}\} \equiv \{x^\mu\}$ let there correspond a two-dimensional hermitean matrix X which is constructed as follows:

$$X_{ik} := x^0 \delta_{ik} + \boldsymbol{x}(\boldsymbol{\sigma})_{ik} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{1.23}$$

The symbol $\boldsymbol{\sigma}$ denotes the three Pauli matrices[3]

$$\boldsymbol{\sigma}^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^{(2)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.24}$$

Equation (1.23) can be written in a somewhat more compact notation if we define

$$\boldsymbol{\sigma}^0 := \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.25}$$

and introduce the contravariant or covariant set of four matrices, respectively,

$$\sigma^\mu := \{\boldsymbol{\sigma}^0, -\boldsymbol{\sigma}\}. \tag{1.26a}$$

$$\sigma_\mu := g_{\mu\nu}\sigma^\nu = \{\boldsymbol{\sigma}^0, \boldsymbol{\sigma}\}, \tag{1.26b}$$

so that (1.23) reads

$$X = x^0 \mathbb{1} + \boldsymbol{x} \cdot \boldsymbol{\sigma} = x_\mu \sigma^\mu = x^\mu \sigma_\mu. \tag{1.27}$$

A first observation is that the invariant norm of x^μ, $x^2 = (x^0)^2 - (\boldsymbol{x})^2$, is identical with the determinant of X.

$$\det X = (x^0)^2 - (x)^2. \tag{1.28}$$

[3] We write the "number" of any Pauli matrix (1.24) in parentheses to avoid confusion with upper and lower indices as in (1.26).

The prescription (1.23) or (1.27) establishes an *isomorphism* between the space of the vectors x^μ (Minkowski space) and the space of the two-dimensional hermitean matrices $\mathscr{H}(2)$.

Let us consider a Lorentz transformation $\Lambda \in \mathrm{L}_+^\uparrow$, (i.e., a proper, orthochronous Lorentz transformation) that takes vector x into vector y,

$$\Lambda : x \mapsto y, \qquad y = \Lambda x.$$

The connection between the corresponding matrices $Y = y_\mu \boldsymbol{\sigma}^\mu$ and $X = X_\nu \sigma^\nu$ must be of the form, recalling that X, Y are hermitean,

$$Y = AXA^\dagger, \tag{1.29}$$

where A is a nonsingular complex 2×2 matrix. $A^\dagger$ denotes the hermitean conjugate matrix. In order to ensure invariance of the norm $y^2 = x^2$, we impose the condition[4]

$$\det A = 1. \tag{1.30}$$

The 2×2 complex matrices with determinant 1 form a group: the *special linear group* (or: *unimodular group*) in two complex dimensions, denoted by SL $(2, \mathbb{C})$. The matrix A, (1.29) is a function of the Lorentz transformation Λ. Actually, if $A(\Lambda)$, is a representative of Λ in $\mathscr{H}(2)$ then so is $-A(\Lambda)$, since the relation (1.29) is invariant under this change of sign and since $-A(\Lambda)$ belongs to SL$(2, \mathbb{C})$, to. The precise correspondence between the group of proper orthochronous Lorentz transformations $\mathrm{L}_+^\uparrow$ and SL$(2, \mathbb{C})$ is established by the following two theorems.

Theorem (II): For any $\Lambda \in \mathrm{L}_+^\uparrow$ there exists a matrix $A(\Lambda) \in \mathrm{SL}(2, \mathbb{C})$ such that if

$$y = \Lambda x \text{ then} \quad Y = A(\Lambda) X A^\dagger(\Lambda),$$

A is determined up to a sign.

Conversely,

Theorem (III): To any $A \in \mathrm{SL}(2, \mathbb{C})$ corresponds a unique $\Lambda \in \mathrm{L}_+^\uparrow$. This Λ is the image of both A and $-A$.

These theorems express the fact that the mapping of SL$(2, \mathbb{C})$ onto $\mathrm{L}_+^\uparrow$ is a homomorphism. Furthermore, as the set of elements $Z = \{\mathbb{1}, -\mathbb{1}\}$ forms an invariant subgroup of SL$(2, \mathbb{C})$, i.e.

$$AZA^{-1} = Z \quad \forall A \in \mathrm{SL}(2,\mathbb{C}), \tag{1.31}$$

[4]Equation (1.29) only requires $|\det \mathrm{A}| = 1$. However, as $\mathrm{A}(\Lambda)$ must be deformable continuously into $\pm$ the unit matrix, we must require condition (1.30).

it is useful to introduce the factor group denoted SL(2, $\mathbb{C}$)/Z. This is the group of the cosets $\{AZ\}$ of the invariant subgroup Z. Here is a short reminder of these notions: A given group G has an invariant subgroup H if for all $g \in G$ one has $gHg^{-1} = H$. That is to say, to every $h_1 \in H$ and every $g \in G$ there corresponds another element $h_2 \in H$ such that $h_1 g = g h_2$. One considers the cosets (gH) and shows that these also form a group. This group is called the *factor group* and is denoted by G/H. Indeed, the composition is defined to be multiplication of cosets,

$$(g_1 H)(g_2 H) = g_1(Hg_2)H = g_1(g_2 H)H = (g_1 g_2)H;$$

Associativity is guaranteed by the associativity of group multiplication within G. The subgroup H as a whole plays the role of the unity, viz.

$$H(gH) = (Hg)H = (gH)H = g(HH) = gH.$$

Finally, the inverse of the element gH is $g^{-1}H$. Thus, theorems (II) and (III) establish the isomorphism

$$\mathrm{L}_+^{\uparrow} = \mathrm{SL}(2,\mathbb{C})/\{1, -1\} \tag{1.32}$$

between the proper orthochronous Lorentz group and the factor group SL(2, $\mathbb{C}$)/Z.

Theorem III is obvious from (1.29, 1.30). Indeed, any $A \in \mathrm{SL}(2, \mathbb{C})$ induces a transformation $X \to Y$ such that $y^2 = x^2$. This means that x and y are related by a Lorentz transformation $\Lambda(A)$. As A can be deformed continuously into the unit element, Λ must belong to $\mathrm{L}_+^{\uparrow}$.

The explicit calculation of Λ from a given A is somewhat technical and shall not be worked out here.[5]

The proof of theorem (II) proceeds by explicit construction of $A(\Lambda)$ for given $\Lambda \in \mathrm{L}_+^{\uparrow}$. According to theorem (I) Λ can be decomposed, in a unique way, into a boost $L(\boldsymbol{v})$, followed by a rotation R, $\Lambda = R \cdot L(\boldsymbol{v})$. Thus it is sufficient to consider these two kinds of transformations separately. In the following we denote the image of R in SL(2, $\mathbb{C}$) by $U(R)$, the image of $L(\boldsymbol{v})$ by $H(\boldsymbol{v})$, in order to underline the fact that $U(R)$ will turn out to be unitary, while $H(\boldsymbol{v})$ will be found to be hermitean.

(i) *Rotations*. It is useful to parametrize the rotation R by a set of Euler angles ϕ, θ, ψ which we define as indicated in Fig. 1.1. One then verifies, through explicit calculation, that the matrices

$$R \leftrightarrow \pm U(R), \tag{1.33}$$

with

$$U(R) = \mathrm{e}^{(\mathrm{i}/2)\psi\sigma^{(3)}} \quad \mathrm{e}^{(\mathrm{i}/2)\theta\sigma^{(2)}} \quad \mathrm{e}^{(\mathrm{i}/2)\phi\sigma^{(3)}} \tag{1.34}$$

[5]This construction can be found for example in [RUE70].

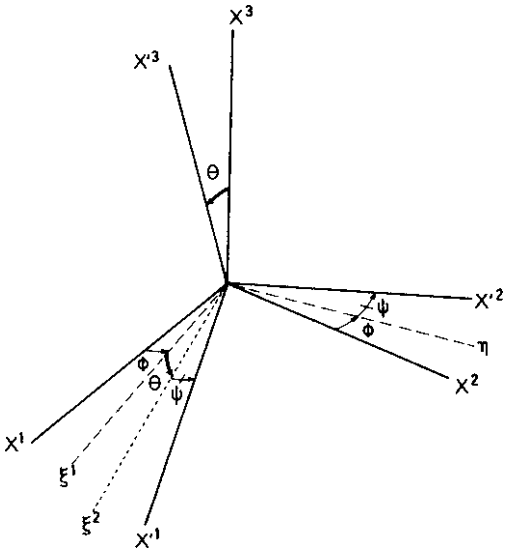

Fig. 1.1 Definition of Euler's angles. First a rotation by the angle ϕ about the x^3-axis, then a rotation by the angle θ about the η-axis, and finally a rotation by the angle ψ about the x'^3-axis

do indeed represent the rotation R. For instance, for a rotation about the 3-axis one has

$$Y = AXA^\dagger = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$
$$= \begin{pmatrix} x^0 + x^3 & e^{i\phi}(x^1 - ix^2) \\ e^{-i\phi}(x^1 + ix^2) & x^0 - x^3 \end{pmatrix},$$

which can be rewritten in the more familiar form

$$y^0 = x^0, \quad y^1 = x^1 \cos\phi + x^2 \sin\phi,$$
$$y^3 = x^3 \quad y^2 = -x^1 \sin\phi + x^2 \cos\phi.$$

Similarly, for a rotation about the 2-axis we use

$$e^{(i/2)\theta\sigma^{(2)}} = \mathbb{1} \cos\theta/2 + i\sigma^{(2)} \sin\theta/2 = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

and verify that $\exp\{(i/2)\theta\sigma^{(2)}\}\ X\ \exp\{-(i/2)\theta\sigma^{(2)}\}$, when written out in terms of Cartesian coordinates, does indeed yield the expected result:

$$y^0 = x^0, \quad y^1 = x^1 \cos\theta - x^3 \sin\theta,$$
$$y^2 = x^2, \quad y^3 = x^3 \cos\theta + x^1 \sin\theta.$$

(The reader should verify this,)

As the composition of three Euler rotations $R_3(\psi)R_2(\theta)R_3(\phi)$ is the most general case, we have thus shown that for any rotation $\Lambda = R$ there exists a matrix $U(R) \in \mathrm{SL}(2, \mathbb{C})$. This matrix is given by expression (1.34).

In fact, $U(R)$ is *unitary* and, therefore, belongs to SU(2), the special group of unitary transformations U in two dimensions. SU(2) is a subgroup of SL(2, $\mathbb{C}$). This is no surprise when one recalls that the rotation group SO(3) is a subgroup of $\mathrm{L}_+^\uparrow$. Thus we recover the well-known isomorphism

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\mathbb{1}, -\mathbb{1}\} \tag{1.35}$$

that is the basis for constructing the representations of the rotation group.[6]

(ii) *Special Lorentz transformations*. Let $\boldsymbol{v}$ be the velocity vector characterizing the "boost" $L(\boldsymbol{v})$. Then $\hat{\upsilon} := \boldsymbol{v}/|\boldsymbol{v}|$ is the unit vector in the direction of the boost, and according to (1.11)

$$\boldsymbol{v} = \hat{\upsilon} \operatorname{tgh} \lambda, \tag{1.36}$$

where λ is the rapidity parameter. The image of $L(\boldsymbol{v})$ is

$$L(\boldsymbol{v}) \leftrightarrow \pm H(\boldsymbol{v}) \tag{1.37}$$

with

$$H(\boldsymbol{v}) = \mathrm{e}^{(1/2)\lambda\sigma\cdot\hat{\upsilon}}$$

This can be seen as follows: By an appropriate rotation of the space coordinate, choose the 3-direction to coincide with $\widehat{\upsilon}$. Then

$$H(\boldsymbol{v}) = \begin{pmatrix} \mathrm{e}^{\lambda/2} & 0 \\ 0 & \mathrm{e}^{-\lambda/2} \end{pmatrix}$$

and

$$Y = HXH^\dagger = \begin{pmatrix} \mathrm{e}^{\lambda/2} & 0 \\ 0 & \mathrm{e}^{-\lambda/2} \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\lambda/2} & 0 \\ 0 & \mathrm{e}^{-\lambda/2} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{e}^{\lambda}(x^0 + x^3) & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & \mathrm{e}^{-\lambda}(x^0 - x^3) \end{pmatrix},$$

[6] See e.g. [HAM62].

so that

$$y^1 = x^1, \quad y^0 = x^0 \cosh\lambda + x^3 \sinh\lambda,$$
$$y^2 = x^2, \quad y^3 = x^0 \sinh\lambda + x^3 \cosh\lambda,$$

as expected. So $\pm H(\boldsymbol{v})$ of (1.37) is indeed the image of the given boost $L(\boldsymbol{v})$.

Using the known properties of the Pauli matrices we can also write

$$H(\boldsymbol{v}) = \mathbb{1}\cosh(\lambda/2) + \boldsymbol{\sigma}\cdot\hat{v}\sinh(\lambda/2). \tag{1.38}$$

Note, in particular, that $H(\boldsymbol{v})$ is a linear combination of the hermitean matrices $\{\sigma^\mu\}$, with real coefficients, and, therefore, is *hermitean*.

In summary: Any Lorentz transformation $\Lambda \in \mathrm{L}_+^\uparrow$ can be decomposed uniquely into a rotation followed by a boost,

$$\Lambda = L(\boldsymbol{v})\cdot\mathscr{R},$$

with $\mathscr{R}$ and $\boldsymbol{v}$ obtained as described in theorem (I). Its image under the isomorphism (1.32) is $\pm A(\Lambda) \in \mathrm{SL}(2,\mathbb{C})$, where

$$A(\Lambda) = H(\boldsymbol{v})U(R), \tag{1.39}$$

and where $U(R)$ and $H(\boldsymbol{v})$ are given by (1.34) and (1.38), respectively. Note that we have recovered the fact that any matrix of SL(2, $\mathbb{C}$) can be written as a product of a hermitean matrix with determinant 1 and a unitary matrix. This is the generalization of the well-known decomposition of a complex number into its modulus (which is a real number) and a phase factor.

1.2 Spinor Representations and Spinor Fields, "Dotted" and "Undotted" Spinors

We can now proceed to an explict construction of spinor representations of SL(2, $\mathbb{C}$) and, thereby, spinor representations of the Lorentz group. By definition, a spinor is a two component object which transforms under rotations of the space coordinates with the unitary matrix $D^{(1/2)}(\psi,\ \theta,\ \phi)$. More precisely, let χ_m ($m = +1/2$ or $-1/2$) denote the basis of an irreducible representation of the rotation group with $j = 1/2$. Any state vector ψ in this space can be expanded in terms of this basis,

$$\psi = \sum_m c_m \chi_m. \tag{1.40}$$

In accordance with our conventions, the transformation law for the expansion coefficients is

$$c'_\mu = \sum_m D^{(1/2)}_{\mu m}(\psi, \theta, \phi) c_m, \tag{1.41}$$

while the base transforms with the transformation matrix $(D^{\mathrm{T}})^{-1}$, contragredient to D[7]. As $(D^{\mathrm{T}})^{-1} = D^*$, we have

$$\chi'_\mu = \sum_m D^{(1/2)*}_{\mu m} \chi_m. \tag{1.42}$$

Thus, invariance of the physical state vector ψ (1.40) is guaranteed, as verified by the explicit calculation,

$$\sum c'_\mu \chi'_\mu = \sum_{m,m'} \sum_\mu D^{(1/2)}_{\mu m} (D^{(1/2)})^\dagger_{m'\mu} c_m \chi_{m'} = \delta_{mm'} c_m \chi_{m'} = c_m \chi_m .$$

With respect to SL(2, $\mathbb{C}$) we already noted that there are two inequivalent spinor representations of SL(2, $\mathbb{C}$). It is not difficult to construct these representations on the basis of the results of the preceding paragraph. Let $A(\Lambda)$ be the image of an arbitrary Lorentz transformation $\Lambda \in \mathrm{L}^{\uparrow}_{+}$. One sees at once that there are four types of spinors, whose behaviour under a given Lorentz transformation is indicated in the following table[8]:

Spinors	Transformation matrix under L.T.	
$\{c_a\} = \{c_1, c_2\}$	$A(\Lambda)$	(1.43a)
$\{c^a\} = \{c^1, c^2\}$	$(A^T)^{-1}(\Lambda)$	(1.43b)
$\{c_A\} = \{c_{\mathrm{I}}, c_{\mathrm{II}}\}$	$A^*(\Lambda)$	(1.43c)
$\{c^A\} = \{c^{\mathrm{I}}, c^{\mathrm{II}}\}$	$\hat{A} := (A^{*T})^{-1} = (A^\dagger(\Lambda))^{-1}$	(1.43d)

As is evident, spinors $\{c^a\}$ with the transformation law (1.43b) are contragredient to spinors $\{c_a\}$ with the transformation law (1.43a). Similarly, spinors $\{c^A\}$ with transformation law (1.43d) and spinors $\{c_A\}$ with transformation law (1.43c) are contragredient to each other.

[7]The term *contragredience* is a general term in linear algebra which helps to distinguish *co*variant and *contra*variant objects. For example, in the decomposition $\mathbf{v} = \sum_v a^v e_v$ of a vector in terms of the set of base vectors $\mathbf{e} = \{e_v\}$, this set is covariant, while the set of expansion coefficients $\mathbf{a} = \{a^v\}$ is contravariant, i.e. in the example

$$\mathbf{e}' = \mathbf{A}\mathbf{e}, \quad \mathbf{a}' = (\mathbf{A}^T)^{-1}\mathbf{a}, \quad \text{so that}$$
$$\mathbf{v} = \mathbf{a}' \cdot \mathbf{e}' = \big((\mathbf{A}^T)^{-1}\mathbf{a}\big) \cdot (\mathbf{A}\mathbf{e}) = \mathbf{a}(\mathbf{A}^{-1}\mathbf{A})\mathbf{e} = \mathbf{a} \cdot \mathbf{e}$$

stays invariant. Expansion coefficients and base vectors are said to be contragredient to each other.

[8]We recall: A^{T} is the transposed, A^* the complex conjugate of A, and $A^\dagger = A^{\mathrm{T}*}$ is the hermitean conjugate.

If we take Λ to be a *rotation* R, i.e. if Λ belongs to the subgroup SO(3) of $\mathrm{L}_+^\uparrow$, then $A(\Lambda = R) \equiv U(R)$ is a unitary matrix. In this case we have $\hat{A} = A$, and $(A^{\mathrm{T}})^{-1} = A^*$. Therefore, we find that with respect to rotations

$$\{c_a\} \sim \{c^A\}, \quad \{c^a\} \sim \{c_A\}, \tag{1.44}$$

where the symbol $\sim$ means "transforms as". Obviously, this is not true for those Lorentz transformations Λ which include boosts and for which $\hat{A}$ is not equal to A.

From the theory of the rotation group[9] it is known that any irreducible spherical tensor of arbitrary rank k can be transformed to a contragredient one by applying to it a rotation of an angle π about the 2-axis. With standard phase conventions for the representation of angular momentum operators, the general transformation matrix to contragredience is

$$(\mathrm{e}^{i\pi J_2})_{\mu m} = D^{(\kappa)}_{\mu m}(0, \pi, 0) = (-)^{\kappa-\mu}\delta_{\mu,-m}. \tag{1.45}$$

For integer k this matrix is symmetric, for half-integer it is antisymmetric.

Specifically, for spinors ($k = 1/2$) the relation is[10]

$$c^a = (\mathrm{e}^{i\pi\sigma^{(2)}/2})^{ab}\, c_b = (\mathrm{i}\sigma^{(2)})^{ab} c_b. \tag{1.46}$$

Let us introduce the notation

$$\varepsilon = \mathrm{e}^{i\pi\sigma^{(2)}/2} = \mathrm{i}\sigma^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.47}$$

and let $\varepsilon^{ab} \equiv \varepsilon$, so that $c^a = \varepsilon^{ab} c_b$. Then evidently $\varepsilon_{ab} = \varepsilon^{-1}$ with

$$\varepsilon^{-1} = \varepsilon^T = -\varepsilon. \tag{1.48}$$

So far, the relationship (1.46) is proven only with respect to rotations. It is not difficult, however, to show that it is true also for boosts, and therefore for any $\Lambda \in \mathrm{L}_+^\uparrow$ (see exercise 1.12). In the case of SL(2, $\mathbb{C}$) spinors there is another and perhaps more direct way of seeing this, by noting that both

$$(b_1 c^1 + b_2 c^2) \quad \text{and} \quad (b_1 c_2 - b_2 c_1) = \det \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$$

[9] Se e.g. [FAR59] where this is called the "U-transformation".

[10] Summation over repeated, contragredient indices is implied.

are SL(2, $\mathbb{C}$) invariants.[11] From this observation we conclude that

$$c^1 \sim c_2 \quad \text{and} \quad c^2 \sim -c_1$$

under any $\Lambda \in \mathrm{L}_+$. Thus transition to contragredience is indeed effected by the transformation (1.46).

It is clear that there must be analogous relationships for spinors of the second kind, viz.

$$c_A = \varepsilon_{AB} c^B. \tag{1.49}$$

As c^a and c_A have the same behaviour under rotations, it is convenient to fix relative signs by taking

$$\varepsilon_{AB} = \varepsilon^{ab} \equiv \varepsilon, \tag{1.50}$$

so that the inverse transformation is $\varepsilon^{AB} = \varepsilon_{ab} = \varepsilon^{-1} = \varepsilon^T = -\varepsilon$. Using the fact that the transposed of ε is the same as its inverse, we note the relationships

$$(A^T)^{-1} = \varepsilon A \varepsilon^T, \tag{1.51a}$$

$$\hat{A} = \varepsilon A^* \varepsilon^T. \tag{1.51b}$$

From these relations one derives, in particular,

$$\varepsilon = A \varepsilon A^T, \quad \varepsilon = A^* \varepsilon A^\dagger. \tag{1.52}$$

These relations show explicitly that ε is *invariant* under all SL(2, $\mathbb{C}$) transformations. In fact, one verifies that ε is the only tensor which is invariant under SL(2, $\mathbb{C}$). We shall make use of this important fact below, in connection with the quantization of the Dirac field, cf. Sect. 1.7.

This spinor calculus was introduced by van der Waerden (van der Waerden, 1929). Following van der Waerden, spinors of the second kind are often denoted thus: $\{c_{\dot{a}} = c_{\dot{1}}, c_{\dot{2}}\}$, i.e. by giving them dotted indices, in order to distinguish them from spinors of the first kind which carry undotted indices. We find it less confusing to label them with capital letters and roman numerals instead. Thus our notation is: *small* indices and *arabic* numerals refer to spinors of the first kind ("undotted spinors"); *capital* indices and *roman* numerals refer to spinors of the second kind ("dotted spinors").

The following rather obvious remarks are useful for the sequel.

(i) Indices of a spinor or, more generally, of a tensor carrying spinor indices, can be "raised" or "lowered" in the standard manner, by multiplying the spinor (or tensor) with (appropriate products of) ε and ε^{T}, respectively.

[11] SL(2, $\mathbb{C}$) is characterized by the invariant skew-symmetric scalar product $b_a \varepsilon^{ab} c_b$ and hence is isomorphic to Sp(2, $\mathbb{C}$), the symplectic group in two complex dimensions.

(ii) Any equation containing spinors that is to be Lorentz invariant (more precisely: invariant with respect to $\mathrm{L}_+^\uparrow$) must contain the same number of small indices and the same number of capital indices on either side.

(iii) Complex conjugation means exchanging small and capital indices (compare (1.43a) with (1.43c), and (1.43b) with (1.43d)).

(iv) Only the summation over *contragredient* indices, either two small or two capital (contraction or, in German, Verjüngung) is an invariant operation. Summation over two unlike indices cannot be invariant. As a consequence tensors of type $t^{ab\cdots m;\,AB\cdots M}$ which are separately symmetric in their small as well as in their capital indices, are irreducible. Indeed all tensors of the type $\varepsilon_{ab}\, t^{ab\cdots m;\,ABcM}$ and $\varepsilon_{AB}\, t^{ab\cdots m;\,AB\cdots M}$ vanish if t has the indicated symmetry.

For a tensor it is irrelevant in which order the group of small indices and the group of capital indices are written, relative to each other. For example $t^{aB} = t^{Ba}$, $t^{ab,C} = t^{C,ab} = t^{aCb}$.

(v) Returning to the definition (1.27) we started from, it is clear that $X = \sigma_\mu x^\mu$ is such a tensor with one small and one capital index. Indeed, when

$$x \mapsto x' = \Lambda x \quad \text{then} \quad X \mapsto X' = A(\Lambda) X A^\dagger(\Lambda),$$

which means that X, when written out explicitly, carries indices as indicated here:

$$X \equiv X_{aB} = (\sigma_\mu)_{aB} x^\mu = (\sigma^\mu)_{aB} x_\mu. \tag{1.53}$$

Equipped with this knowledge we can now define two classes of *spinor fields*[12] by means of the transformation behaviour under SL (2, $\mathbb{C}$).

Spinor fields of the *first* kind are denoted by

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}. \tag{1.54a}$$

Under $\Lambda \in \mathrm{L}_+^\uparrow$ they transform according to

$$\phi(x) \mapsto \phi'_a(x' = \Lambda x) = A(\Lambda)_a{}^b \phi_b(x). \tag{1.54b}$$

Spinor fields of the *second* kind are denoted by

$$\chi(x) = \begin{pmatrix} \chi^{\mathrm{I}}(x) \\ \chi^{\mathrm{II}}(x) \end{pmatrix} \tag{1.55a}$$

and transform according to the law

[12] A spinor field is a spinor with respect to SL (2, $\mathbb{C}$) whose entries are complex functions instead of complex numbers.

$$\chi(x) \mapsto \chi'^{A}(x' = \Lambda x) = (A^*(\Lambda))^{A}{}_{B}\,\chi^{B}(x). \tag{1.55b}$$

(Note that $(A*)^{A}{}_{B} = \varepsilon^{AM} A^{*N}_{M} \varepsilon_{NB} = (A^{T*})^{-1} \equiv \hat{A}$).

The spinor fields have identical transformation behaviour under rotations R. Indeed, for $\Lambda = R$ they transform with the matrix $D^{(1/2)}$ $(\psi,\ \theta,\ \phi)$, as indicated by (1.41). They behave differently, however, under more general transformations $\Lambda \in \mathrm{L}^{\uparrow}_{+}$, for which $(A^T)^{-1}$ and A^* are not the same.

For later purpose and in order to get some practice in working with SL (2, $\mathbb{C}$) spinor fields of first and second kind, we recapitulate the remarks (i) to (v) above and write out some examples in applying them to spinor fields. According to remark (iii) complex conjugation transforms small into capital, and capital into small indices, i.e.

$$\begin{aligned} (\phi_a(x))^* &= \phi^*_A(x), \\ (\chi^A(x))^* &= \chi^{*a}(x). \end{aligned} \tag{1.56a}$$

This applies whenever the index has its normal position, i.e. is a *lower*, *small* index (i.e., a spinor of first kind), or an *upper*, *capital* index (i.e. a spinor of second kind). If that index was raised, or lowered, by means of the ε-tensor (1.50), then there is an extra minus sign which follows from the rules but should not be forgotten:

$$\begin{aligned} (\phi^a)^* &= (\varepsilon^{ab}\phi_b)^* = \varepsilon^{ab}\phi^*_B = -\varepsilon^{AB}\phi^*_B = -\phi^{*A}, \\ (\chi_A)^* &= (\varepsilon_{AB}\chi^B)^* = \varepsilon_{AB}\chi^{*b} = -\varepsilon_{ab}\chi^{*b} = -\chi^*_a. \end{aligned} \tag{1.56b}$$

This minus sign is a consequence of the convention (1.50), according to which $\varepsilon^{AB} = -\varepsilon^{ab}$. [Had we chosen the convention $\varepsilon^{AB} = \varepsilon^{ab}$ instead, there would be no minus sign here but it would appear in other places; cf. the discussion of parity, charge conjugation and time reversal in sect. 1.5 below].

Products of the kind $\phi_a\varepsilon^{ab}\phi_b$, $\chi^A\varepsilon_{AB}\chi^B$ etc. are Lorentz invariant. One shows easily that they can be written in several, equivalent ways, for instance,

$$\phi_a\varepsilon^{ab}\phi_b = \phi_a\phi^a = -\phi^b\phi_b, \quad \chi^A\varepsilon_{AB}\chi^B = \chi^A\chi_A = -\chi_B\chi^B, \tag{1.57a}$$

and similarly

$$\phi_a\chi^{*a} = -\phi^a\chi^*_a, \quad \phi^*_A\chi^A = -\phi^{*A}\chi_A. \tag{1.57b}$$

Before closing this section, we shall show that the two kinds of spinors can be related by means of space reflexion P, or equivalently, by time reversal T. By definition we have

$$P\{x^0, \boldsymbol{x}\} = \{x^0, -\boldsymbol{x}\}, \tag{1.58a}$$

$$T\{x^0, \boldsymbol{x}\} = \{-x^0, \boldsymbol{x}\}. \tag{1.58b}$$

Let $X = \sigma^\mu\, x_\mu = \sigma_\mu x^\mu$ and $X^P = \sigma_\mu\ (P\ x)^\mu$, then it follows that

$$XX^P = (x^0\mathbb{1} + \boldsymbol{x}\cdot\boldsymbol{\sigma})(x^0\mathbb{1} - \boldsymbol{x}\cdot\boldsymbol{\sigma}) = (x^0)^2 - \boldsymbol{x}^2 = x^2. \tag{1.59}$$

One verifies easily that, if we define the set $\hat{\sigma}^{\mu}$ in analogy to $\hat{A}$, (1.51)[13], then

$$\hat{\sigma}^{\mu} = \varepsilon \sigma^{\mu *} \varepsilon^{-1} = \sigma_{\mu} = (\mathbb{1}, \boldsymbol{\sigma}) \tag{1.60}$$

and, therefore, that

$$\varepsilon X^{*} \varepsilon^{-1} = X^{P}.$$

Under a Lorentz transformation Λ_A, we have $Y = \sigma_{\mu}(\Lambda_A x)^{\mu} = A\ X\ A^{\dagger}$, and, since $\hat{A} = \varepsilon A * \varepsilon^{-1}$, $Y^{P} = \hat{A} X^{P} \hat{A}^{\dagger}$. This last relation, which can also be written as follows (using $P^2 = 1$).

$$\sigma_{\mu}((P\Lambda_A P)Px)^{\mu} = \hat{A}\sigma_{\mu}(Px)^{\mu}\hat{A}^{\dagger},$$

tells us that the Lorentz transformation $\Lambda_{\hat{A}}$ pertaining to $\hat{A}$, and the original Lorentz transformation Λ_A are related by the equation

$$\Lambda_{\hat{A}} = P\Lambda_A P. \tag{1.61}$$

Finally, by defining $X^{T} := \sigma_{\mu}(TX)^{\mu}$ and observing that $X^{T} = -X^{P}$, we find in a similar manner

$$\Lambda_{\hat{A}} = T\Lambda_A T. \tag{1.62}$$

An important remark is the following. These relations show that space reflexion and time reversal relate spinors of the first kind to spinors of the second kind and vice versa. On the other hand, we know that we can reach any homogeneous Lorentz transformation by multiplying $\mathrm{L}_{+}^{\uparrow}$ by *P, T* and *PT*. Therefore, in order to construct a spinor equation that is to be covariant under the *full* Lorentz group, we shall need spinor fields of *both* kinds.

The exact relationship between $\phi_a(x)$ and $\chi^{B}(x)$ under space reflection, a priori, is fixed only within a phase factor $\mathrm{e}^{i(\pi/2)n}$. This is so because we have always the freedom to combine the operation of space reflection with a complete rotation by 2π. Whilst this makes no difference for integer spin, it yields an extra minus sign in the case of half-integral spin. Therefore, space reflection applied twice to a spinor field can be chosen to yield plus or minus that same field. The only restriction is that the same phase convention must be chosen for all spinors, in order to ascertain the correct transformation behaviour of bilinears in the spinor fields.

These remarks illustrate the fact that it is not meaningful to assign an *absolute* intrinsic parity to a fermion. Only relative parities can be physically relevant. We come back to this below.

[13]Note, however, that whilst $\sigma^0 = \mathbb{1}$ belongs to SL (2, $\mathbb{C}$) the Pauli matrices $\boldsymbol{\sigma}$ do not.

1.3 Dirac Equation for Free Particles

As a more technical preparation to what follows, we recall the definitions (1.25, 1.26, 1.60):

$$\sigma^\mu := (\sigma^0, -\boldsymbol{\sigma}), \quad \hat{\sigma}^\mu := \varepsilon \sigma^{\mu *} \varepsilon^{-1} = (\sigma^0, \boldsymbol{\sigma}),$$

where σ^0 is the two-dimensional unit matrix and $\boldsymbol{\sigma} = \{\sigma^{(i)}\}$ is a shorthand notation for the Pauli matrices (1.24). With the aid of the well-known relation

$$\sigma^{(i)}\sigma^{(j)} = \delta^{ij} + i \sum_k \varepsilon^{ijk}\sigma^{(k)} \tag{1.63}$$

we verify the important relationship

$$(\sigma^\mu)_{aB}(\hat{\sigma}^\nu)^{Bc} + (\sigma^\nu)_{aB}(\hat{\sigma}^\mu)^{Bc} = 2g^{\mu\nu}\delta_a^c. \tag{1.64}$$

Furthermore, we can define derivatives of spinors, in analogy to (1.53), by introducing[14]

$$\partial_{aB} := (\sigma_\mu \partial^\mu)_{aB} = (\sigma^\mu \partial_\mu)_{aB} = 1\partial_0 - \boldsymbol{\sigma} \cdot \nabla. \tag{1.65}$$

The nature and position of indices are the same as in (1.53) since $\partial^\mu = \partial/\partial x_\mu$ behaves like x^μ under Lorentz transformations and, therefore, with $x' = \Lambda x$, we have

$$(\sigma^\mu \partial'_\mu) = A(\Lambda)(\sigma^\mu \partial_\mu)A^\dagger(\Lambda),$$

We can also construct the matrix $\hat{\sigma}^\mu \partial_\mu$ by multiplying the complex conjugate of (1.65) from the left by ε and from the right by ε^{-1}. Complex conjugation converts small into capital indices and vice versa, while ε raises indices. As a result, it follows that $\hat{\sigma}^\mu \partial_\mu$ carries indices as indicated here:

$$\partial^{Ab} := (\hat{\sigma}^\mu \partial_\mu)^{Ab} = \mathbb{1}\partial^0 + \boldsymbol{\sigma} \cdot \nabla. \tag{1.66}$$

Making use of relation (1.64) we note that

$$\partial^{Ab}\partial_{bC} = \delta^A_C \partial_\mu \partial^\mu = \delta^A_C \Box \tag{1.67}$$

and similarly

$$\partial_{aB}\partial^{Bc} = \delta^c_a \Box, \tag{1.67'}$$

where $\Box = \partial_\mu \partial^\mu = (\partial^0)^2 - \Delta$ is the d'Alembert or four-dimensional Laplace operator (1.20).

Equipped with these tools we can now proceed to derive the Dirac equation.

[14] Note the minus sign in front of $\boldsymbol{\sigma} \cdot \nabla$ which is due to $\partial_\mu = (\partial^0, \nabla)$ being the covariant derivative (17).

A relativistic wave equation (more precisely: a system of equations) that is to provide a quantum-mechanical description of free spin-1/2 particles has to meet the following conditions:

(i) The equation must be linear, homogeneous and of first order in the time derivative. Linearity is imposed by the superposition principle; homogeneity is relevant for the force-free case: the equation does not contain external sources (inhomogeneous terms); first-order time derivatives are necessary if a probability interpretation is to be possible.

(ii) The equation must be covariant under the full Lorentz group, including space and time reversal. As we have noted above, this implies that it must contain both spinor fields of the first kind $\phi_a(x)$ and, spinor fields of the second kind $\chi^A(x)$.

(iii) In order to ensure the correct relativistic energy–momentum relation, both kinds of fields must obey the Klein–Gordon equation for the mass m of the particles

$$(\Box + m^2)\phi_a(x) = 0, \tag{1.68a}$$

$$(\Box + m^2)\chi^B(x) = 0. \tag{1.68b}$$

The simplest system of linear homogeneous differential equations of first order which meets these conditions reads

$$\mathrm{i}(\sigma^\mu \partial_\mu)_{aB}\chi^B(x) = m\phi_a(x), \tag{1.69a}$$

$$\mathrm{i}(\hat{\sigma}^\mu \partial_\mu)^{Bc}\phi_c(x) = m\chi^B(x). \tag{1.69b}$$

This set of four coupled differential equations constitutes what is called the *Dirac equation*(*s*) for the force-free case. It describes the free motion of spin-1/2 particles of arbitrary mass m, including the limit of mass zero, in which case (1.69) reduce to what are called the Weyl equations.

Let us then verify that the Dirac equation does indeed obey the conditions listed above. First, it is evident that (1.69) are covariant under $\mathrm{L}_+^\uparrow$, by construction. They are also invariant under parity P and time reversal T, as these operations transform χ into ϕ, $(\sigma^\mu \partial_\mu)$ into $(\hat{\sigma}^\mu \partial_\mu)$ and vice versa. (The details are worked out below in Sect. 1.5.) Thus they are covariant with respect to the *full* Lorentz group.

Second, it is easily verified that each component $\phi_a(x)$ satisfies the Klein–Gordon equation. Indeed, applying the operator $\mathrm{i}(\sigma^\mu \partial_\mu)$ to (1.69b) we obtain for the left-hand side, using (1.67'),

$$-(\sigma^\mu \partial_\mu)_{aB}(\hat{\sigma}^\nu \partial_\nu)^{Bc}\phi_c(x) = -\delta_a^c \Box \phi_c(x) = -\Box \phi_a(x).$$

For the right-hand side we make use of (1.69a),

$$m\mathrm{i}(\sigma^\mu \partial_\mu)_{aB}\chi^B(x) = m^2\phi_a(x).$$

Thus we obtain $\Box\phi_a(x) + m^2\,\phi_a(x) = 0$.

Similary, by applying the operator $\mathrm{i}(\hat{\sigma}_\mu\partial^\mu)$ to (1.69a) we verify that the components $\chi^B(x)$ also satisfy the Klein–Gordon equation.

It is customary and useful to group the two spinors $\phi_a(x)$ and $x^B(x)$ together into one four-component spinor, a so-called Dirac spinor,

$$\psi(x) := \begin{pmatrix} \phi_a(x) \\ \chi^B(x) \end{pmatrix} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \chi^{\mathrm{I}}(x) \\ \chi^{\mathrm{II}}(x) \end{pmatrix}. \tag{1.70}$$

Equations (1.69) can then be written in a more compact form, upon introduction of a set of four 4×4 matrices called Dirac matrices,

$$\gamma^\mu := \begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix}, \tag{1.71}$$

$$(\mathrm{i}\gamma^\mu\partial_\mu - m\mathbb{1}_4)\psi(x) = 0, \tag{1.72}$$

where $\mathbb{1}_4$ is now the four-dimensional unit matrix. For later purposes it will be convenient to define one further matrix

$$\gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.73}$$

(which is also written γ^5, without distinction as to the position of the index 5).

With σ^μ and $\hat{\sigma}^\mu$ as defined before, we have

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & -\sigma^{(i)} \\ \sigma^{(i)} & 0 \end{pmatrix},\ \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{1.74}$$

From (1.64) we obtain the important anticommutator relations for Dirac matrices

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}_4, \tag{1.75}$$

which imply, in particular, that

$$(\gamma^0)^2 = \mathbb{1}_4, \quad (\gamma^i)^2 = -\mathbb{1}_4 \quad (i = 1, 2, 3).$$

Writing out explicity the spinor indices, the matrices γ^μ of (1.71) have the structure

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{aA} \\ (\hat{\sigma}^\mu)^{Aa} & 0 \end{pmatrix}. \tag{1.71'}$$

In particular if we multiply the four γ-matrices to form γ_5, as indicated in the definition (1.73), we find

$$\gamma_5 = \mathrm{i}\begin{pmatrix} (\sigma^0\hat{\sigma}^1\sigma^2\hat{\sigma}^3)_a^b & 0 \\ 0 & (\hat{\sigma}^0\sigma^1\hat{\sigma}^2\sigma^3)^A{}_B \end{pmatrix} = \begin{pmatrix} \delta_a^b & 0 \\ 0 & -\delta_B^A \end{pmatrix}. \tag{1.73'}$$

This explicit representation (1.73') of γ_5 leads us to two important remarks:

(i) According to (1.73') γ_5 maps the spinor $(\phi_a,\ \chi^A)^T$ onto the spinor $(\phi_a, -\chi^A)^T$. If we compare this to the action of the unit matrix

$$\mathbb{1} = \begin{pmatrix} \delta_a^b & 0 \\ 0 & \delta_B^A \end{pmatrix}$$

we see that the combinations

$$\frac{1}{2}(\mathbb{1} + \gamma_5) = \begin{pmatrix} \delta_a^b & 0 \\ 0 & 0 \end{pmatrix},\ \frac{1}{2}(\mathbb{1} - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_B^A \end{pmatrix} \tag{1.76}$$

project onto spinors of the first kind and spinors of the second kind, respectively.

(ii) From what we said above it is clear that it would not be meaningful to distinguish between γ_5 with upper or lower index.

From (1.71') and (1.73') we see that the structure of a general matrix M that maps the spinor $\psi = (\phi_b,\ \chi^B)^T$ onto another spinor $\psi' = \left(\phi'_a,\ \chi'^A\right)^T$ must be the following:

$$M = \begin{pmatrix} M_a{}^b & M_{aB} \\ M^{Ab} & M^A{}_B \end{pmatrix}.$$

This is important to know when we wish to form invariants, or Lorentz covariants (i.e. vectors, tensors etc.), in terms of ψ and its hermitean conjugate $\psi^\dagger$. Actually, $\psi^\dagger$ is not the appropriate, conjugate spinor because of its spinorial structure which is

$$\psi^\dagger(x) = (\phi^*{}_A(x) \qquad \chi^{*a}(x))$$

and which cannot be contracted with $M\psi(x)$. On the other hand, we know that, for instance, $\chi^{*a}\phi_a$ and $\phi^*{}_{\mathrm{A}}\ \chi^A$ are invariants, cf. (1.57). The correct position of the indices is achieved if instead of $\psi^\dagger$ we introduce

$$\begin{aligned} \overline{\psi(x)} &:= (\phi_A^*\ \chi^{*a}) \begin{pmatrix} 0 & \delta_B^A \\ \delta_a^b & 0 \end{pmatrix} \\ &= (\chi^{*b}(x)\ \phi_B^*(x)), \end{aligned} \tag{1.77}$$

This can also be written as follows:

$$\begin{aligned} \overline{\psi(x)} &= (\phi_A^*(\hat{\sigma}^0)^{Ac} \quad \chi^{*a}(\sigma^0)_{aC}) \begin{pmatrix} 0 & (\sigma^0)_{cB} \\ (\hat{\sigma}^0)^{Cb} & 0 \end{pmatrix} \\ &\equiv (\phi^{*C} \quad \chi^{*C})\gamma^0. \end{aligned}$$

We then see that

$$\overline{\psi(x)}\psi(x) = \chi^{*a}\phi_a + \phi^*_A\chi^A,$$
$$\overline{\psi(x)}\gamma_5\psi(x) = \chi^{*a}\phi_a - \phi^*_A\chi^A$$

are invariant under proper, orthochronous Lorentz transformations. We note that $\overline{\psi(x)}\psi(x)$ is even, $\overline{\psi(x)}\gamma_5\psi(x)$ is odd under space reflection (see the discussion at the end of the previous section), so that the first is a Lorentz *scalar*, the second a Lorentz *pseudoscalar*. Similarly

$$\overline{\psi(x)}\gamma^\alpha\psi(x) = \chi^{*a}(\sigma^\alpha)_{aB}\chi^B + \phi *_A (\hat{\sigma}^\alpha)^{Ab}\phi_b,$$

is a Lorentz vector, etc. We shall elaborate on this in somewhat more detail in Sects. 3.2, 4.1.

The form (1.71), or (1.74), of γ_μ is just one possible representation of the Dirac matrices – the so-called *high-energy representation*. This is a *natural* choice because in this representation the reduction of the full Dirac field into irreducible (two-component) spinors is complete, see (1.70). There are, of course, other, equivalent, representations of the γ-matrices two of which we now discuss.

Suppose we subject $\psi(x)$ to a linear, nonsingular substitution S:

$$\psi(x) \to \psi'(x) = S\psi(x),$$

so that the Dirac equation (1.72) becomes

$$(\mathrm{i}\gamma'^\mu\partial_\mu - m\mathbb{1}_4)\psi'(x) = 0,$$

with $\gamma'^\mu = S\,\gamma^\mu S^{-1}$. Obviously, the anticommutators (1.75) are invariant under any such substitution. For instance, the choice

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} = S^{-1}$$

leads to the representation

$$\gamma'^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma'^i = \begin{pmatrix} 0 & \sigma^{(i)} \\ -\sigma^{(i)} & 0 \end{pmatrix}, \quad \gamma'_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{1.78}$$

This representation which is called the *standard representation* is particularly useful for weakly relativistic situations in electron and muon physics.

Both representations (1.74) and (1.78) belong to the class of representations in which the Dirac matrices have the following hermiticity properties,

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \tag{1.79a}$$

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu. \tag{1.79b}$$

These may be verified by explicit calculation.

In this book we use only this class of representations; there are, however, other representations in which these properties do not hold[15].

The following relation is always true (and follows from (1.75)):

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0. \tag{1.80}$$

Thus γ_5 *anticommutes* with any product of an *odd* number of Dirac matrices, but *commutes* with any product of an *even* number of Dirac matrices.

One other representation is useful in the discussion of relativistic single particle problems such as bound states in an external potential, scattering of a particle in external fields etc. We start from the standard representation (1.78) (omitting the primes) and set

$$\begin{aligned} \beta = \gamma^0 &= \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}, \\ \alpha^{(i)} = \beta\gamma^i &= \begin{pmatrix} 0 & \sigma^{(i)} \\ \sigma^{(i)} & 0 \end{pmatrix}. \end{aligned} \tag{1.81}$$

Then $\gamma^i = \beta\alpha^{(i)}$ and, upon multiplication of the Dirac equation by β, we obtain

$$\mathrm{i}\frac{\partial}{\partial t}\psi(x) = (-i\boldsymbol{\alpha}\cdot\nabla + m\beta)\psi(x). \tag{1.82a}$$

When written in this form, the Dirac equation shows the closest formal resemblance to the nonrelativistic Schrödinger (or Schrödinger–Pauli) equation, with the matrix

$$\text{“}H_0\text{”} = -\mathrm{i}\alpha\cdot\nabla + m\beta \tag{1.82b}$$

playing the role of the force-free Hamiltonian. For this reason the form (1.82b) of the equation may be called *Hamiltonian form* of the Dirac equation. This analogy must, however, be understood with great care. Indeed H_0, eq. (1.82b) does not have the properties required by quantum mechanics of a single particle. Specifically, the spectrum of H_0 is not bounded from below; H_0 has arbitrarily large negative eigenvalues (see below). This is a first hint to the fact that Dirac theory is not a single particle theory and can only be interpreted consistently in its second-quantized form. It is true, however, that the operator (1.82b) appears in the Hamiltonian density of the Dirac field. When integrated over all space, this density yields the

[15]More on representations of Dirac matrices can be found in [QP07], Sect. 9.1. For instance, in the description of neutrinos the class of Majorana representations is particularly relevant, i.e. the representations in which all γ-matrices are pure imaginary.

correct Hamiltonian of the Dirac field and does have the properties required by the principles of quantum mechanics.

Before we turn to the derivation of explicit solutions of the Dirac equation, we add a few further comments on these results. The explicit two-component formulation (1.69) is essential in discussing the basic principles of Dirac theory. In the literature on elementary particle physics one uses mostly the four-component form (1.72) of the Dirac equation. This form is indeed useful for most practical calculations of processes involving spin-1/2 particles and we make use of it in many chapters of this book.[16] It is not so useful, however, for questions about the principles of the theory. For instance, the covariance of (1.72), if it is not derived as done here, is not obvious and must be proven explicitly by making use of a theorem of Pauli.[17] Questions like: why do we have to describe relativistic spin-1/2 particles by (at least) four-component wave functions? why do we have to quantize Dirac theory by means of anticommuting field operators? etc., do not have straightforward answers in the four-component theory. In the two-component formulation the covariance of eqs (1.69) is evident.

In general, the answers to questions about fundamental properties of the theory are obtained in a more direct and transparent manner than in the four-component formulation.

1.4 Plane Wave Solutions of the Dirac Equation

In this section we derive plane wave solutions of the Dirac equation. For definiteness, we take (1.72) in the standard representation (1.78). We make the ansatz

$$\psi_\alpha(x) = u_\alpha^{(r)}(\boldsymbol{p})\mathrm{e}^{-ipx} + \upsilon_\alpha^{(r)}(\boldsymbol{p})\mathrm{e}^{ipx}, \tag{1.83}$$

where $px = p^0x^0 - \boldsymbol{px}$ with $p^0 = (m^2 + \boldsymbol{p}^2)^{1/2}$; u and υ are spinor amplitudes characterized by three-momentum $\boldsymbol{p}$ and spin orientation $r = 1$ or 2 with respect to some arbitrary quantization axis. The first term on the r.h.s. of (1.83) is called *positive frequency part* of $\psi(x)$; its time dependence is $\mathrm{e}^{-iEt/\hbar}$. The second term, correspondingly, is called *negative frequency part*. This is in accord with the Schrödinger equation, where stationary states have the time dependence $\mathrm{e}^{-iEt/\hbar}$. The physical interpretation of the latter will become clear in the context of the quantization of the Dirac field.

[16] We note, however, that quantum electrodynamics and other field theories can equivalently be formulated in the two-component formalism discussed above. This has been worked out, for the case of QED, by L. M. Brown, Proc. of Colorado Theor. Physics Institute, Colorado (1961). In some applications this formalism is simpler than the standard one, see also (Kersch et al., 1986).

[17] The theorem says that for any two sets of Dirac matrices which fulfill the anticommutation relations (1.75) there is a nonsingular matrix S which transforms one set into the other.

It follows from (1.72) that the spinors u and υ obey the equations

$$(\gamma^\mu p_\mu - m\mathbb{1})u^{(r)}(\boldsymbol{p}) = 0, \tag{1.84}$$

$$(\gamma^\mu p_\mu + m\mathbb{1}) = \upsilon^{(r)}(\boldsymbol{p}) = 0, \tag{1.85}$$

It is useful to consider also the equations which follow for the hermitean conjugate spinors. These equations take a particularly simple form if the definition (1.77) is introduced[18]

$$\overline{\psi(x)} = \psi^\dagger(x)\gamma^0, \tag{1.86a}$$

$$\overline{u(\boldsymbol{p})} = u^\dagger(p)\gamma^0, \quad \overline{\upsilon(\boldsymbol{p})} = \upsilon^\dagger(p)\gamma^0. \tag{1.86b}$$

Note that both $\psi^\dagger$ and $\bar{\psi}$, as well as $u^\dagger$, $\bar{u}$, $\upsilon^\dagger$, $\bar{\upsilon}$ are "row vectors", so that forms like $\bar{u}u$ are simple numbers, while $u\bar{u}$ etc. are four-by-four matrices. From (1.72, 1.84, 1.85) one easily derives

$$\overline{\psi(x)}\,(\mathrm{i}\gamma^\mu \overleftarrow{\partial}_\mu + m\mathbb{1}) = 0, \tag{1.72'}$$

where the arrow is meant to indicate that the derivative applies to $\bar{\psi}$ on the left, i.e. $\bar{\psi}\gamma^\mu \overleftarrow{\partial}_\mu \equiv \partial_\mu \bar{\psi}\gamma^\mu$. Furthermore,

$$\overline{u(\boldsymbol{p})}(\gamma^\mu p_\mu - m\mathbb{1}) = 0, \tag{1.84'}$$

$$\overline{\upsilon(\boldsymbol{p})}\,(\gamma^\mu p_\mu + m\mathbb{1}) = 0. \tag{1.85'}$$

Equations (1.84, 1.85), for the case of massive particles $m \neq 0$, can be solved in two steps:

1. Go to the rest system of the particle where $p = (m, \mathbf{0})$. As $p^0 = m$, (1.84) reduces to

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_4(0) \end{pmatrix} = 0,$$

while (1.85) reduces to

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \upsilon_1(0) \\ \upsilon_2(0) \\ \upsilon_3(0) \\ \upsilon_4(0) \end{pmatrix} = 0,$$

[18] See the discussion in Sect.3.2.1.

the solutions of which are

$$u^{(r)}(0) = \begin{pmatrix} \chi^{(r)} \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \upsilon^{(r)}(0) = \begin{pmatrix} 0 \\ 0 \\ \chi^{(r)} \end{pmatrix},$$

respectively. Here $\chi^{(r)}$ is a two-component object which, in fact, must be a Pauli spinor, well-known from nonrelativistic quantum mechanics.

2. The solutions for arbitrary momentum $\boldsymbol{p}$ can then be expressed in terms of the rest-frame solutions by utilizing the relation

$$\gamma^\mu p_\mu \gamma^\nu p_\nu = \frac{1}{2} p_\mu p_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = p_\mu g^{\mu\nu} p_\nu = p^2,$$

which follows from (1.75). It is seen that

$$u^{(r)}(\boldsymbol{p}) = N(\gamma^\mu p_\mu + m\mathbb{1})u^{(r)}(0), \tag{1.87}$$

$$\upsilon^{(r)}(\boldsymbol{p}) = -N(\gamma^\mu p_\mu - m\mathbb{1})\upsilon^{(r)}(0) \tag{1.88}$$

satisfy (1.84) and (1.85) respectively. The normalization constant N is conveniently chosen so as to obtain

$$u^{(r)\dagger}(\boldsymbol{p})\upsilon^{(s)}(\boldsymbol{p}) = \upsilon^{(r)\dagger}(\boldsymbol{p})\upsilon^{(s)}(\boldsymbol{p}) = 2p_0\delta_{rs}. \tag{1.89}$$

This is a covariant normalization of one-particle states.[19] It is achieved by taking $N = (p^0 + m)^{-1/2}$. Inserting the explicit representation of γ-matrices into (1.87) and (1.88) (here in the standard representation), we have

$$u^{(r)}(\boldsymbol{p}) = \sqrt{p^0 + m} \begin{pmatrix} \chi^{(r)} \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p_0 + m} \chi^{(r)} \end{pmatrix}, \tag{1.90}$$

$$\upsilon^{(r)}(\boldsymbol{p}) = \sqrt{p^0 + m} \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p_0 + m} \chi^{(r)} \\ \chi^{(r)} \end{pmatrix}. \tag{1.91}$$

One verifies by explicit calculation that

$$\overline{u^{(r)}(\boldsymbol{p})}\, u^{(s)}(\boldsymbol{p}) = 2m\delta_{rs}, \tag{1.92}$$

$$\overline{\upsilon^{(r)}(\boldsymbol{p})}\, \upsilon^{(s)}(\boldsymbol{p}) = -2m\delta_{rs}, \tag{1.93}$$

[19] In some textbooks the normalization of one-fermion states is taken to be p_0/m instead of $2p_0$. Our normalization is the same for fermions and for bosons.

and also

$$u^{(r)\dagger}(\boldsymbol{p})v^{(s)}(-\boldsymbol{p}) = 0. \tag{1.94}$$

Obviously these spinors can be transformed to any other representation of the Dirac matrices by applying the corresponding transformation matrix S to them. (Example: Transformation to high-energy representation by means of S, see above).

Mass zero case: We have derived the spinors in momentum space (1.90, 1.91) by "boosting" the Pauli spinors from the particle's rest system to the appropriate momentum $\boldsymbol{p}$, see (1.87, 1.88). For a massless particle there is no rest system. Nevertheless, we obtain the plane wave solutions describing the force-free motion of a massless spin-1/2 particle by simply taking the limit $m \to 0$ of the solutions (1.90) and (1.91).[20] We then have, of course, $p^0 = |p|$ and the operator

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p^0} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} = h \tag{1.95}$$

becomes the helicity operator.

For the sake of illustration, let us transform the $m = 0$ solutions back to the "high-energy" representation (1.74), by means of the transformation matrix S. We obtain from (1.90)

$$u^{(r)}(\boldsymbol{p}, m = 0) = \sqrt{\frac{|\boldsymbol{p}|}{2}} \begin{pmatrix} (\mathbb{1} + h)\chi^{(r)} \\ (\mathbb{1} - h)\chi^{(r)} \end{pmatrix}. \tag{1.96}$$

Taking for $\chi^{(r)}$ magnetic substates in the direction of $\boldsymbol{p}$, or opposite to it, means taking the eigenvalues of h to be $+1$ or -1, respectively. Thus, either the lower two components or the upper two components vanish.

Obviously, these solutions can be obtained directly from our equations (1.69): In the limit $m = 0$, these equations decouple. With the ansatz $\mathrm{e}^{\pm ipx}u(p)$, they are seen to lead to the eigenvalue equation of the helicity operator (1.95).

1.5 A Few More Properties of Dirac Spinors

We return to the two-component form (1.69) of the Dirac equation and derive a few more properties of its solutions. First we note that if $\psi(x) = (\phi_a(x),\ \chi^A(x))^T$ is a solution then so is the spinor

$$\psi_P(x) = \begin{pmatrix} \chi_a(Px) \\ \phi^A(Px) \end{pmatrix} = \begin{pmatrix} (\sigma^0)_{aB} & \chi^B(Px) \\ (\hat{\sigma}^0)^{Ab} & \phi_b(Px) \end{pmatrix} = \gamma^0\psi(Px), \tag{1.97}$$

Here $Px = (x^0,\ -\boldsymbol{x})$ is the parity transform of x, cf. (1.58a); γ^0 is given by (1.71).

[20] This limiting procedure is only applicable for spin 1/2. For higher spin $J > 1/2$ the limit $m \to 0$ is discontinuous: Whilst a massive particle with spin J has $(2J+1)$ magnetic substates, a massless particle can have only two helicity states $h = (\boldsymbol{J}\cdot\boldsymbol{p})/|\boldsymbol{p}| = \pm J$.

Similarly, one shows that if $\psi(x)$ is a solution of the Dirac equation (1.69) then also

$$\psi_C(x) = \begin{pmatrix} \chi_a^*(x) \\ -\phi^{*B}(x) \end{pmatrix} = \mathrm{i}\gamma^2\psi^*(x) \tag{1.98}$$

is a solution.[21] $\psi_C(x)$ is called the *charge conjugate* spinor of $\psi(x)$. For the moment, of course, this is only a formal definition. Charge conjugation becomes physically relevant only when the particle interacts, e.g. with an external electromagnetic field or with other particles. It can then be shown that the transformation (1.98) does indeed lead to the wave functions of the corresponding antiparticle.

It is not difficult to prove that ψ_C, as defined by (1.98) is a solution of the Dirac equation. Take the complex conjugate of (1.69a), multiply with ε from the left and insert $\varepsilon \cdot \varepsilon^{-1}$ between the operators $\sigma^\mu \partial_\mu$ and χ (remember rule (iii) above).

$$-\mathrm{i}\varepsilon^{AA'}(\sigma^{\mu *}\partial_\mu)_{A'b'}\varepsilon^{b'b}\varepsilon_{bb''}\chi^{*b''}(x) = m\varepsilon^{AA'}\phi^*_{A'}(x).$$

From (1.60) and with our convention (1.50) we can write this as

$$\mathrm{i}(\hat{\sigma}^\mu \partial_\mu)^{Ab}\chi^*_b(x) = -m\phi^{*A}(x),$$

where χ_B and ϕ^a are defined according to the rules (1.49) and (1.46), respectively. The second equation (1.69b) is treated in an analogous way:

$$-\mathrm{i}\varepsilon_{bb'}(\hat{\sigma}^{\mu *}\partial_\mu)^{b'A'}\varepsilon_{A'A}\varepsilon^{AA''}\phi^*_{A''}(x) = m\varepsilon_{bb'}\chi^{*b'}(x),$$

which can be written as

$$\mathrm{i}(\sigma^\mu \partial_\mu)_{bA}(-\phi^{*A}(x)) = m\chi^*{}_b(x).$$

Thus, we have shown that

$$\psi_C(x) := \mathrm{i}\gamma^2\psi^*(x) = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}\begin{pmatrix} \phi_a \\ \chi^A \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & \varepsilon_{ab} \\ -\varepsilon^{BA} & 0 \end{pmatrix}\begin{pmatrix} \phi^*_A \\ \chi^{*b} \end{pmatrix} = \begin{pmatrix} \chi^*_a \\ -\phi^{*B} \end{pmatrix} = \begin{pmatrix} -(\chi^{II})^* \\ (\chi^I)^* \\ (\phi_2)^* \\ -(\phi_1)^* \end{pmatrix}$$

is also a solution of (1.69).

If one prefers to write the charge conjugate spinor in terms of $\bar{\psi}$, cf. (1.86), relation (1.98) reads

[21] Recall that complex conjugation converts small into capital, capital into small indices, $(\chi^B)^* = \chi^{*b}$, $(\phi_a)^* = \phi^*_A$, and that indices are lowered and raised by means of ε-matrices.

$$\psi_C(x) = \mathrm{i}\gamma^2\gamma^0\overline{(\psi(x))}^T. \tag{1.98'}$$

That the matrix

$$C = \mathrm{i}\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T \tag{1.99}$$

does indeed convert particle wave functions into antiparticle wave functions and vice-versa can be seen by considering the Dirac equation for a fermion with charge e in presence of some *external* electromagnetic fields A_μ,

$$(\mathrm{i}\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\mathbb{1})\psi(x) = 0. \tag{1.100}$$

Take the complex conjugate of this equation, transform the wave function according to (1.98) or [equivalently (1.98')] and observe that

$$(C\gamma^0)(\gamma^\mu)^*(C\gamma^0)^{-1} = -\gamma^\mu. \tag{1.101}$$

One thereby obtains from (1.100)

$$(\mathrm{i}\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m\mathbb{1})\psi_C(x) = 0, \tag{1.102}$$

which is seen to be the Dirac equation for the antiparticle (charge – e) in the same *external* fields.[22]

Note that applying the transformation (1.98) twice leads back to $\psi(x)$,

$$(\psi_{\mathrm{c}}(x))_{\mathrm{c}} = \mathrm{i}\gamma^2(\mathrm{i}\gamma^2\psi^*(x))^* = \gamma^2\gamma^{2*}\psi(x) = \psi(x).$$

Also, C could yield (1.98) multiplied by a phase η. Without loss of generality one can impose the condition $\eta^2 = 1$.

The class of spinors for which $\psi_C(x) = \psi(x)$, are called *Majorana spinors*. Note that these are not *real* fields (as would be the case for a spin zero field). From what we said above it is clear that such a spinor can only describe a neutral particle, $e = 0$.

Finally, one proves by similar means that with $\psi(x)$ the time reversed spinor

$$\psi_T(x) := \begin{pmatrix} (\sigma^0)_{aA}\varepsilon^{AA'}\phi^*_A(T_X) \\ (\hat{\sigma}^0)^{Bb}\varepsilon_{bb'}\chi^{*b'}(T_X) \end{pmatrix} = \begin{pmatrix} \phi^*_a(Tx) \\ \chi^{*B}(Tx) \end{pmatrix} = \begin{pmatrix} -\phi^*_2(Tx) \\ \phi^*_1(Tx) \\ -\chi^{*II}(Tx) \\ \chi^{*I}(Tx) \end{pmatrix} \tag{1.103}$$

is also a solution of the Dirac equation (1.69).

[22] Note that here we apply charge conjugation C only to $\psi(x)$, but not to the sources of the external fields. Had we done so, we would have found (1.100) to be invariant under C. The electromagnetic interaction is invariant under C.

The relationship (1.103) thus reads

$$\psi_T(x) = \begin{pmatrix} 0 & \sigma^0_{aA}\varepsilon^{AB} \\ \hat{\sigma}^{0Bb}\varepsilon_{bc} & 0 \end{pmatrix} \begin{pmatrix} \chi^{*c}(Tx) \\ \phi^*{}_B(Tx) \end{pmatrix} \equiv T \begin{pmatrix} \chi^{*c}(Tx) \\ \phi^*{}_B(Tx) \end{pmatrix}. \qquad (1.103')$$

The spinor on the r.h.s. is nothing but the transposed of $\overline{\psi(Tx)}$, see (1.77), while the matrix T, in the high-energy representation, is equal to

$$T = \gamma_5(\mathrm{i}\gamma^2), \qquad (1.104)$$

One verifies that $T := \mathrm{i}\gamma_5\gamma^2$ has the properties

$$T = \mathrm{i}\gamma_5\gamma^2 = -T^{-1} = -T^\dagger = -T^T. \qquad (1.105)$$

1.6 Quantization of Majorana Fields

There are many indications for the fact that the Dirac field cannot be a classical field. One indication for this is that a spinor field ϕ or χ changes sign when a complete rotation by 2π is performed on it. Therefore, such a field cannot be a classical observable. Such observables which, of course, must be invariant under complete rotations of the coordinate system, can only depend on bilinear forms in the spinor fields and their conjugates (so-called sesquilinear forms). Another indication is this: when one computes the total energy, i.e. the Hamilton density of the unquantized Dirac field integrated over all space, this energy is found to be zero. (See exercise 1.13, for the case of Majorana spinors.) Furthermore, if one wants to interpret the Dirac equation in the framework of a single particle theory, in a spirit like in nonrelativistic quantum mechanics of one particle dynamics, one runs into two major difficulties: The energy of a free fermion can assume arbitrarily large negative values. Also, particle and antiparticle appear in an asymmetric way: while free particles are states with positive energy, antiparticle states appear as "holes" in particle levels of negative energy. The interpretation in terms of "particles" and "holes" avoids the negative energies but does not repair the apparent asymmetry in the treatment of particles and antiparticles.[23]

All these difficulties disappear if Dirac theory is interpreted in the framework of second quantization. This is what we are going to show next. For the sake of clarity, we start with the case of Majorana fields, but we will see that the case of more general Dirac fields is no more difficult to treat.

A Majorana field of mass m is defined here by the condition

$$\psi_C(x) = \psi(x),$$

[23] In a system of a *finite* number of fermions the hole theory is a prefectly consistent and useful approach. This is not so in a field theory with *infinitely* many degrees of freedom and in which genuine antiparticles occur.

which implies

$$\phi_a(x) = \varepsilon_{ab}(\chi^B(x))^* = \varepsilon_{ab}\chi^{*b}(x) = \chi^*{}_a(x). \tag{1.106}$$

In this case the Dirac equation (1.69) reduces to

$$\mathrm{i}(\sigma^\mu\partial_\mu)_{aB}\chi^B(x) = m\chi^*{}_a(x), \tag{1.107a}$$

or, equivalently

$$\mathrm{i}(\hat{\sigma}^\mu\partial_\mu)^{Bc}\phi_c(x) = -m\phi^{*B}(x). \tag{1.107b}$$

These equations could have been derived from the following Lagrangian density (c.f. exercise 1.1):

$$\mathscr{L}_M = \frac{\mathrm{i}}{2}\phi^*{}_B(\hat{\sigma}^\mu\overleftrightarrow{\partial}_\mu)^{Bb}\phi_b + \frac{m}{2}[\phi^*{}_A\varepsilon^{AB}\phi^*{}_B + \phi_a\varepsilon^{ab}\phi_b], \tag{1.108}$$

where $\overleftrightarrow{\partial}_\mu$ acts on the right and (with a minus sign) on the left.

From this one derives the "momentum" canonically conjugate to ϕ_a,

$$\pi^a := \frac{\partial\mathscr{L}_M}{\partial(\partial^0\phi_a)} = \frac{\mathrm{i}}{2}\phi^*{}_B(\hat{\sigma}^0)^{Ba}. \tag{1.109}$$

Therefore, in quantizing the Majorana field we should discuss commutation rules of the operators ϕ as well as ϕ^*. However, as ϕ and ϕ^* are related by eq. (1.107b), it is sufficient to discuss the commutation of $\phi_a(x)$ and $\phi_b(y)$. The commutation rules for ϕ and π can be derived from these by applying the operator on the left-hand side of eq. (1.107b) to them.

Similary, the commutation rules for the χ field and its conjugate will then follow by means of the relation (1.106). Let us first consider the *commutator* of $\phi_a(x)$ and $\phi_b(y)$, for which we write

$$[\phi_a(x), \phi_b(y)] = t_{ab}\,f(x-y). \tag{1.110}$$

Obviously, the conditions that the right-hand side of (1.110) must fulfill, are these:

(i) t_{ab} must be an invariant tensor with respect to SL(2, $\mathbb{C}$), i.e. $t_{ab} = A_a{}^m A_b{}^n t_{mn}$.
(ii) $f(x-y)$ must be Lorentz invariant and must satisfy the Klein – Gordon equation for mass m, both in x and y.
(iii) The product $t_{ab}\,f(x-y)$ must be antisymmetric under the simultaneous interchange of a with b, and x with y.

The only SL(2, $\mathbb{C}$) invariant tensor is $t_{ab} = \varepsilon_{ab}$ which, as we know, is *anti*-symmetric in its indices. As for $f(x-y)$ we note that there are precisely two linearly independent, Lorentz invariant solutions of the Klein–Gordon equation for mass

m.[24] These can be chosen as follows:

$$\Delta_0(z;m) = -\frac{\mathrm{i}}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2\omega_k}(\mathrm{e}^{-\mathrm{i}kz} - \mathrm{e}^{\mathrm{i}kz}), \tag{1.111}$$

$$\Delta_1(z;m) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2\omega_k}(\mathrm{e}^{-\mathrm{i}kz} + \mathrm{e}^{\mathrm{i}kz}). \tag{1.112}$$

with $\omega_k = (k^2 + m^2)^{1/2}$ and $z = x - y$. We note, in particular, that $\Delta_0(z;m)$ is *antisymmetric* when z is replaced by $-z$ and vanishes for spacelike separation of x and y. Thus, Δ_0 is a causal distribution.

$\Delta_1(z;m)$ on the other hand, is symmetric in z and does not vanish for $z^2 < 0$. Thus, Δ_1 cannot be a causal distribution.

We are now in a difficulty: The quantization rule should be *causal*, i.e. the commutator $[\phi_a(x),\ \phi_b(y)]$ should vanish whenever $(x-y)$ is spacelike. Therefore we must take the right-hand side of (1.110) to be

$$\varepsilon_{ab}\Delta_0(x-y;m). \tag{1.113}$$

This, however, cannot be correct as this last expression is *symmetric* under $(a \leftrightarrow b, x \leftrightarrow y)$. Therefore, if we insist on quantizing the theory by means of commutators, we have no other choice than to take the commutator (1.110) to be

$$[\phi_a(x), \phi_b(y)] = c_1\varepsilon_{ab}\Delta_1(x-y;m). \tag{1.114}$$

Such a theory is not acceptable on physical grounds: it is Lorentz invariant but is in conflict with causality, and therefore, will necessarily lead to physically unacceptable consequences (see also Sect. 1.8).

On the other hand, there is really no compelling reason why one should try to impose commutators on fermion fields. Fermion fields are not observable. Commutators are relevant only for quantum mechanical observables which, as we said before, must be bilinear in fermion fields.

With this remark in mind, it is not difficult to resolve the puzzle. In contrast to (1.114) the expression (1.113) is perfectly acceptable: It is Lorentz invariant and vanishes for spacelike $(x-y)$. As it is symmetric under simultaneous interchange of a and b, x and y, it is natural to consider the *anticommutator*

$$\{\phi_a(x), \phi_b(y)\} := \phi_a(x)\phi_b(y) + \phi_b(y)\phi_a(x) \tag{1.115}$$

of these fields, instead of their commutator, and to require the quantization rule

$$\{\phi_a(x), \phi_b(y)\} = c_0\varepsilon_{ab}\Delta_0(x-y;m), \tag{1.116}$$

[24] See Appendix A.

where c_0 is a constant to be determined from (1.107). For this purpose, apply the operation $\mathrm{i}\left(\hat{\sigma}^\mu \partial_\mu^x\right)^{Aa}$ onto (1.116) and sum over a. This gives

$$\{\phi^{*A}(x), \phi_b(y)\} = -\frac{\mathrm{i}c_0}{m}(\hat{\sigma}^\mu \partial_\mu^x)^{Aa} \varepsilon_{ab} \Delta_0(x-y;m).$$

If we take $x^0 = y^0$, then

$$\begin{aligned}\{\phi^{*A}(x), \phi_b(y)\}_{x^0=y^0} &= -\frac{\mathrm{i}c_0}{m}\delta^{Aa}\varepsilon_{ab}\partial_0^x \Delta_0(x-y;m)|_{x^0=y^0} \\ &= \frac{\mathrm{i}c_0}{m}\delta^{Aa}\varepsilon_{ab}\delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned}$$

Multiplying this equation with ε_{BA}, one obtains

$$\{\phi_B^*(x), \phi_b(y)\}_{x^0=y^0} = \frac{\mathrm{i}c_0}{m}\delta_{Bb}\delta(\boldsymbol{x}-\boldsymbol{y}).$$

The left-hand side is a positive hermitean operator. Thus, c_0 must be negative, pure imaginary. For dimensional reasons c_0 must have the dimension of an energy. Thus we take

$$c_0 = -\mathrm{i}m. \tag{1.117}$$

This quantization rule by means of anticommutators leads to a consistent interpretation of the theory. This is shown below for the case of the general Dirac field of which the Majorana field is a special case.

1.7 Quantization of Dirac Field

The reasoning of the preceding paragraph is readily applied to the more general case of arbitrary Dirac fields. Here, the possibility of quantizing via *commutators* is excluded for the following reason: Suppose we would require the causal commutator

$$[\phi_a(x), \chi^*{}_b(y)] = c_3\varepsilon_{ab}\Delta_0(x-y;m), \tag{1.118}$$

where c_3 is some number still to be determined. Applying the charge conjugation to this equation[25], we would obtain the following sequence of equations,

[25] Recall that $C^{-1} = -C$, so that

$$\begin{aligned} C\phi_a C^{-1} &= \chi*_a, & C\chi^B C^{-1} &= -\phi^{*B}, \\ C^{-1}\chi *_a C &= \phi_a, & C^{-1}\phi^{*B} C &= \chi^B. \end{aligned}$$

$$\begin{aligned}[\chi^*{}_a(x), \phi_b(y)] &= c_3 \varepsilon_{ab} \Delta_0(x-y;m) \\ &= -[\phi_b(y), \chi^*{}_a(x)] = -c_3 \varepsilon_{ba} \Delta_0(y-x;m) \\ &= -c_3 \varepsilon_{ab} \Delta_0(x-y;m).\end{aligned}$$

The last step follows from the symmetry of expression (1.113). Obviously, $c_3 = 0$, and the ansatz (1.118) is seen to be inconsistent. Here again, we could repair this inconsistency by replacing the antisymmetric distribution Δ_0 by the symmetric Δ_1. However, this would again lead to an acausal theory. Very much like in the case of Majorana fields the only other possibility is to quantize the theory by means of *anticommutators*. Thus we postulate

$$\{\phi_a(x), \chi^*{}_b(y)\} = -\mathrm{i}m \varepsilon_{ab} \Delta_0(x-y;m). \tag{1.119}$$

This quantization rule leads to a consistent interpretation of Dirac fields in terms of particles and antiparticles with the correct (positive) energy spectrum. These particles obey Fermi–Dirac statistics, so that the Pauli principle is a consequence of the theory.[26]

We could prove these statements right away by expanding the two-component fields ϕ and χ in terms of plane wave solutions and in terms of the corresponding creation and annihilation operators. In order to establish the connection with other, more conventional presentations of this subject, we prefer instead to reformulate first the quantization rule (1.119) in terms of four-component field operators $\psi(x)$, as defined above in (1.70). We have

$$\psi(x) = \begin{pmatrix} \phi_a(x) \\ \chi^B(x) \end{pmatrix}, \quad \psi^\dagger(y) = (\phi^*{}_C(y) \chi^{*d}(y)),$$

their anticommutator being

$$\{\psi(x), \psi^\dagger(y)\} = \begin{pmatrix} \{\phi_a(x), \phi^*{}_C(y)\} & \{\phi_a(x), \chi^{*d}(y)\} \\ \{\chi^B(x), \phi^*{}_C(y)\} & \{\chi^B(x), \chi^{*d}(y)\} \end{pmatrix}.$$

Each of the anticommutators of two-component fields is a two dimensional matrix. All four of them can be derived from (1.119) by applying the ε-tensors and/or by making use of the Dirac equation (1.69). One finds

[26] This is a special case of the famous spin-statistics theorem (Fierz 1938, Pauli 1940).

$$\begin{aligned}\{\phi_a(x), \phi^*{}_C(y)\} &= -\frac{\mathrm{i}}{m}\left(\sigma^{\mu *}\partial_\mu^y\right)_{Cb'}\{\phi_a(x), \chi^{*b'}(y)\}\\ &= -\frac{\mathrm{i}}{m}\left(\sigma^{\mu *}\partial_\mu^y\right)_{Cb'}\varepsilon^{b'b}\{\phi_a(x), \chi_b^*(y)\}\\ &= \left(\sigma^{\mu *}\partial_\mu^y\right)_{Ca}\Delta_0(x-y;m)\\ &= \left(\sigma^{\mu}\partial_\mu^y\right)_{aC}\Delta_0(x-y;m).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\{\phi_a(x), \chi^{*d}(y)\} &= \varepsilon^{db}\{\phi_a(x), \chi^*{}_b(y)\}\\ &= \varepsilon^{db}\varepsilon_{ab}(-\mathrm{i}m)\Delta_0(x-y;m) = \delta_a^d \mathrm{i}m\Delta_0(x-y;m),\\ \left\{\chi^B(x), \phi^*{}_C(y)\right\} &= \left\{\phi_c(y), \chi^{*b}(x)\right\}^* = \delta_C^B \mathrm{i}m\Delta_0(x-y;m),\\ \{\chi^B(x), \chi^{*d}(y)\} &= \frac{\mathrm{i}}{m}\left(\hat{\sigma}^\mu\partial_\mu^x\right)^{Ba}\varepsilon^{db}\{\phi_a(x), \chi^*{}_b(y)\}\\ &= -\left(\hat{\sigma}^\mu\partial_\mu^x\right)^{Bd}\Delta_0(x-y;m),\end{aligned}$$

so that we find for the anticommutator of ψ and $\psi^\dagger$,

$$\{\psi(x),\ \psi^\dagger(x)\} = \mathrm{i}\begin{pmatrix}\mathrm{i}\sigma^\mu\partial_\mu^x & m\mathbb{1}\\ m\mathbb{1} & \mathrm{i}\hat{\sigma}^\mu\partial_\mu^x\end{pmatrix}\Delta_0(x-y;m),$$

where we have used that $\partial_\mu^y\Delta_0 = -\partial_\mu^x\Delta_0$.

The matrix that appears in this last expression can be written as follows, cf. (1.71)

$$(m\gamma^0 + \mathrm{i}\gamma^\mu\gamma^0\partial_\mu^x).$$

By multiplying with γ^0 from the right we obtain, finally,

$$\{\psi(x),\ \overline{\psi(y)}\} = \mathrm{i}(m\mathbb{1} + \mathrm{i}\gamma^\mu\partial_\mu^x)\ \Delta_0(x-y;m). \tag{1.120}$$

Thus we have arrived at a compact notation of the quantization rule (1.119), formulated in terms of the field operator $\psi(x)$ and its adjoint.

Let us now derive a few consequences of the quantization of the Dirac field by means of anticommutators. Firstly, let us look at (1.120) for the special case of equal times, $x^0 = y^0$. With the aid of formulae (A.4) for the covariant causal distribution Δ_0, we find

$$\{\psi_a(x),\ \psi_\beta^\dagger(y)\}_{x^0=y^0} = \delta_{\alpha\beta}\,\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{1.121}$$

Secondly, we expand $\psi(x)$ and $\overline{\psi(x)}$, as usual, in terms of "normal oscillations", i.e. in terms of plane wave solutions, very much like in (1.83),

$$\psi_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{2E_p} \left[a^{(r)}(\boldsymbol{p}) u_\alpha^{(r)}(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px} + b^{(r)\dagger}(\boldsymbol{p}) \upsilon_\alpha^{(r)}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px}\right], \tag{1.122a}$$

$$\overline{\psi_\alpha(x)} = \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{2E_p} \left[a^{(r)\dagger}(\boldsymbol{p}) \overline{u_\alpha^{(r)}(\boldsymbol{p})} \mathrm{e}^{\mathrm{i}px} + b^{(r)}(\boldsymbol{p}) \overline{\upsilon_\alpha^{(r)}(\boldsymbol{p})} \mathrm{e}^{-\mathrm{i}px}\right]. \tag{1.122b}$$

Here, $u^{(r)}(\boldsymbol{p}), \upsilon^{(r)}(\boldsymbol{p})$, are the force-free solutions in momentum space (1.90, 1.91) that we constructed above in sec. 1.4. r is the spin index, $\boldsymbol{p}$ the momentum and $E_p = (m^2 + \boldsymbol{p}^2)^{1/2}$ the corresponding energy. The inverse formulae expressing the operators $a^{(r)}(\boldsymbol{p})$ and $b^{(r)}(\boldsymbol{p})$ in terms of the field operators are easily derived by using the orthogonality relations (1.92–1.94). One finds

$$a^{(r)}(\boldsymbol{p}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \mathrm{e}^{\mathrm{i}px} u_\alpha^{(r)\dagger}(\boldsymbol{p}) \psi_\alpha(x), \tag{1.123a}$$

$$b^{(r)}(\boldsymbol{p}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x \mathrm{e}^{\mathrm{i}px} \overline{\psi_\alpha(x)}\, \gamma^0_{\alpha\beta}\, \upsilon_\beta^{(r)}(\boldsymbol{p}). \tag{1.123b}$$

In (1.122, 1.123) we have written out the Dirac spinor indices, for the sake of clarity. Thus, in (1.122) the spinors $u(\boldsymbol{p})$ and $\upsilon(\boldsymbol{p})$ in momentum space carry the spinor index of the field operators $\psi(x)$. In (1.123) one has to sum over the spinor indices as indicated, the operators $a^{(r)}(\boldsymbol{p})$ and $b^{(r)}(\boldsymbol{p})$ carrying no such index.

In going over from two-component spinors to four-component spinors (1.70) we lose the clear distinction between co- and contragredient spinor indices of first and second kind. Also the covariance properties of the theory become less transparent than in the two-component formulation. What we gain, however, is a very compact notation that is useful for almost all practical calculations involving Dirac fields. As long as the order of a product of Dirac spinors and Dirac matrices is respected, we need not write out the spinor indices at all. The sum over first and second kind spinors is automatically contained.

From (1.123, 1.121), making use of relations (1.92–1.94) one derives the following anticommutation relations:

$$\{a^{(r)}(\boldsymbol{p}),\ a^{(s)\dagger}(\boldsymbol{q})\} = 2E_p\, \delta_{rs}\, \delta(\boldsymbol{p} - \boldsymbol{q}), \tag{1.124a}$$

$$\{b^{(r)}(\boldsymbol{p}),\ b^{(s)\dagger}(\boldsymbol{q})\} = 2E_p\, \delta_{rs}\, \delta(\boldsymbol{p} - \boldsymbol{q}), \tag{1.124b}$$

$$\{(a^{(r)}(\boldsymbol{p}),\ a^{(s)}(\boldsymbol{p})\} = \{b^{(r)}(\boldsymbol{p}),\ b^{(s)}(\boldsymbol{q})\} = \{a^{(r)}(\boldsymbol{p}),\ b^{(s)}(\boldsymbol{q})\} = \{a^{(r)}(\boldsymbol{p}),\ b^{(s)\dagger}(\boldsymbol{q})\} = 0. \tag{1.124c}$$

These anticommutation rules show that we deal here with creation and destruction operators for two kinds of particles that obey Fermi–Dirac statistics.[27] In more detail,

$a^{(r)\dagger}(\boldsymbol{p})$ creates a *particle* state with four-momentum $p = (E_p, \boldsymbol{p})$ and spin projection r,
$b^{(r)\dagger}(\boldsymbol{p})$ creates a *antiparticle* state with four-momentum $p = (E_p, \boldsymbol{p})$ and spin projection r,
$a^{(r)}(\boldsymbol{p})$ and $b^{(r)}(\boldsymbol{p})$ are the corresponding annihilation operators.

Thus, for example, applying $a^{(r)\dagger}(\boldsymbol{p})$ to the vacuum yields a one-particle state

$$a^{(r)\dagger}(\boldsymbol{p})|0\rangle = |\boldsymbol{p}, r\rangle,$$

which is normalized according to the covariant prescription

$$\langle \boldsymbol{p}', r' \,|\boldsymbol{p},\, r\,\rangle = 2E_p\, \delta_{rr'}\, \delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{1.125}$$

Some of these statements are proven in the following section.

1.8 Lagrange Density of Dirac Field, Charge, Energy, Momentum and Spin of Dirac Particles

It is not difficult to find a Lagrange density whose Euler equations are the Dirac equation for $\psi(x)$ and its adjoint. In (1.108) we already found a Lagrange density for the case of Majorana spinors. For the more general case of unrestricted Dirac fields a Lagrangian is

$$\begin{aligned}\mathscr{L}_D = \frac{\mathrm{i}}{2}\,&[\phi^*{}_C(x)\,(\hat{\sigma}^\mu \overleftrightarrow{\partial}_\mu)^{Ca}\,\phi_a(x) + \chi^{*d}\,(\hat{\sigma}^\mu \overleftrightarrow{\partial}_\mu)_{dB}\,\chi^B(x)]\\ &-m_{\mathrm{D}}\,[\phi^*{}_B(x)\chi^B(x) + \chi^{*d}(x)\,\phi_d(x)],\end{aligned} \tag{1.126}$$

where m_{D} is a mass parameter and the quantity in square brackets is called Dirac mass term.[28] One verifies easily that $\mathscr{L}_{\mathrm{D}}$ leads to the correct equations (1.69). When written in terms of four-spinors this Lagrangian takes a very simple form, viz.

$$\mathscr{L}_{\mathrm{D}} = \overline{\psi(x)}\left[\frac{\mathrm{i}}{2}\gamma^\mu \overleftrightarrow{\partial}^\mu - m_{\mathrm{D}}\mathbb{1}\right]\psi(x). \tag{1.126$'$}$$

The field variables being $\psi(x)$ and $\bar{\psi}(x)$, their conjugate momenta are

[27] See e.g. [SAK67].

[28] The most general case is treated below in Sect. 1.8.4.

$$\Pi(x) := \frac{\partial \mathscr{L}_{\mathrm{D}}}{\partial(\partial_0 \psi)} = \frac{\mathrm{i}}{2}\psi^\dagger(x), \tag{1.127a}$$

$$\overline{\Pi(x)} = \frac{\partial \mathscr{L}_{\mathrm{D}}}{\partial(\partial_0 \bar{\psi})} = -\frac{\mathrm{i}}{2}\gamma^0\psi(x). \tag{1.127b}$$

1.8.1 Charge of Particles and Antiparticles

The Lagrangian (1.126) is invariant under *global* gauge transformations, also called gauge transformations of the first kind,

$$\psi(x) \to \mathrm{e}^{\mathrm{i}\alpha}\psi(x), \quad \overline{\psi(x)} \to \mathrm{e}^{-\mathrm{i}\alpha}\,\overline{\psi(x)}. \tag{1.128}$$

Taking α to be infinitesimal this means that $\mathscr{L}_D$ is invariant with respect to variations $\delta\psi = \mathrm{i}\alpha\psi$, $\delta\mathscr{L}_{\mathrm{D}} = 0$. When this is worked out (and making use of the equations of motion) one obtains the conservation condition

$$\partial_\mu j^\mu(x) = 0 \tag{1.129}$$

for the "current density" operator

$$j^\mu(x) = \overline{\psi(x)}\,\gamma^\mu\,\psi(x). \tag{1.130}$$

Note that, at this point, we do not know which physical current density is to be represented by the operator (1.130) nor do we know whether the divergence condition (1.129) actually corresponds to a physical conservation law. These questions cannot be answered until we know what the interactions of the fermion field are and how these interactions behave under the same gauge transformations. For a world *with* interactions we will have to consider simultaneous global gauge transformations of all fields that enter the theory, possibly also *local* gauge transformations (i.e. gauge transformations where α becomes dependent on space and time coordinates).

For the moment it may suffice to say that for *leptons*, i.e. for particles which have only electromagnetic and weak (and gravitational) interactions, the operator (1.130) multiplied with e represents the electromagnetic current density. If the ψ are taken to be *free* fields (1.122), then $j^\mu(x)$ is the current operator in the sense of perturbation theory, i.e. $ej^\mu(x)A_\mu(x)$ is the interaction with the Maxwell field, represented by the potentials A_μ. In this case we can compute single particle matrix elements of $j^\mu(x)$ and of the corresponding charge operator Q. Indeed, if the fields are sufficiently well-behaved such that $j^\mu(x)$ vanishes at infinity,[29] we conclude

[29] In fact the fields (1.122) are not well-behaved and, strictly speaking, we should smooth them out with appropriate weight functions.

from the divergence condition (1.129) that the integral of $j^0(x)$ over all space is a constant of the motion:

$$Q := \int \mathrm{d}^3x j^0(\boldsymbol{x};t),\ \frac{\mathrm{d}}{\mathrm{d}t}Q = 0. \tag{1.129′}$$

This follows from

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x j^0(x) = -\int \mathrm{d}^3x \sum_{i=1}^{3}\frac{\partial}{\partial x^i} j^i(x) = 0.$$

It is not difficult to compute Q using (1.122) and the relations (1.92–1.94). One finds

$$Q = \sum_{r=1}^{2}\int \frac{\mathrm{d}^3p}{2E_p}\,[a^{(r)\dagger}(\boldsymbol{p})\ a^{(r)}(\boldsymbol{p}) + b^{(r)}(\boldsymbol{p})\,b^{(r)\dagger}(\boldsymbol{p})], \tag{1.131}$$

or, making use of the anticommutators (1.124b),

$$Q = \sum_{r=1}^{2}\int \frac{\mathrm{d}^3p}{2E_p}\,[a^{(r)\dagger}(\boldsymbol{p})\ a^{(r)}(\boldsymbol{p}) - b^{(r)\dagger}(\boldsymbol{p})\ b^{(r)}(\boldsymbol{p})]. \tag{1.131′}$$

We now calculate one-particle matrix elements of Q, again making use of the anticommutation rules (1.124).

For particles created by $a^{(r)\dagger}(\boldsymbol{p})$ we find

$$Qa^{(a)\dagger}(\boldsymbol{p})|0\rangle = +a^{(r)\dagger}(\boldsymbol{p})|0\rangle. \tag{1.132a}$$

Similarly, for "b"-type particles we have

$$Qb^{(a)\dagger}(\boldsymbol{p})|0\rangle = -b^{(r)\dagger}(\boldsymbol{p})|0\rangle. \tag{1.132b}$$

These results show that "a"-type particles and "b"-type particles have opposite "charge". This "charge" can be the electric charge, but it can also be any other charge-like quantum number that is respected by the fermion's interactions (lepton number, baryon number etc.) Thus, "a" and "b" particles are charge-conjugate to each other, or, in other terms, they are *antiparticles* of each other. We note, however, the complete symmetry of the theory in the two types of particles. It is only a matter of convention which of them is called particle and which is called antiparticle. This is in contrast to the old "hole theory" where particles are accepted as such but antiparticles are doomed to live in the shadow world of "holes" in particle states with "negative energy" (see also the remark at the end of this section).

The operator Q can also be applied to a Fock-state of many free fermions, say N particles and M antiparticles. Any such state is an eigenstate of Q with eigenvalue

("charge") $(N - M)$. If in every reaction of these fermions the "charge" Q is conserved, we say that Q is "additively conserved". This means that

$$\sum_i Q(i) = \sum_f Q(f),$$

where i counts the particles and antiparticles in the initial state, f counts the particles and antiparticles in the final state.

1.8.2 Energy and Momentum

In classical field theory the energy–momentum tensor density is given by the general expression

$$T^{\mu\nu}(x) := \sum_n \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_n)} \partial^\nu \phi_n - g^{\mu\nu} \mathscr{L}. \tag{1.133}$$

It satisfies the continuity equation

$$\partial_\mu T^{\mu\nu}(x) = 0 \tag{1.134}$$

(use the equations of motion to verify this).

Specifically, the energy density of the Dirac field is given by

$$\mathscr{H}(x) \equiv T^{00}(x) = -\overline{\psi(x)} \left(\frac{\mathrm{i}}{2} \gamma \cdot \overleftrightarrow{\nabla} - m \mathbb{1} \right) \psi(x) \tag{1.135}$$

and the momentum density is given by

$$T^{0m}(x) = \frac{\mathrm{i}}{2} \overline{\psi(x)} \gamma^0 \overleftrightarrow{\partial}^m \psi(x) = -\frac{\mathrm{i}}{2} \psi^\dagger \overleftrightarrow{\nabla}_m \psi. \tag{1.136}$$

The divergence condition (1.134) implies that the four-vector

$$(H, \boldsymbol{P}) := \left(\int \mathrm{d}^3 x \; T^{00}, \int \mathrm{d}^3 x T^{0i} \right)$$

is a constant of the motion. The operators H and $\boldsymbol{P}$ are easily calculated by inserting the expansions (1.122) of the fields in terms of creation and annihilation operators. Consider first H,

$$H = \int \mathrm{d}^3 x \, \mathscr{H}(x) = \int \mathrm{d}^3 x \psi^\dagger(x)(-\mathrm{i}\boldsymbol{\alpha} \cdot \nabla + m\beta)\psi(x),$$

where we have integrated the derivative on $\bar{\psi}$ by parts and have inserted the definitions (1.81). With (1.122) and making use again of the orthogonality properties of spinors in momentum space one finds

$$H = \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{2E_p} E_p \, [a^{(r)\dagger}(\boldsymbol{p})\, a^{(r)}(\boldsymbol{p}) - b^{(r)}(\boldsymbol{p})\, b^{(r)\dagger}(\boldsymbol{p})]. \tag{1.137a}$$

Upon using the anticommutation rule (1.124b) this can be written, up to an infinite constant,

$$H = \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{2E_p} E_p \, [a^{(r)\dagger}(\boldsymbol{p})\, a^{(r)}(\boldsymbol{p}) + b^{(r)\dagger}(\boldsymbol{p})\; b^{(r)}(\boldsymbol{p})]. \tag{1.137b}$$

The infinite constant should not worry us as only energy *differences* are physically relevant. Whenever the differences of the energies of any two states is taken, this constant drops out.

Applying the operator H to a single particle state shows that $E_p = (m^2 + \boldsymbol{p}^2)^{1/2}$ is the energy of this state, independently of whether the state contains an "a"-type or "b"-type particle. For calculating the action of H onto a more general Fock state the following commutators are helpful:

$$[a^{(r)}(\boldsymbol{p}),\; H] \;=\; E_p a^{(r)}(\boldsymbol{p}), \tag{1.138a}$$

$$[b^{(r)}(\boldsymbol{p}),\; H] \;=\; E_p b^{(r)}(\boldsymbol{p}), \tag{1.138b}$$

$$[a^{(r)\dagger}(\boldsymbol{p}),\; H] \;=\; -E_p\, a^{(r)\dagger}(\boldsymbol{p}), \tag{1.138c}$$

$$[b^{(r)\dagger}(\boldsymbol{p}),\; H] \;=\; -E_p b^{(r)\dagger}(\boldsymbol{p}). \tag{1.138d}$$

They show that applying $a^\dagger(\boldsymbol{p})$ or $b^\dagger(\boldsymbol{p})$ to any eigenstate ψ_0 of H increases its energy by the amount E_p: Indeed, let ψ_α be an eigenstate of H with energy E_α,

$$H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle.$$

Then $a^{(r)\dagger}(\boldsymbol{p})|\psi_\alpha\rangle$ is also eigenstate of H,

$$\begin{aligned} H a^{(r)\dagger}(\boldsymbol{p})|\psi_\alpha\rangle &= [a^{(r)\dagger}(\boldsymbol{p})\, H + E_p\, a^{(r)\dagger}(\boldsymbol{p})]|\psi_\alpha\rangle \\ &= (E_\alpha + E_p)\, a^{(r)\dagger}(\boldsymbol{p})|\psi_\alpha\rangle. \end{aligned}$$

Its energy is seen to be $E_\alpha + E_p$. Exactly the same argument applies to the state $b^{(r)\dagger}(\boldsymbol{p})|\psi_\alpha\rangle$.

Similarly, application of $a(\boldsymbol{p})$ or $b(\boldsymbol{p})$ leads to a new eigenstate of H whose energy is reduced by the amount E_p.

Quite similarly, the total momentum operator may be calculated from (1.136) and (1.122), and is found to be

$$\boldsymbol{P} = \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{2E_p} \boldsymbol{p} [a^{(r)\dagger}(\boldsymbol{p})\, a^{(r)}(\boldsymbol{p}) + b^{(r)\dagger}(\boldsymbol{p}) b^{(r)}(\boldsymbol{p})] \tag{1.139}$$

By using arguments completely analogous to the above, one shows that the operators $a^{(r)\dagger}(\boldsymbol{p})$ or $b^{(r)\dagger}(\boldsymbol{p})$, when applied to a Fock state, add an additional three-momentum $\boldsymbol{p}$, independently of the *type* of particle. Similarly, the corresponding annihilation operators take away three-momentum $\boldsymbol{p}$, independently of the type of particle.

To summarize, we have convinced ourselves that $a^{(r)\dagger}(\boldsymbol{p})$ and $b^{(r)\dagger}(\boldsymbol{p})$ [$a^{(r)}(\boldsymbol{p})$ and $b^{(r)}(\boldsymbol{p})$] must be interpreted as creation [annihilation] operators for free fermions with four-momentum $p = (E_p = (p^2 + m^2)^{1/2}, \boldsymbol{p})$ and spin projection r. The corresponding "wave functions" in momentum space are given by the spinors $u^{(r)}(\boldsymbol{p})$ and $v_c^{(r)}(\boldsymbol{p})$, respectively, see (1.90) and (1.91). The two kinds of particle are distinguished through their "charge", cf. (1.131) and (1.132). The two types of particle are said to be antiparticles of each other. The formalism is completely symmetric in the two kinds of particle.

Remark. Let us return, for a moment, to the Hamiltonian H of (1.137a) and note the minus sign in front of the second term. In passing from (1.137a) to (1.137b) it was essential that b and $b^\dagger$ *anti*commuted, in order to obtain the integrand of (1.137b) with the two plus signs. Had we taken *commutators* instead of the anticommutators (1.121), the second term of (1.137b) would still be negative. This would mean not only that antiparticle states had negative kinetic energies but also that H, the total energy of the field, had arbitrarily large negative eigenvalues. Both consequences, quite obviously, are not tenable on physical grounds: We know that free electrons and positrons have positive energy, and we know that a physical Hamiltonian must have a spectrum which is bounded from below. In many introductions to Dirac theory these difficulties are quoted as the primary motivation for rejecting commutators of fermion fields and for using anticommutators instead. This procedure, although acceptable, is not satisfactory for it does not reveal the real origin of the difficulty. As we have seen above, enforcing commutators leads unavoidably into a theory which is in conflict with causality. This is the deeper reason why the theory must be quantized by means of anticommutators. The spectrum of H coming out unbounded from below, when commutators are used instead, is a symptom rather than the fundamental cause of the difficulties.

1.8.3 Spin Properties of Dirac Particles

As is well-known relativistic motion mixes spin and orbital angular momentum degrees of freedom in a complicated way. In general, when the *total* angular

momentum (sum of spin and orbital angular momentum) is conserved in a reaction, neither the total spin nor the total orbital momentum are conserved separately. Therefore, for a *massive* particle, one must go to the rest frame if one wishes to know its spin. In the rest system the orbital angular momentum is zero, so that when we perform a rotation R in that system the particle state will transform with $D^{(S)}(R)$ where S is its spin. From the very construction of force-free solutions of the Dirac equation, we know that massive fermions carry spin 1/2.

Remark. The general case can be treated in several ways. The most transparent is perhaps the method used above which consists in "boosting" the particle state back to its rest system and perform a rotation there.[30] Another approach makes use of the Pauli-Lubanski four-vector which is defined by

$$W_\mu = -\frac{1}{2}\,\varepsilon_{\alpha\beta\lambda\mu}\,M^{\alpha\beta}\,P^\lambda$$

and where P^λ and $M^{\alpha\beta}$ are the generators of infinitesimal Lorentz transformations. This set of operators generates the *little* group (i.e. the set of all Lorentz transformations that leave the eigenvalues of the energy–momentum operator P^μ invariant) and yields a relativistic description of arbitrary spins. It is also very useful in discussing the spin properties of *massless* particles. In this case one finds that (excluding continuous spin) massless particles are characterized by helicity states rather than spin states and that the helicity can only have two values $\pm\lambda.\lambda$, which must be integer or half-integer is said to be the "spin" of the particle.

We do not go into these general matters here and refer the reader to the extensive literature on this subject. Instead, we turn to the more practical question of how to handle the spin and polarization properties of spin 1/2-particles.

(a) The case of massive fermions

Let us first recall the nonrelativistic description of spin and polarization of a massive fermion (in other words we go to the rest system of the particle first). Consider a statistical mixture of spin-1/2 particles polarized parallel or opposite to a given direction

$$\boldsymbol{n} = (\sin\theta\ \cos\phi,\ \sin\theta\ \sin\phi,\ \cos\theta)$$

in space. The angles θ and ϕ specify the vector $\boldsymbol{n}$ with respect to a given frame of reference K_0.

Had we used, instead, a frame K whose 3-direction coincides with $\boldsymbol{n}$, the density matrix describing this state would be given by

$$\rho|_K = \lambda_+|m_n = +\frac{1}{2}\rangle\ \langle m_n = +\frac{1}{2}| + \lambda_-|\,m_n = -\frac{1}{2}\rangle\,\langle m_n = -\frac{1}{2}| = \begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}. \tag{1.140}$$

[30] See e.g. [OMN70]

Here $\lambda_+,\ \lambda_-$ are the statistical weights of the fraction of particles polarized along the positive or negative $\boldsymbol{n}$-direction, respectively. These weights have the properties

$$0 \leq \lambda_\pm \leq 1, \quad \lambda_+ + \lambda_- = 1.$$

The states with $(\lambda_+ = 1,\ \lambda_- = 0)$, $(\lambda_+ = 0, \lambda_- = 1)$ are pure states corresponding to full polarization parallel and opposite to $\boldsymbol{n}$, respectively. The same density matrix, but written out with respect to the original frame, is obtained from (1.140) through a rotation by the Euler angles $(\psi,\ \theta,\ \phi)$

$$\rho|_{\kappa^0} = D^{(1/2)\dagger}(\psi,\ \theta,\ \phi)\, \rho|_\kappa\, D^{(1/2)}(\psi,\ \theta,\ \phi), \tag{1.141}$$

with

$$\begin{aligned} D^{(1/2)}(\psi,\ \theta,\ \phi) &= \mathrm{e}^{(\mathrm{i}/2)\psi\sigma^{(3)}}\, \mathrm{e}^{(\mathrm{i}/2)\theta\sigma^{(2)}}\, \mathrm{e}^{(\mathrm{i}/2)\phi\sigma^{(3)}} \\ &= \begin{pmatrix} \cos(\theta/2)\, \mathrm{e}^{(\mathrm{i}/2)(\psi+\phi)} & \sin(\theta/2)\mathrm{e}^{(\mathrm{i}/2)(\psi-\phi)} \\ -\sin(\theta/2)\mathrm{e}^{-(\mathrm{i}/2)(\psi-\phi)} & \cos(\theta/2)\mathrm{e}^{-(\mathrm{i}/2)(\psi+\phi)} \end{pmatrix}. \end{aligned}$$

The angles $\theta,\ \phi$ are the same as before, ψ is arbitrary, but drops out in the density matrix. One finds by straightforward calculation,

$$\rho|_{\kappa^0} = \frac{1}{2}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\lambda_+ - \lambda_-) \begin{pmatrix} \cos\theta & \sin\theta\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\, \mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix} \right\},$$

which is, of course, the same as

$$\rho = \frac{1}{2}\{\mathbb{1} + (\lambda_+ - \lambda_-)\, \boldsymbol{\sigma}\cdot\boldsymbol{n}\} \equiv \lambda_+ P_+ + \lambda_- P_-, \tag{1.141$'$}$$

where $P_\pm$ are the projectors onto "spin up" and "spin down" states along the direction $\boldsymbol{n}$, respectively,

$$P_\pm := \frac{1}{2}(\mathbb{1} + \boldsymbol{\sigma}\cdot\boldsymbol{n}). \tag{1.142}$$

It is convenient to define a polarization vector

$$\boldsymbol{\zeta} := (\lambda_+ - \lambda_-)\boldsymbol{n}, \tag{1.143}$$

so that

$$\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{\zeta}\cdot\boldsymbol{\sigma}). \tag{1.144}$$

This vector has the norm

$$\boldsymbol{\zeta}^2 = (\lambda_+ - \lambda_-)^2 = (\mathbb{1} - 2\lambda_-)^2 \leq 1.$$

It lies within a sphere of unit radius, the so-called *Poincaré* sphere.[31] Its direction gives the direction of predominant polarization, its magnitude the degree of polarization. Thus, if ζ lies *inside* the sphere, the state is a statistical mixture, if ζ lies *on* the sphere, the state is pure. Indeed, if we compute the polarization $\boldsymbol{P}$ in the state described by the density matrix (1.141′), we find

$$\boldsymbol{P} = \langle \boldsymbol{\sigma} \rangle = \mathrm{tr}(\rho \boldsymbol{\sigma}) = (\lambda_+ - \lambda_-)\boldsymbol{n} \equiv \zeta. \tag{1.145a}$$

The *degree* of polarization is given by

$$P \equiv |\boldsymbol{P}| = |\zeta| = \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-} = \lambda_+ - \lambda_-. \tag{1.145b}$$

Thus far, our description of polarization applies to the rest system of the particle, or, in an approximate way, to weakly relativistic motion. How do these notions generalize when we deal with truly *relativistic* motion of the particle?

For this purpose we must (i) transform the polarization vector $\boldsymbol{\zeta}$ to arbitrary Lorentz frames, (ii) find a covariant four-vector which is the generalization of the spin operator $\boldsymbol{\sigma}/2$, by constructing the covariant form of the spin projection operators (1.142), and (iii) find a covariant (Lorentz invariant) expression for the density matrix.

Point (i) is easy to carry out: $\boldsymbol{\zeta}$ being a classical quantity, we simply have to "boost" the vector $(0,\ \boldsymbol{\zeta})$ to the particle momentum $\boldsymbol{p}$, cf.(1.15):

$$s = L(\boldsymbol{p})(0, \boldsymbol{\zeta}) = \left(\frac{1}{m} \boldsymbol{p} \cdot \boldsymbol{\zeta}, \boldsymbol{\zeta} + \frac{\boldsymbol{p} \cdot \boldsymbol{\zeta}}{m(E_p + m)} \boldsymbol{p} \right). \tag{1.146}$$

This four-vector has the following properties:

$$s^2 = -\zeta^2, \quad \text{i.e.} \quad 0 \leq (-s^2) \leq 1, \tag{1.147a}$$

$$s \cdot p = 0. \tag{1.147b}$$

This is verified as before, cf. (1.16), the case of maximal polaritation.

In particular, the degree of polarization (which obviously is a Lorentz scalar) is $P = \sqrt{(-s^2)}$. As ζ is a spin expectation value, cf (1.145a) it must be even under parity operation in the particle's rest frame. From this observation we see that s^μ, (1.146) is an *axial* four-vector.

Concerning (ii): The construction of the covariant spin projection operators is somewhat complicated by the fact that the Dirac equation admits solutions of positive and negative frequencies (particles and antiparticles) each of which can have

[31] This sphere is originally defined in the description of polarized or partially polarized light. The formalism describing polarized electromagnetic waves (or photons) is the same as for spin-1/2 particles. The real quantities $\zeta_1,\ \zeta_2,\ \zeta_3$ are called Stokes parameters in electrodynamics.

two polarization states. Therefore matrices like $u_\alpha^{(r)}(\boldsymbol{p})\overline{u_\beta^{(r)}(\boldsymbol{p})}$ (or $\upsilon_\alpha^{(r)}(\boldsymbol{p})\overline{\upsilon_\beta^{(r)}(\boldsymbol{p})}$), which are the analogues of the non-relativistic projector $P_r = |\chi^{(r)}\rangle\langle\chi^{(r)}|$, when taken separately, will project out the spin direction r only for positive (negative) frequency solutions. In other words, both $u\bar{u}$ and $\upsilon\bar{\upsilon}$ will contain the covariant spin projection operator but multiplied by projectors onto solutions with positive and negative frequency, respectively. With these remarks in mind we proceed as follows:

Let $u(\boldsymbol{p})$ be a particle spinor of momentum $\boldsymbol{p}$, mass m, polarized along an arbitrary direction $\boldsymbol{n}$ in the rest frame of the particle. Let $\upsilon(\boldsymbol{p})$ be the corresponding antiparticle spinor with the same three-momentum and let it also be polarized along the same direction $\boldsymbol{n}$. $\boldsymbol{n}$ is a unit vector, $\boldsymbol{n}^2 = 1$. In the rest system of these particles we have (using the standard representation),

$$u(0)\overline{u(0)} = \frac{1}{2}\begin{pmatrix} \mathbb{1} + \boldsymbol{\sigma}\cdot\boldsymbol{n} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\upsilon(0)\overline{\upsilon(0)} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} - \boldsymbol{\sigma}\cdot\boldsymbol{n} \end{pmatrix}.$$

Expressing these matrices in terms of γ-matrices one verifies easily that

$$u(0)\overline{u(0)} = \frac{1}{2}(\mathbb{1} + \gamma^0)\frac{1}{2}(\mathbb{1} - \gamma_5 \boldsymbol{n}\cdot\gamma),$$

$$\upsilon(0)\overline{\upsilon(0)} = -\frac{1}{2}(\mathbb{1} + \gamma^0)\frac{1}{2}(\mathbb{1} - \gamma_5 \boldsymbol{n}\cdot\gamma).$$

We know from (1.87,1.88) that $u(\boldsymbol{p}) = N(\not{p} + m)u(0)$ and $\upsilon(\boldsymbol{p}) = -N(\not{p} - m)\upsilon(0)$ with $N = (E_p + m)^{-1/2}$. Using this we calculate the sum[32]

$$\begin{aligned} u(\boldsymbol{p})\overline{u(\boldsymbol{p})} + \upsilon(\boldsymbol{p})\overline{\upsilon(\boldsymbol{p})} &= \frac{1}{E_p + m}\{(\not{p} + m)u(0)\overline{u(0)}(\not{p} + m) \\ &\quad + (\not{p} - m)\upsilon(0)\overline{\upsilon(0)}(\not{p} - m)\} \\ &= \frac{1}{4(E_p + m)}\{(\not{p} + m)(\mathbb{1} + \gamma_0)(\mathbb{1} - \gamma_5 n^i\gamma^i)(\not{p} + m) \\ &\quad - (\not{p} - m)(\mathbb{1} - \gamma_0)(\mathbb{1} - \gamma_5 n^i\gamma^i)(\not{p} - m)\} \\ &= \frac{1}{4(E_p + m)}\{[4m\not{p} + 2\not{p}\gamma^0\not{p} + 2m^2\gamma^0] - \gamma_5 n^i[2m(\gamma^i\not{p} \\ &\quad - \not{p}\gamma^i) - 2m^2\gamma^0\gamma^i + 2\not{p}\gamma^0\gamma^i\not{p}]\}. \end{aligned}$$

[32] We use the "slash" notation, $\not{a} \equiv a^\mu\gamma_\mu$.

By commuting the γ-matrices such as to move $\not{p}$ to the left in each term and using $\not{p}\not{p} = m^2$ this can be transformed to

$$
\begin{aligned}
&= \left\{ \not{p} - \gamma_5 n^i \left[-\not{p}\gamma^i + \not{p}\gamma^0 \frac{p^i}{E_p + m} + \frac{m}{E_p + m} p^i \right] \right\} \\
&= \not{p} \left\{ \mathbb{1} + \gamma_5 \left[-n^i \gamma^i + n^i p^i \left(\frac{1}{E_p + m} \gamma^0 + \frac{1}{m(E_p + m)} \not{p} \right) \right] \right\} \\
&= \not{p} \left\{ \mathbb{1} + \gamma_5 \left[\frac{\boldsymbol{n} \cdot \boldsymbol{p}}{m} \gamma^0 - \left(n^i + \frac{\boldsymbol{n} \cdot \boldsymbol{p}}{m(E_p + m)} p^i \right) \gamma^i \right] \right\} \\
&= \not{p}(\mathbb{1} + \gamma_5 \not{n}),
\end{aligned}
$$

where n is defined in analogy to (1.146) and is nothing but the four-vector $(0; \boldsymbol{n})$ "boosted" to the system where the particles have momentum $\boldsymbol{p}$. In exactly the same manner one computes the difference

$$u(\boldsymbol{p})\overline{u(\boldsymbol{p})} - \upsilon(\boldsymbol{p})\overline{\upsilon(\boldsymbol{p})} = m(\mathbb{1} + \gamma_5 \not{n}).$$

From these results follow the important relations

$$u_\alpha(\boldsymbol{p})\overline{u_\beta(\boldsymbol{p})} = \frac{1}{2}\{(\not{p} + m\mathbb{1})(\mathbb{1} + \gamma_5 \not{n})\}_{\alpha\beta}, \tag{1.148a}$$

$$\upsilon_\alpha(\boldsymbol{p})\overline{\upsilon_\beta(\boldsymbol{p})} = \frac{1}{2}\{(\not{p} - m\mathbb{1})(\mathbb{1} + \gamma_5 \not{n})\}_{\alpha\beta}. \tag{1.148b}$$

We recall the definition of n^μ:

$$n^\mu = \left(\frac{\boldsymbol{p} \cdot \boldsymbol{n}}{m}, \boldsymbol{n} + \frac{\boldsymbol{n} \cdot \boldsymbol{p}}{m(E_p + m)} \boldsymbol{p} \right), \tag{1.149}$$

which satisfies $n^2 = -1,\ (np) = 0$.

Let us comment on these results. Obviously, the spin projection operator onto the positive $\boldsymbol{n}$ direction is

$$\pi_n = \frac{1}{2}(1 + \gamma_5 \gamma^\mu n_\mu) \tag{1.150}$$

for both particles and antiparticles. In the rest system it reduces to the familiar form

$$\pi_{\boldsymbol{n}}|_{\boldsymbol{p}=0} = \frac{1}{2} \begin{pmatrix} 1 + \boldsymbol{\sigma} \cdot \boldsymbol{n} & 0 \\ 0 & 1 - \boldsymbol{\sigma} \cdot \boldsymbol{n} \end{pmatrix}. \tag{1.151}$$

The second term in (1.150) is the scalar product of n^μ and $\gamma_5 \gamma^\mu$. We already know that n^μ is an axial vector; $\gamma_5 \gamma^\mu$, when taken between Dirac fields, is also an axial vector, so that $(\bar{\psi} \gamma_5 \gamma^\mu \psi) n_\mu$ is a scalar.

As to the other factor in (1.148) it should be clear that

$$\Omega_\pm = \pm\frac{1}{2m}(\not{p} \pm m\mathbb{1}) \tag{1.152}$$

are the projectors onto positive and negative frequency solutions, respectively. The normalization follows from the requirement

$$\Omega_\pm^2 = \Omega_\pm.$$

From $\not{p}\not{n} = 2(pn) - \not{n}\not{p} = -\not{n}\not{p}$ and from (1.80) we see that $\not{p}$ commutes with $\gamma_5\gamma^\mu n_\mu$ and, therefore, that

$$[\pi_n, \Omega_+] = 0 = [\pi_n, \Omega_-],$$

which repeats our statement that π_n is the spin projection operator, independently of whether we deal with a particle or an antiparticle.

It is now easy to answer question (iii): The covariant density matrix that describes an ensemble of *particles* with partial polarization $\boldsymbol{P} = \zeta$, cf. (1.145), is[33]

$$\rho = 2m\Omega_+\frac{1}{2}(\mathbb{1} + \gamma_5\not{s}) = \frac{1}{2}(\not{p} + m\mathbb{1})(\mathbb{1} + \gamma_5\not{s}), \tag{1.153}$$

where s is defined in (1.146) and has the norm

$$\sqrt{-s^2} = |\zeta| \equiv P = \lambda_+ - \lambda_-.$$

We verify that

$$\rho^2 = 4m^2\Omega_+\frac{1}{2}\left(\frac{1+\zeta^2}{2}\mathbb{1} + \gamma_5\not{s}\right),$$

which is equal to $2m\rho$ only for $|\zeta| = 1$, and that

$$\mathrm{tr}\left(\frac{\rho^2}{4m^2}\right) = \frac{1}{2}(1+\zeta^2) \leq \mathrm{tr}\left(\frac{\rho}{2m}\right) = 1. \tag{1.154a}$$

Note that ρ is not hermitean but that instead

$$\gamma^0\rho^\dagger\gamma^0 = \rho. \tag{1.154b}$$

In eq. (1.154a) the equality sign holds if $|\zeta| = 1$, i.e. for a pure state. In that case $\rho^2 = 2m\rho$.

[33] We have normalized the density matrix in accordance with the covariant normalization (1.92,1.89), i.e. $\mathrm{tr}\,\rho = 2m$.

Similarly, the density matrix describing antiparticles with polarization ζ is given by[34]

$$\rho = -2m\Omega_- \frac{1}{2}(\mathbb{1} + \gamma_5 \not{s}) = \frac{1}{2}(\not{p} - m\mathbb{1})(\mathbb{1} + \gamma_5 \not{s}). \tag{1.155}$$

It is easy to see that $\rho^\dagger$ describes the parity-mirror state of ρ, i.e. $\boldsymbol{p} \to -\boldsymbol{p}$ but $\boldsymbol{\zeta} \to \boldsymbol{\zeta}$. Since fermions always appear in bilinear forms in any observable, there is no harm in having a non-hermitean density matrix (see exercise 1.9).

If one insists on having a hermitean density matrix one may use, instead of (1.153):

$$P := \gamma^0 \rho = \frac{1}{2}(E_p \mathbb{1} - \boldsymbol{p} \cdot \boldsymbol{\alpha} \pm m\beta)(\mathbb{1} + \gamma_5 \not{s}). \tag{1.156}$$

This matrix has the properties (for particles *and* antiparticles),

$$P^\dagger = P, \quad \mathrm{tr}(P) = 2E_p.$$

Thus, its trace is not a Lorentz scalar [the normalization is the covariant one of (1.89)].

It is covenient to express the polarization vector ζ on (or inside) the Poincaré sphere in terms of its component ζ_l along the particle's three-momentum $\boldsymbol{p}$ and its components perpendicular to $\boldsymbol{p}$,

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_\mathrm{l} \boldsymbol{p}/|\boldsymbol{p}| + \boldsymbol{\zeta}_\mathrm{t} \quad \text{with} \quad \boldsymbol{\zeta}_\mathrm{l} = \frac{1}{|\boldsymbol{p}|}\boldsymbol{p} \cdot \boldsymbol{\zeta}. \tag{1.157a}$$

The four-vector s, (1.146), is then given by the components

$$s^0 = \frac{1}{m}|\boldsymbol{p}|\boldsymbol{\zeta}_\mathrm{l}, \quad \boldsymbol{s}_\mathrm{l} = \frac{E_p}{m}\zeta_\mathrm{l}\frac{\boldsymbol{p}}{|\boldsymbol{p}|}, \quad \boldsymbol{s}_\mathrm{t} = \boldsymbol{\zeta}_\mathrm{t}. \tag{1.157b}$$

The special cases of longitudinal and transverse polarizations (with respect to the momentum $\boldsymbol{p}$) can be read off these formulae.

A case of special interest is the case of extreme relativistic motion which we discuss separately.

(*b) Extreme relativistic motion and the neutrinos*

Suppose first that we deal with a massive particle whose energy is very large as compared to its mass, i.e.

$$E \gg m, \quad |\boldsymbol{p}| \simeq \boldsymbol{E}.$$

Let us take the 3-direction in the direction of the particle's momentum $\boldsymbol{p}$. Then from (1.153, 1.155)

[34] Normalization and sign in agreement with (1.93).

$$\rho \simeq \frac{1}{2}\{E(\gamma^0-\gamma^3)\pm m\mathbb{1}\}\left\{\mathbb{1}+\gamma_5\left[\frac{E}{m}\zeta_1(\gamma^0-\gamma^3)-\zeta_t^1\gamma^1-\zeta_t^2\gamma^2\right]\right\}$$
$$\simeq \frac{1}{2}E(\gamma^0-\gamma^3)\{\mathbb{1}+\gamma_5[\mp\zeta_1-\zeta_t^1\gamma^1-\zeta_t^2\gamma^2]\}.$$

Here we have used $(\gamma^0-\gamma^3)^2=(\gamma^0)^2+(\gamma^3)^2-\{\gamma^0,\gamma^3\}=0$ and we have neglected $m\mathbb{1}$ against $E(\gamma^0-\gamma^3)$. The result can also be written as follows:

$$\rho \simeq \frac{1}{2}\not{p}\{\mathbb{1}-\gamma_5[\pm\zeta_1+\zeta_t^1\gamma^1+\zeta_t^2\gamma^2]\}; \tag{1.158}$$

the positive sign holds for particles, the negative sign for antiparticles.

The expression (1.158) which can be used to describe, for example, electrons and positrons at ultra-relativistic energies, shows that such particles can have any partial or full polarization, along their momentm or transverse to it. For instance, a statistical mixture of electrons with positive helicity (statistical weight λ_+) and electrons with negative helicity (statistical weight λ_-) at very high energy is described by

$$\rho \simeq \frac{1}{2}\not{p}\{1-(\lambda_+-\lambda_-)\gamma_5\}.$$

Expression (1.158) is also applicable to massless fermions, i.e. to neutrinos. However, there is one essential restriction. The only possible spin states are the ones with positive or negative helicity $h=\pm\lambda$. This can be understood very qualitatively as follows: Massless particles have no rest system. For a *massive* particle we can always go back to its rest system and rotate its spin into any direction we wish by means of the full rotation group. For a *massless* particle, the particle's momentum $\boldsymbol{p}$ singles out a specific spatial direction; the only Lorentz transformations that may remain "good" symmetries are the ones which leave this direction invariant (so-called *little group*): rotations about the spatial direction $\boldsymbol{p}$ and reflections with respect to any plane in the three- dimensional space that contains this direction. Regarding the spin properties of photons, we know that "right"- and "left"-circularly polarized plane waves are described by the polarization vectors (taking the photon momentum $\boldsymbol{q}$ in the 3-direction)

$$e_{\pm1}:=\mp\frac{1}{\sqrt{2}}(e_1\pm \mathrm{i}e_2).$$

Rotations about the three-axis leave these quantities invariant (except for multiplication by a phase); while reflection with respect to the plane spanned by the 1-axis and the 3-axis transforms one state into the other. As both kinds of transformations are symmetries which leave the Maxwell equations invariant (in homogeneous and isotropic space), also any other transverse photon polarization is possible: any linear superposition of the two helicity states $e_{\pm1}$ is acceptable (linear polarization, elliptic polarization).

For neutrinos and antineutrinos the possible spin states are the states with helicity $\lambda = \pm\frac{1}{2}$. It appears that *neutrinos* which are produced in weak interactions at moderate energies always have *negative* helicity (they are said to be "left-handed"), whilst *antineutrinos* always have *positive* helicity (they are said to be "right-handed"). Thus, the density matrix for neutrinos [(1.158) with upper sign and $\zeta_1 = -1$] as well as for antineutrinos [(1.158) with lower sign and $\zeta_1 = +1$] reads

$$\rho^{(\nu)} = \frac{1}{2}\not{p}(\mathbb{1} + \gamma_5). \tag{1.159}$$

It describes a pure state: a neutrino state of negative helicity or an antineutrino state of positive helicity. A priori there is no reason why the massless neutrino (antineutrino) couples to other particles only in left-handed (right-handed) states. In principle massless fermions could have either helicity, or could be in states which are superpositions of the two helicities $\lambda = \pm\frac{1}{2}$. As we have seen above, the density matrix

$$\rho(m = 0) = \frac{1}{2}\not{p}(\mathbb{1} - 2\lambda\gamma_5) \tag{1.160}$$

describes particles with helicity λ, or antiparticles with helicity $-\lambda$. For electrons at ultra-high energies these are indeed possible states.

The fact that for neutrinos and antineutrinos (1.159) is the only possibility, is of dynamical origin: In connection with the photon's helicity we have said above that the symmetry operations relevant to a massless particle of a given momentum $\boldsymbol{p}$ are rotations about $\boldsymbol{p}$ and reflections with respect to planes containing the vector $\boldsymbol{p}$. These reflections, in particular, convert positive into negative helicity and vice versa (exercise 1.14). Whilst electromagnetic interactions are invariant under both rotations and reflections, this is not so for weak interactions. Weak interactions are invariant under rotations but not under parity or, for our purpose, under reflections with respect to planes that contain $\boldsymbol{p}$. Therefore, the two possible helicity states are not degenerate dynamically. Actually parity violation in the leptonic sector is found to be maximal: one helicity state ($\lambda = +1/2$ for ν, $\lambda = -1/2$ for $\bar{\nu}$) seems to decouple completely from the physical world of particles.

As a consequence the parity transform (1.160) of $\rho(m = 0)$:

$$\rho_P(m = 0) = \gamma^0\rho(m = 0;\ \boldsymbol{p} \to -\boldsymbol{p})\gamma^0 = \frac{1}{2}\not{p}(1 + 2\lambda\gamma_5) \tag{1.161}$$

describes states which do not couple in weak interactions. By the same token charge conjugation cannot be a symmetry of weak interactions, for it transforms a left-handed neutrino (right-handed antineutrino) into a left-handed antineutrino (right-handed neutrino) which decouples from other particles.

As an exercise the reader is invited to show that the charge conjugate of (1.160) is given by

$$\rho_C(m = 0) = \frac{1}{2}\not{p}(\mathbb{1} + 2\lambda\gamma_5). \tag{1.162}$$

If, on the other hand, we consider the combined operation of parity and charge conjugation, PC, then we see from (1.161, 1.162) that $\rho(m = 0)$ and, more specifically, $\rho^{(\nu)}$ are invariant under PC. Thus the neutrino and the antineutrino are PC-partners of each other.

Note. For spin-1/2 particles it is customary to denote the helicity states by $h := 2\lambda$, i.e. $h = \pm 1$ instead of $\lambda = \pm 1/2$. We shall adopt this convention in Chaps. 2 and 3.

1.8.4 Dirac and Majorana Mass Terms

In (1.108) and (1.126) we have encountered real mass terms for the case of one single Majorana and one single Dirac field, respectively. These are special cases of the most general mass term that is compatible with Lorentz invariance. As the general case is instructive and provides further insight into the structure of the theory we work it out in some detail.

Let $\phi_a(x)$ and $\chi^A(x)$ be spinors of first and second kind, respectively, without any condition (such as, e.g., (1.106)) imposed on them. The most general, Lorentz invariant and hermitean Lagrangian (without interaction terms) that can be constructed on the basis of these fields, is the following:

$$\begin{aligned}\mathscr{L} = &\frac{\mathrm{i}}{2}\left\{\phi^*{}_A(x)(\hat{\sigma}_\mu \overleftrightarrow{\partial}_\mu)^{Ab}\phi_b(x) + \chi^{*a}(x)(\sigma^\mu \overleftrightarrow{\partial}_m)_{aB}\chi^B(x)\right\}\\ &+ \left\{m_{\mathrm{D}}\chi^*{}_a\varepsilon^{ab}\phi_b - m^*_{\mathrm{D}}\phi^*{}_A\varepsilon^{AB}\chi_B\right\}\\ &+ \frac{1}{2}\left\{m_1\phi_a\varepsilon^{ab}\phi_b + m_1^*\phi^*{}_A\varepsilon^{AB}\phi^*_B\right\}\\ &- \frac{1}{2}\{m_2\chi^{*a}\varepsilon_{ab}\chi^{*b} + m_2^*\chi^A\varepsilon_{AB}\chi^B\}. \end{aligned} \tag{1.163}$$

Here m_{D}, m_1, m_2 are arbitrary, complex parameters with dimension of masses. Each one of the three mass terms in curly brackets is hermitean by itself, as is easily verified by means of relations (1.56, 1.57). The terms containing m_1, m_2 and their complex conjugates are generalizations of the mass term in the Majorana Lagrangian (1.108), whilst the terms in m_{D}, m^*_{D} generalize the Dirac case of (1.126).

In order to clarify the structure of the mass term in (1.163) we note first that it can be rewritten in a way which eliminates the apparent asymmetry in the spinor fields $\phi(x)$ and $\chi(x)$. Indeed, $\chi^*{}_a = \varepsilon_{ab}\chi^{*b}$ is another spinor of first kind, having the same transformation properties as ϕ_a. Likewise $\phi^{*A} = \varepsilon^{AB}\phi*_B$ is a spinor of second kind, very much like χ^A. It is convenient to introduce the following, more symmetric notation:

$$\phi^{(1)}_a(x) := \phi_a(x), \quad \phi^{(2)}_a(x) := \chi^*_a(x), \tag{1.164}$$

as well as to define the following Majorana fields:

$$\Phi^{(1)}(x) := \begin{pmatrix} \phi_a(x) \\ -\phi^{*A}(x) \end{pmatrix} \equiv \begin{pmatrix} \phi_a^{(1)}(x) \\ -\phi^{(1)*A}(x) \end{pmatrix}, \tag{1.165a}$$

$$\Phi^{(2)}(x) := \begin{pmatrix} \chi^*{}_a(x) \\ \chi^A(x) \end{pmatrix} \equiv \begin{pmatrix} \phi_a^{(2)}(x) \\ -\phi^{(2)*A}(x) \end{pmatrix}. \tag{1.165b}$$

[The signs in (1.165) follow from the convention for ε_{AB}, from (1.98) for $\psi_C(x)$, via the relations (1.56).] By making use of the defining relation $\hat{\sigma}^\mu = \varepsilon(\sigma^\mu) * \varepsilon^{-1}$ and of the hermiticity of the matrices σ^μ one shows that

$$\phi^a(\sigma^\mu \overleftrightarrow{\partial}_\mu)_{aB}\phi^{*B} = \phi^*{}_A(\hat{\sigma}^\mu \overleftrightarrow{\partial}_\mu)^{Ab}\phi_b$$

and an analogous relation for the kinetic energy of the χ-field. Thus, the kinetic energy in the Lagrangian (1.163) can be written in the following form:

$$\mathscr{L}_{\text{kin}} = \frac{\mathrm{i}}{4}\sum_{k=1}^{2}\{\phi^{(k)*}{}_A(x)(\hat{\sigma}^\mu \overleftrightarrow{\partial}_\mu)^{Ab}\phi_b^{(k)}(x) - \phi^{(k)a}(x)(\sigma^\mu \overleftrightarrow{\partial}_\mu)_{aB}\phi^{(k)*B}(x)\}. \tag{1.166}$$

The mass terms in m_1 and m_2 have the structure

$$-\frac{1}{2}m_{ii}\phi^{(i)a}(x)\phi_a^{(i)}(x),$$

with $m_{11} \equiv m_1$, $m_{22} \equiv m_2$. As to the term in m_{D} we note that $\phi^a\chi^*{}_a \equiv \phi^{(1)a}\phi_a^{(2)}$ is the same as $\chi^{*a}\phi_a \equiv \phi^{(2)a}\phi_a^{(1)}$. Therefore all mass terms can be written in the following notation:

$$\mathscr{L}_{\text{mass}} = -\frac{1}{2}\left\{\sum_{i,k=1}^{2} m_{ik}\phi^{(i)a}\phi_a^{(k)} + \text{h.c.}\right\}, \tag{1.167}$$

where

$$m_{ik} = \begin{pmatrix} m_1 & m_{\text{D}} \\ m_{\text{D}} & m_2 \end{pmatrix} =: M. \tag{1.168}$$

The mass matrix M, (1.168), is a symmetric, (in general) complex 2×2 matrix. The physical fermion fields which are described by the free Lagrangian (1.167) are obtained by a transformation W,

$$\phi_a^{(i)'}(x) = \sum_j W_{ij}\phi_a^{(j)}(x) \tag{1.169}$$

of the fields $\phi^{(i)}$ which leaves invariant $\mathscr{L}_{kin}$ and transforms $\mathscr{L}_{mass}$ to diagonal form with real, positive semidefinite eigenvalues. If the kinetic energy $\mathscr{L}_{kin}$ is to be invariant, W must be unitary,

$$WW^\dagger = \mathbb{1}. \tag{1.170}$$

In order to diagonalize L_{mass} of (1.167), we must have

$$W^* M W^\dagger = \overset{\circ}{M}, \tag{1.171}$$

with

$$\overset{\circ}{M} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad \lambda_1, \lambda_2 \geq 0.$$

It is easy to show that λ_1 and λ_2 can indeed be requested to be positive semidefinite, without loss of generality. To see this, suppose that the eigenvalues $\lambda_j = |\lambda_j| \mathrm{e}^{\mathrm{i}\alpha j}$ were indeed complex. Let

$$P = \begin{pmatrix} \mathrm{e}^{i\alpha_1/2} & 0 \\ 0 & \mathrm{e}^{i\alpha_2/2} \end{pmatrix}$$

so that

$$\overset{\circ}{M} = P \overset{\circ}{M'} P \quad \text{with} \quad \overset{\circ}{M'} = \begin{pmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{pmatrix},$$

Multiplying (1.171) by P^{-1}, both from the right and from the left, and observing that $P^{-1} = P^* = P^\dagger$, we obtain

$$(PW)^* M (PW)^\dagger = \overset{\circ}{M'},$$

Now, if W is unitary, so is $W' := PW$, and our assertion is proved. Therefore, from now on we shall assume $\lambda_1, \lambda_2 \geq 0$. As it stands, (1.171) is not a standard diagonalization prescription because M is not hermitean (the equation is multiplied by W^*, not W, from the left). However, by taking its hermitean conjugate which is

$$W M^\dagger W^T = \overset{\circ}{M}, \tag{1.171$'$}$$

by multiplying this equation with (1.171) from the right and using the unitarity of W, we may transform this eigenvalue problem to

$$W(M^\dagger M) W^\dagger = \overset{\circ}{M}^2, \tag{1.172}$$

i.e. to the standard problem of diagonalizing the hermitean matrix $(M^\dagger M)$. This is what we now proceed to do. We set

$$m_1 = \mu_1 \mathrm{e}^{i\phi_1}, \quad m_2 = \mu_2 \mathrm{e}^{i\phi_2}, \quad m_{\mathrm{D}} = \mu_{\mathrm{D}} \mathrm{e}^{i\phi_{\mathrm{D}}}$$

with $\mu_1,\ \mu_2,\ \mu_\mathrm{D}$ real and positive. By transforming, in a first step, the fields $\phi_a^{(i)} \to \phi_a^{(i)} \mathrm{e}^{\mathrm{i}\phi_\mathrm{i}/2}$, the matrix M becomes

$$M \to M' = \begin{pmatrix} \mu_1 & \mu_\mathrm{D}\mathrm{e}^{\mathrm{i}\phi} \\ \mu_\mathrm{D}\mathrm{e}^{\mathrm{i}\phi} & \mu_2 \end{pmatrix} \tag{1.168'}$$

with

$$\phi = \varphi_\mathrm{D} - \frac{1}{2}(\varphi_1 + \varphi_2). \tag{1.173}$$

This gives

$$M'^{\dagger} M' = \begin{pmatrix} \mu_1^2 + \mu_\mathrm{D}^2 & \mu_\mathrm{D}(\mu_1 \mathrm{e}^{\mathrm{i}\phi} + \mu_2 \mathrm{e}^{-\mathrm{i}\phi}) \\ \mu_\mathrm{D}(\mu_1 \mathrm{e}^{-\mathrm{i}\phi} + \mu_2 \mathrm{e}^{\mathrm{i}\phi}) & \mu_2^2 + \mu_\mathrm{D}^2 \end{pmatrix},$$

whose eigenvalues are easily calculated,

$$\begin{aligned} (\lambda_{1/2})^2 = \frac{1}{2}\{&\mu_1^2 + \mu_2^2 + 2\mu_\mathrm{D}^2 \\ &\pm \sqrt{\left(\mu_1^2 - \mu_2^2\right)^2 + 4\mu_\mathrm{D}^2 \left(\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2 \cos(2\phi)\right)}\} \end{aligned} \tag{1.174}$$

Let us now interpret these results. After diagonalization, as described above, the mass term in the Lagrangian (1.167) becomes

$$\mathscr{L}_\mathrm{mass} = -\frac{1}{2}\sum_{k=1}^{2} \lambda_k \{\phi^{(k)'a} \phi_a^{(k)'} + \phi^{(k)'*}{}_A \phi^{(k)'*A}\}. \tag{1.175}$$

Here, $\phi_a^{'(i)}$ is given by (1.169). When expressed in terms of four-component spinors (1.165), the mass eigenstates are

$$\Phi^{'(k)}(x) = \begin{pmatrix} \sum\limits_l W_{kl} \phi_a^{(l)}(x) \\ -\sum\limits_l W_{kl}^* \phi^{(l)*A}(x) \end{pmatrix}. \tag{1.176}$$

When written in terms of these Majorana fields the mass term becomes simply

$$-\frac{1}{2}\sum_{k=1}^{2} \lambda_k \overline{\Phi'^{(k)}(x)} \Phi'^{(k)}(x)$$

and the transformed Lagrangian reads

$$\mathscr{L} = \frac{1}{2}\sum_{k=1}^{2} \left\{ \frac{i}{2} \overline{\Phi'^{(k)}(x)} \gamma^\alpha \overleftrightarrow{\partial}_\alpha \Phi'^{(k)}(x) - \lambda_k \overline{\Phi'^{(k)}(x)} \Phi'^{(k)}(x) \right\}, \tag{1.177}$$

i.e. it has the familiar form (1.126′)

Cases of special interest are the ones where λ_1 and λ_2 are degenerate. From (1.174) we see that this happens *either*

(i) if $\mu_1 = \mu_2,\ \mu_{\mathrm{D}} = 0$; *or*
(ii) if $\mu_1 = \mu_2 =: \mu$ and

$$\phi = \varphi_{\mathrm{D}} - \frac{1}{2}(\varphi_1 + \varphi_2) = (2n+1)\pi/2.$$

The first case is trivial and gives $\lambda_1 = \lambda_2 = \mu_1 = \mu_2$. The second is particularly interesting and has $\lambda_1 = \lambda_2 = \left(\mu^2 + \mu_{\mathrm{D}}^2\right)^{1/2}$. In either case we can introduce a Dirac field by combining $\Phi^{(1)}$ and $\Phi^{(2)}$:

$$\psi(x) := \frac{1}{\sqrt{2}}(\Phi^{(1)}(x) + \mathrm{i}\Phi^{(2)}(x)). \tag{1.178}$$

The orthogonal combination

$$\frac{1}{\sqrt{2}}(\Phi^{(1)}(x) - \mathrm{i}\Phi^{(2)}(x)) = \psi_C(x)$$

is then the charge conjugate of ψ. The Lagrangian (1.177) reduces to the Lagrangian of a single Dirac field.

This discussion shows that, in some sense, *Majorana fields are more fundamental than Dirac fields*. The general Lagrangian (1.163) describes two Majorana fields, (1.176), with mass eigenvalues $\lambda_1,\ \lambda_2$, (1.174), respectively. Only if these eigenvalues are equal can the two fields be combined to a Dirac field and its charge conjugate.[35]

The essential difference between these two cases is the following:

(1) If $\lambda_1 \neq \lambda_2$, the basic fields are Majorana fields and, therefore, cannot carry any additively conserved charge quantum number. In other words, if the theory with interactions admits an absolutely and additively conserved charge Q, then the two Majorana fields must belong to the eigenvalue zero of that charge. If this is not so, then the mass terms conserve that charge Q only modulo 2 because they connect states which differ by $\Delta Q = \pm 2$.
(2) If $\lambda_1 = \lambda_2$ the basic fields are one Dirac field and its charge conjugate ψ_C. These fields being different, ψ can carry any nonvanishing charge Q, ψ_C then carrying charge $-Q$.

[35] It is interesting to remark that these considerations can be generalized to an arbitrary number of fields.

1.8.5 Neutrino Masses: The Seesaw Mechanism

The analysis of mass terms of fermion fields as described by the Lagrangian (1.163) has an interesting application in the physics of neutrinos. We have seen above that charged leptons such as the electron can have only Dirac mass terms, i.e. terms of type m_{D} (or μ_{D}) in the formulae of the preceding section, unless one or the other of the exceptional relations (i) or (ii) of Sect. 1.8.4 is fulfilled. As noted above, this is due to the observation that a genuine Majorana mass term does not conserve any additive property such as the electric charge of a fermion. In contrast to this, a neutrino such as the electron's partner ν_{e}, being electrically neutral, could well have all three types of mass terms, including the Majorana terms μ_1 and μ_2. Experiment tells us, however, that the upper limit for the mass of ν_{e} is very small as compared to the electron mass m_{e}, say, of the order of $10^{-6}\, m_{\mathrm{e}}$, i.e. $m(\nu_{\mathrm{e}}) < 4 \cdot 10^{-6}\, m_{\mathrm{e}}$. A nontrivial mass matrix (1.168') for the neutral partner(s) of the electron could be the following. Assume the Dirac mass term μ_{D} to be of the order of m_{e} : $\mu_{\mathrm{D}} \approx m_{\mathrm{e}}$. In order to keep track of the upper limit just quoted, assume $\mu_1 \ll \mu_{\mathrm{D}}$. In contrast to this term the other Majorana mass term μ_2 could be very large, the idea being that it is a signal for some new physics that would emerge at a high mass scale $\mu_2 \gg \mu_{\mathrm{D}} \approx m_{\mathrm{e}}$. In this situation the eigenvalues (1.174) of the squared mass matrix can be approximated by neglecting μ_1, as compared to μ_{D} and μ_2, i.e.

$$(\lambda_{1/2})^2 \approx \frac{1}{2}\mu_2^2\{1 + 2\varepsilon \pm \sqrt{1 + 4\varepsilon}\}, \quad \text{with} \quad \varepsilon := \frac{\mu_{\mathrm{D}}^2}{\mu_2^2}.$$

Keeping only the lowest nonvanishing order in each case, one finds $\lambda^2{}_1 \approx \mu_2^2$, and $\lambda^2{}_2 \approx \mu_2^2\varepsilon^2$. Taking the square roots, one has

$$\lambda_1 \approx \mu_2, \quad \lambda_2 \approx \frac{\mu_{\mathrm{D}}^2}{\mu_2}. \tag{1.179}$$

The result (1.179) is interpreted as follows. The mass matrix (1.168) with $\mu_{\mathrm{D}} \approx m_{\mathrm{e}}$ and a large energy scale $\mu_2 \gg m_{\mathrm{e}}$ gives rise to a heavy neutral particle with mass $\lambda_1 \approx \mu_2$, and a light neutral particle with mass $\lambda_2 \approx m_{\mathrm{e}}^2/\mu_2$. Both are Majorana particles. The first one is a new, heavy, neutrino-like particle; the second is a candidate for ν_{e}, the light neutrino that accompanies the electron.

This way of generating a small but nonvanishing neutrino mass, $m(\nu_{\mathrm{e}}) \ll m_{\mathrm{e}}$ is called *seesaw mechanism* (Gell-Mann et al., 1979): the higher up (in energy scale) the object (λ_1) is on one end of the plank, the further down is the second object (λ_2) on the other end. Indeed, many models of massive neutrinos make use of this pattern to explain the smallness of the measured upper limits on the masses of ν_{e}, ν_μ and ν_τ. At the same time such models predict the existence of heavy neutrinos which are partners of the light ones. The smaller the masses of ν_{e}, ν_μ, ν_τ, the larger the masses of the partners.

1.9 Charged Fermion Fields in Interaction with Electromagnetic Fields

1.9.1 External Field Case

In many situations electromagnetic interactions of charged leptons with some other charged system can be treated in the *external field approximation*. If the system with which the lepton interacts is very heavy, it will be able to absorb or to provide three-momentum in reactions with the lepton without altering its own state. If, in adition, it is dynamically inert its internal structure will not intervene in such reactions. In those cases the effect of the system on a charged lepton can be represented, to a certain approximation, by classical external vector potentials $A_\mu^{\text{ext}}(x)$. A case of special relevance for atomic, nuclear and particle physics is the electromagnetic interaction of electrons and muons with nuclei. As far as the *kinematics* is concerned, the nucleus is so much heavier than the electron or the muon that in a reaction with these leptons its recoil is usually not important. The nucleons act like an external macroscopic source of electric and magnetic fields which can absorb or produce any mismatch of three-momentum that there may be in a given reaction. Well-known examples for such situations are: Bremsstrahlung in matter; atomic bound states of electrons or muons; elastic and inelastic scattering of electrons or muons on nuclei at intermediate energies.

Regarding the *dynamics*, the internal structure of the nucleus (other than its initial and final state involved in the reaction) is usually unimportant. This is so because the virtual intermediate excitation of higher states of the nucleus necessitates (at least) a two-step process and, therefore, is of higher order in the fine structure constant. Such effects, which are called *nuclear polarizability shift* in atoms, and *dispersion corrections* in electron scattering, are generally small and may be added as a correction to the results of the external field approximation from which on started.

Let $A_\mu^{\text{ext}}(t, x)$ be an external, classical four-vector potential describing a given set of classical electric and magnetic fields. The coupling of a lepton of charge Q to these fields is found by the substitution

$$\partial_\mu \to \partial_\mu + \mathrm{i}QA_\mu^{\text{ext}} \tag{1.180}$$

in the particle's equations of motion. This is the so-called "minimal coupling" rule on which we shall comment below. If we apply this prescription to the free Dirac equation (1.72), we obtain

$$(\mathrm{i}\gamma^\mu\partial_\mu - Q\gamma^\mu A_\mu^{\text{ext}} - m\mathbb{1})\psi(x) = 0. \tag{1.181}$$

For example, let us take a stationary external electric field and let us consider stationary solutions of this equation (positive frequency), viz.

$$\begin{aligned} &A_\mu^{\text{ext}}(x) = (\phi(x); \mathbf{0}), \\ &\psi(x) = \mathrm{e}^{-\mathrm{i}Et}\psi(\boldsymbol{x}), \\ &[(E - V(\boldsymbol{x}))\gamma^0 + \mathrm{i}\boldsymbol{\gamma}\cdot\nabla - m\mathbb{1}]\psi(\boldsymbol{x}) = 0, \end{aligned} \tag{1.181'}$$

where we have set $V(\boldsymbol{x}) = Q\phi(\boldsymbol{x})$.

For an electron or muon $Q = -e$, while for a point-like nucleus of charge number Z, $\phi(\boldsymbol{x}) = Ze/|\boldsymbol{x}|$, so that

$$[(E + Ze^2/|\boldsymbol{x}|)\gamma^0 + \mathrm{i}\boldsymbol{\gamma}\cdot\nabla - m1]\psi(\boldsymbol{x}) = 0.$$

Equivalently, by multiplying (1.181') with $\gamma^0 \equiv \beta$ [in the standard representation (1.78)] from the left, one has

$$E\psi(\boldsymbol{x}) = [-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla + V(\boldsymbol{x}) + m\beta]\psi(\boldsymbol{x}), \tag{1.182}$$

where, for a point-like nucleus,

$$V(\boldsymbol{x}) = -Ze^2/|\boldsymbol{x}|. \tag{1.183}$$

If the spatial extension of the nuclear charge density $\rho_c(\boldsymbol{x})$ cannot be neglected as compared to the typical size of the electron or muon state, we have instead

$$V(x) = -Ze^2 \int \mathrm{d}^3x' \frac{\rho_c(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{1.184}$$

We take the nuclear charge density $\rho_c(\boldsymbol{x})$ normalized to unity,

$$\int \mathrm{d}^3\boldsymbol{x}\rho_c(\boldsymbol{x}) = 1. \tag{1.185}$$

In particular, if this density is spherically symmetric, the potential is also spherically symmetric and is given by[36]

$$V(r) = -4\pi Ze^2 \left[\frac{1}{r}\int_0^r \rho_c(r')r'^2\mathrm{d}r' + \int_r^\infty \rho_c(r')r'\mathrm{d}r'\right]. \tag{1.186}$$

It is (1.182) that is the most convenient one in treating atomic bound states of charged leptons as well as scattering off nuclei at low energies. The representation (1.81) is particularly well adapted in this case as it divides the Dirac spinors naturally

[36] For a point-like nucleus, placed at the origin,

$$\rho_c(\boldsymbol{x}) = \delta(\boldsymbol{x}) = (1/r^2)\delta(r)\delta(\cos\theta - 1)\delta(\phi),$$

in which case both (1.184) and (1.186) go over into (1.183).

into "large" (i.e. nonrelativistic) and "small" (i.e. relativistic) components. Also, in this form, (1.182) comes closest to the nonrelativistic Schrödinger–Pauli equation.[37] This is so because β which multiplies the dominant mass term, is an "even" matrix, that is, has the structure $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, with entries in the diagonal 2×2 blocks, whilst the α_i which multiply the components of the three-momentum are "odd", that is, have the structure $\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$. Thus, β connects upper with upper, as well as lower with lower two-spinors, whilst the α_i connect upper with lower, and lower with upper two-spinors. For particle solutions the lower two-spinor is of order $p/m \sim v/c$ relative to the upper. In the limit $\boldsymbol{p} \to 0$ the upper two-spinor goes over into the nonrelativistic Schrödinger wave function multiplied by a Pauli spinor.

The other extreme situation is the one where the particle's motion is highly relativistic, that is where $E \gg m$. In that case the mass term in eq. (1.181') is negligibly small. It is then more convenient to write eq. (1.181') in a representation where both γ^0 and the γ^i are "odd"[38], as in this case the equations of motion of upper and lower two-spinors decouple from each other. The "high-energy representation" (1.74) has the required property. Writing

$$\psi(\boldsymbol{x}) = \begin{pmatrix} \phi(\boldsymbol{x}) \\ \chi(\boldsymbol{x}) \end{pmatrix}$$

in this representation, (1.181') becomes

$$[\mathrm{i}\boldsymbol{\sigma} \cdot \nabla + (E - V(\boldsymbol{x}))]\phi(\boldsymbol{x}) = m\chi(\boldsymbol{x}), \tag{1.187a}$$

$$[-\mathrm{i}\boldsymbol{\sigma} \cdot \nabla + (E - V(\boldsymbol{x}))]\chi(\boldsymbol{x}) = m\phi(\boldsymbol{x}). \tag{1.187b}$$

When the mass term is neglected, these equations decouple completely. Quite obviously, they follow directly from the Dirac equation (1.69) if harmonic time dependence $\mathrm{e}^{-\mathrm{i}Et}$ is introduced.

We shall encounter both situations below and shall make use of either representation in our treatment of bound state problems and of relativistic scattering.

1.9.2 *Interaction with the Quantized Maxwell Field*

We start from the combined Lagrange densities of the *free* Dirac field and the *free* Maxwell field

$$\mathscr{L}_0(x) = \mathscr{L}_\mathrm{D}(x) + \mathscr{L}_\gamma(\boldsymbol{x}), \tag{1.188}$$

[37] See e.g. [SCH68].

[38] Obviously, γ^0 and γ^i cannot be even simultaneously.

where $\mathscr{L}_D(x)$ is given by (1.126), whilst $\mathscr{L}_\gamma(x)$ is

$$\mathscr{L}_\gamma(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x), \tag{1.189}$$

where $F^{\mu\nu} = f^{\mu\nu}$ and

$$f^{\mu\nu}(x) := \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \tag{1.190}$$

is the covariant electromagnetic field tensor,

$$F^{\mu\nu} = f^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{1.190'}$$

so that $E^i = -f^{0i}$, $B^i = -\frac{1}{2}\varepsilon^{ijk} f^{jk}$,

$$f^{lm} = -\varepsilon^{lmn} B^n,$$

$A_\mu(x)$ is the quantized Maxwell field.

We remind the reader of the convention in numbering rows and columns of the field tensor (1.190'). As in (1.1) and (1.7), to take an example, rows and columns are numbered 0,1,2,3, with Greek indices standing for all four values, Latin indices for the spatial components 1, 2, 3 only. For example, the element f^{lm} with $l = 1$ and $m = 2$ of (1.190') is $-B^3$. We distinguish between the symbols $F^{\mu\nu}$ for the field strength tensor and $f^{\mu\nu}$ as defined by (1.190), even though they are identical in the case of Maxwell theory. We do this because in gauge theories based on non-Abelian Lie groups the field strength tensor $F^{\mu\nu}$ contains a term bilinear in the vector potential, in addition to the component (1.190) (Sect. 3.3.4). Maxwell theory is based on U(1), i.e. an Abelian group, and no bilinear terms occur.

Note that we use here a system of natural units for electric charges, currents, electric and magnetic fields (cf. Notation and Conventions p. XV) in which, e.g., $\boldsymbol{E}_{\text{nat}} = E/\sqrt{4\pi}$, $\boldsymbol{B}_{\text{nat}} = B/\sqrt{4\pi}$, so that the numerical factor in (1.189) is 1/4 instead of the $1/16\pi$ which the reader may be used to. In contrast to this term in the Lagrange density, there is no change in the product of a charge Q and a vector potential A_μ, or in the product of a current density j^μ and the vector potential, the former being multiplied by $\sqrt{4\pi}$, the latter being divided by that same factor.

The minimal substitution rule,

$$\partial_\alpha \psi \to \partial_\alpha \psi + \mathrm{i} Q A_\alpha \psi, \tag{1.191a}$$

$$\partial_\alpha \bar{\psi} \to \partial_\alpha \bar{\psi} - \mathrm{i} Q A_\alpha \bar{\psi}, \tag{1.191b}$$

when introduced into $\mathscr{L}_0(x)$, (1.188), leads to the interaction Lagrangian

$$\mathscr{L}_0 \to \mathscr{L}(x) = \mathscr{L}_{\mathrm{D}}(x) + \mathscr{L}_{\gamma}(x) - Q\overline{\psi(x)}\gamma^{\alpha}\psi(x)A_{\alpha}(x). \tag{1.192}$$

The interaction term is seen to be the scalar product of the four-vector potential and the particle's electromagnetic current

$$j^{\alpha}_{\mathrm{e.m.}}(x) = Q\overline{\psi(x)}\gamma^{\alpha}\psi(x), \tag{1.193}$$

a result which is well-known from classical electrodynamics. As it stands the Lagrangian density (1.192) implies that the free particle described by the field $\psi(x)$ has a "normal" magnetic moment

$$\mu = g\frac{Q}{2m}\frac{1}{2} = \frac{Q}{2m}, \tag{1.194}$$

so that its g-factor is equal to 2. Any deviation from this so-called Dirac value comes about through radiative corrections and, therefore, is of order $O(\alpha)$.

This is not difficult to prove. Consider the interaction of the particle with a stationary magnetic field $\boldsymbol{B}$, which we describe by a three-vector potential $\boldsymbol{A}$. For simplicity we consider scattering of the particle from an initial momentum $\boldsymbol{p}$ to a final momentum $\boldsymbol{p}'$, through the interaction

$$\langle \boldsymbol{p}'|j_{\mathrm{e.m.}}(0)|\boldsymbol{p}\rangle \cdot \boldsymbol{A}(\boldsymbol{q}). \tag{1.195}$$

We consider the matrix element in the Breit system which is defined by the requirement

$$\boldsymbol{p} + \boldsymbol{p}' = 0.$$

This system of reference is particularly convenient since the limit of taking the squared four-momentum transfer $q^2 = (p'-p)^2 \to 0$ to zero leads us automatically to the rest system of the particle. At the same time (1.195) gives the corresponding nonrelativistic expression. It is not difficult to verify (in the standard representation) the following:

$$\begin{aligned}
&\langle -\boldsymbol{p}|j^{i}_{\mathrm{e.m.}}(0)|\boldsymbol{p}\rangle \\
&= \frac{1}{(2\pi)^3}Q\overline{u(-\boldsymbol{p})}\gamma^{i}u(\boldsymbol{p}) \\
&= \frac{Q}{(2\pi)^3}u^{\dagger}(-\boldsymbol{p})\begin{pmatrix} 0 & \sigma^{(i)} \\ \sigma^{(i)} & 0 \end{pmatrix}u(\boldsymbol{p}) \\
&= \frac{2Q}{(2\pi)^3}\mathrm{i}\varepsilon^{ijk}p^{j}(\chi^{\dagger}\sigma^{k}\chi).
\end{aligned}$$

Therefore, the scattering amplitude which describes magnetic back scattering is

$$T = \frac{\mathrm{i}}{(2\pi)^3}2Q\varepsilon^{ijk}p^{j}(\chi^{+}\sigma^{k}\chi)A^{i}(\boldsymbol{q} = 2\boldsymbol{p}).$$

Let us compare this to the nonrelativistic scattering amplitude which we would obtain from the well-known interaction of a magnetic moment $\boldsymbol{\mu}$ in a constant external field $\boldsymbol{B}$,

$$T_{\text{n.r.}} := (\psi_{-p}, H_{\text{int}}\psi_p)$$

with

$$H_{int} = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -g\frac{Q}{2m}\boldsymbol{S}\cdot\boldsymbol{B}, \quad B^i = \varepsilon^{ijk}\partial_j A^k(\boldsymbol{x})$$

and

$$\psi_p = \sqrt{\frac{2m}{(2\pi)^3}}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\chi.$$

(The factor $\sqrt{2m}$ is included in order to obtain the correct normalization when the nonrelativistic limit $\langle\boldsymbol{p}'|\boldsymbol{p}\rangle$ is taken.) One finds by partial integration

$$\begin{aligned} T_{\text{n.r.}} &= gQ\frac{1}{2}(\chi^\dagger\sigma^i\chi)\varepsilon^{ijk}2\mathrm{i}p_j A^k\frac{1}{(2\pi)^3} \\ &= \frac{\mathrm{i}gQ}{(2\pi)^3}\varepsilon^{kji}p^j(\chi^\dagger\sigma^i\chi)A^k(2\boldsymbol{p}). \end{aligned}$$

Comparing $T_{\text{n.r.}}$ and T, we see that indeed $g = 2$. In very much the same way we can show that the corresponding antiparticle has the opposite magnetic moment

$$\mu_{\text{antiparticle}} = -Q/2m.$$

As it stands, the Lagrangian $\mathscr{L}(x)$, (1.192), describes the quantum electrodynamics of a fermion and its antiparticle of given physical mass m, charge Q and normal g factor, $g = 2$. Of course, we can generalize it immediately to an arbitrary number of fermions (of the same charge and g-factor but different masses),

$$\mathscr{L}_{\text{QED}} = \sum_{\text{f}}\mathscr{L}_{\text{D}}(x, m_{\text{f}}) + \mathscr{L}_\gamma(x) = \sum_{\text{f}} Q(\text{f})\bar{\psi}_{\text{f}}\gamma^\alpha\psi_{\text{f}}A_\alpha, \tag{1.196}$$

so as to be able to describe electrons, muons, τ-leptons etc. ($Q_{\text{e}} = Q_\mu = Q_\tau = -e$). Appendix C summarizes the Feynman rules for quantum electrodynamics (QED) of spin-1/2 fermions, as defined by the Lagrange density (1.192).

We note here in passing that predictions of this theory for all electromagnetic properties of electrons and muons are in perfect agreement with experiment. In particular, tests of low-energy properties of electrons and muons have been pushed to the *natural limit* of quantum electrodynamics, i.e. up to the point where weak and strong interactions start to interfere with the pure electromagnetic interactions. We shall come back to this statement and quote some examples at various places in this book.

1.9.3 Some Remarks on These Results

(i) The result (1.194) says that the g-factor of electrons and muons is $g = 2$ (except for radiative corrections). This has always been considered a major success of Dirac's theory of electrons and muons. While this is certainly true, it is sometimes said also that $g = 2$ is a consequence of the minimal substitution rule. Unfortunately this latter statement is not correct. It is true that if we postulate the specific form (1.196) for the Lagrangian of leptonic QED then $g = 2$ follows from it, as we have seen above. However, there is nothing that forbids us to add an arbitrary four-divergence $\partial_\mu M^\mu$ to the free Lagrangian, as we know that this will not alter the equations of motion (supposing M^μ sufficiently well-behaved). For instance, we may wish to take $\mathscr{L}'_{\mathrm{D}} = \mathscr{L}_{\mathrm{D}} + \partial_\mu M^\mu$ instead of $\mathscr{L}_{\mathrm{D}}$, with the choice

$$M^\mu = -\mathrm{i}\frac{a}{8m}\overline{\psi(x)}\sigma^{\mu\nu}\overleftrightarrow{\partial}_\nu\psi(x).$$

For a free particle this does not make any difference. $\mathscr{L}'_{\mathrm{D}}$ yields the same equations of motion as $\mathscr{L}_{\mathrm{D}}$. However, if we introduce the coupling to the Maxwell field through minimal substitution, we find $\mathscr{L}_{\mathrm{QED}}$ of (1.196) but supplemented by the term called the Pauli term,[39]

$$\frac{a}{4m}Q\overline{\psi(x)}\sigma^{\mu\nu}\psi(x)\,f_{\mu\nu}. \tag{1.197}$$

This new term describes the interaction of an anomalous magnetic moment of magnitude $2a$. Thus, the new "minimal" theory describes fermions whose g-factor is

$$g' = 2(1 + a). \tag{1.198}$$

In the light of this remark there does not seem to be anything special about the value $g = 2$. The Dirac value $g = 2$ refers to the specific Lagrangian (1.196). The minimal coupling prescription must be applied to the specific Lagrangian (1.126), (1.188).

(ii) If a strongly interacting fermion (that is a baryon) couples to a photon the vertex

$$\langle p'|j^\alpha_{\mathrm{e.m.}}(0)|p\rangle$$

is renormalized through strong interactions and must be analyzed in terms of electric and magnetic form factors $F_i(q^2 = (p - p')^2)$ (see below). However, if the momentum transfer is sufficiently small, we may treat the baryon like a point-like Dirac particle that carries a given anomalous magnetic moment a. In this case we may use the theory sketched above by simply adding to it the Pauli term (1.197).

[39] This is worked out in [GAS66].

Such a situation is encountered, for instance, in the atomic bound states of exotic atoms with antiprotons or Σ^- particles. In these baryonic atoms the fine structure is determined by the term

$$V_{ls} = \frac{1}{2m^2}(1+2a)\boldsymbol{l}\cdot\boldsymbol{s}\frac{1}{r}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}r}, \tag{1.199}$$

where V_c is the Coulomb potential created by the nucleus. The fine structure may in fact be used to measure the anomaly a. This has been done for the case of the Σ^- with the result (Hertzog et al. 1983, 1988),

$$\mu(\Sigma^-) = -1.11 \pm 0.04$$

(iii) One might wonder whether the choice of the plus sign in the minimal substitution rule

$$\partial_\alpha \to \partial_\alpha + \mathrm{i}Q(\mathrm{f})A_a \tag{1.200}$$

is unique or not, $Q(\mathrm{f})$ being the charge of the particle,

$$Q(p) = +|e|, \quad Q(\mathrm{e}) = Q(\mu) = Q(\tau) = -|\mathrm{e}|.$$

Quite obviously this sign determines for example the sign of the potential term in (1.182) and, therefore, is related to the fact that like charges repell, unlike charges attract each other. In fact, the sign is fixed by two standard conventions in electrodynamics: (i) it is customary to let the electric field $\boldsymbol{E}$ of a *positive* point charge point *outward*; (ii) the sign of the four-potential is fixed such that $\boldsymbol{E} = -\nabla A^0$.

(iv) Minimal coupling and the requirement of gauge invariance of QED are intimately connected. Suppose we subject a charged spinor field $\psi(x)$ to a local gauge transformation,

$$\psi(x) \to \mathrm{e}^{\mathrm{i}\Lambda(x)}\psi(x), \ \overline{\psi(x)} \to \mathrm{e}^{-\mathrm{i}\Lambda(x)}\overline{\psi(x)}. \tag{1.201}$$

Whilst the mass term in the free Lagrangian $\mathscr{L}_\mathrm{D}$ is obviously invariant under this transformation, the kinetic energy term

$$\overline{\psi}\frac{i}{2}\gamma^\alpha \overleftrightarrow{\partial}_\alpha \psi \tag{1.202}$$

is not. It goes over into

$$\bar{\psi}\frac{\mathrm{i}}{2}\gamma^\alpha \overleftrightarrow{\partial}_\alpha \psi - \bar{\psi}\gamma^\alpha\psi\partial_\alpha\Lambda(x) \tag{1.202'}$$

If on the other hand we consider the "minimally substituted" kinetic energy instead,

$$\frac{\mathrm{i}}{2}(\bar{\psi}\gamma^{\alpha}(\partial_{\alpha}+\mathrm{i}QA_{\alpha})\psi-((\partial_{\alpha}-\mathrm{i}QA_{\alpha})\bar{\psi})\gamma^{\alpha}\psi), \tag{1.203}$$

then we see that the extra term in (1.202') can be absorbed into the four-vector potential by the substitution

$$A_{\alpha}-\frac{1}{Q}\partial_{\alpha}\Lambda(x)=:\ A'_{\alpha}. \tag{1.204}$$

This, however, is a gauge transformation of the four-vector potential which leaves invariant $\mathscr{L}_M$, the Lagrangian of the Maxwell fields. Thus, the transformations (1.202) and (1.204), taken together, leave invariant the coupled system of Maxwell and Dirac fields. For this it is essential that the modified kinetic energy contain what is called the "covariant derivative"

$$\begin{aligned}\vec{D}_a &:= \vec{\partial}_{\alpha}+\mathrm{i}QA_{\alpha},\\ \overleftarrow{D}_a &:= \overleftarrow{\partial}_{\alpha}-\mathrm{i}QA_{\alpha},\end{aligned} \tag{1.205}$$

We have touched here upon a general feature of gauge fields (A_{μ}) in interaction with matter fields (ψ) to which we shall return below in connection with gauge theories of more general nature.

1.10 Global Symmetries and Fermion Fields

In this section we consider more general, global, gauge transformations (1.128) acting on quantized fermion fields of the kind (1.122). While the transformations (1.128) pertain to a one-parameter group, called U(1),

$$\mathrm{U}(1)=\{\mathrm{e}^{\mathrm{i}\alpha}|\alpha\in\mathbb{R},0\leq\alpha\leq 2\pi\}. \tag{1.206}$$

hence an Abelian group, there are many situations in physics where the transformation groups are non-Abelian. Fermion fields are often classified by nontrivial representations of these groups, say of dimension M. A spinor field $\Psi(x)$ is then an object with $4\times M$ components

$$\Psi(x)=\begin{pmatrix}\psi^{(1)}\\ \psi^{(2)}\\ \vdots\\ \psi^{(M)}\end{pmatrix}, \tag{1.207}$$

where $\psi^{(i)}(x)$ is an ordinary quantized Dirac field (1.122) describing particle number i, itself member of an M-dimensional multiplet. This multicomponent structure of the spinor (1.207) requires a few, though simple, modifications in writing Lagrange densities such as (1.126′) for free fields, or more complicated Lagrange densities for interacting fields. This is what we work out in Sect. 1.10.1. While parity P and time reversal T apply to the multi-component spinor (1.207) in essentially the same way as to any single Dirac field, cf. Sect. 1.5, the action of charge conjugation C introduces nontrivial phases that depend on the multiplet structure of (1.207). This is dealt with in Sect. 1.10.2.

1.10.1 Fermions in Representations of Non-Abelian Groups

Consider a compact Lie group G. Its Lie algebra $Lie(G)$ is spanned by the generators for infinitesimal transformations $T_i,\ i = 1,\ 2,\ \ldots, N$, (also called, somewhat inaccurately, infinitesimal generators, for short) which obey the commutation rules

$$[T_i,\ T_j] = \mathrm{i} \sum_{k=1}^{N} C_{ijk}\, T_k,\ i,\ j = 1, 2, \ldots, N. \tag{1.208}$$

The constants C_{ijk} are the *structure constants* of the group. A priori, the structure constants are antisymmetric only in the first two indices i and j. However, through an appropriate choice of the generators the structure constants can be made antisymmetric in all three indices. This will be assumed in what follows. Following standard conventions in physics the generators are chosen to be hermitean, $T_i^\dagger = T_i$, so that an element $g \in G$ of the group has the form[40]

$$g = \exp\left\{\mathrm{i} \sum_{k=1}^{N} \alpha_k\, T_k\right\} \tag{1.209}$$

with α_k being real constants. An example that we use for the purpose of illustration below is $G = \mathrm{SU}(2)$, the abstract group that is defined by the group of complex 2×2 matrices which are unitary and have determinant 1,

$$\mathrm{SU}(2) = \{M_{2\times 2} | M^\dagger M = \mathbb{1},\ \det\ M = 1\}. \tag{1.210}$$

Other examples relevant for the construction of local gauge theories will be given in Chap. 3. SU(2) has three generators $T_i = \sigma^{(i)}/2$, with $\sigma^{(i)}$ the Pauli matrices (1.24). Its structure constants are $C_{ijk} = \varepsilon_{ijk}$ where ε_{ijk} is the totally antisymmetric

[40] If G has several branches the formula (1.209) generates those elements which can be deformed continuously into the identity element.

Levi–Civita tensor in three real dimensions, $\varepsilon_{ijk} = +1(-1)$ if $\{ijk\}$ is an even (odd) permutation of $\{123\}$, $\varepsilon_{ijk} = 0$ whenever two of its indices are equal. SU(2) is closely related to SO(3), the rotation group in three real dimensions (see e.g. [SCH10]). In particular they have the same Lie algebra. The group parameters are (generalized) angles of rotation and, therefore, belong to compact intervals $[0, \pi]$ or $[0, 2\pi]$. Indeed, as we saw in Sect. 1.1.2, SU(2) is relevant for the description of how spinors transform with respect to rotations. In many instances, however, SU(2), or some other compact Lie group G, refers to some abstract, internal, "charge" space, i.e. describes internal properties of fermionic fields which are independent of space and time.

In this spirit, suppose that we are dealing with M fermions which span a unitary, reducible or irreducible, representation of G. It is convenient to group the spinor fields that describe them into a single column vector $\Psi(x)$, cf. (1.207), of M Dirac fields. In the representation space the generators T_i are represented by $M \times M$ matrices

$$T_i \ \rightarrow \ U(T_i), \quad i = 1 \quad , 2, \ldots, \ N, \tag{1.211}$$

or, more precisely, by the direct product $\mathbb{1}_{4\times 4} \otimes U(T_k)$ where the first factor acts on the Dirac indices of each individual field $\psi^{(i)} = \left(\psi_1^{(i)}, \psi_2^{(i)}, \psi_3^{(i)}, \psi_4^{(i)}\right)^T$, while the second factor acts on the multiplet index (i). Similarly, a finite transformation (1.209) acting on $\Psi(x)$ is a unitary $M \times M$ matrix given by the exponential series in the generators

$$U(g) = \exp\left\{ \mathrm{i} \sum_{k=1}^{N} \alpha_k \, U(T_k) \right\},$$

or, more precisely, by $\mathbb{1}_{4\times 4} \otimes U(g)$. (The letter U is a reminder of the choice of a *unitary* representation, i.e. a representation in which $U(g)$ is unitary, $U(T_i)$ is hermitean). In writing down the kinetic energy for the multicomponent field $\Psi(x)$ as in (1.126′) we must keep track of the requirement that $\gamma^\mu \partial_\mu$ should act on each spinor field $\psi^{(i)}(x)$, irrespective of its place in the multiplet. Therefore, the correct notation would be

$$\overline{\Psi(x)}\, \gamma^\mu \ \otimes \mathbb{1}_{M\times M} \, \partial_\mu \, \Psi(x).$$

Whenever the context is unambigous, and in order to alleviate the notation, one often omits the reference to the representation by writing g instead of $U(g)$, T_k instead of $U(T_k)$, and omits the tensor product of Dirac 4×4 matrices with $M \times M$ matrices in the internal, "charge" space.

Let us illustrate these matters by the following examples. Take $G = \mathrm{SU}(2)$ and $\psi^{(\mathrm{nuc})}(x) = (\psi^{(p)}, \ \psi^{(n)})^T$ the doublet representation ($M = 2$) of SU(2), called nucleonic, to contain the proton and the neutron. Assume the proton and the neutron to have the same mass m. (This is a reasonable assumption because the mass difference $m(n) - m(p) = 1.29\,\mathrm{MeV}$ is small as compared to $m(p) = 938.3\,\mathrm{MeV}$). A Lagrange density describing protons p, neutrons n, and their antiparticles $\bar{p}$ and $\bar{n}$, respectively, is

$$\begin{aligned}\mathscr{L}^{(\mathrm{nuc})} &= \overline{\Psi^{(\mathrm{nuc})}(x)}\left(\frac{\mathrm{i}}{2}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu} - m\mathbb{1}_{4\times4}\right)\otimes\mathbb{1}_{2\times2}\,\Psi^{(\mathrm{nuc})}(x)\\ &= \overline{\Psi^{(\mathrm{nuc})}(x)}\left(\frac{\mathrm{i}}{2}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu} - m\right)\Psi^{(\mathrm{nuc})}(x),\end{aligned} \tag{1.212}$$

the right-hand side being the shorthand notation mentioned above.

An operator describing the transition from a neutron to a proton, for instance, is proportional to

$$\begin{aligned}\overline{\Psi^{(\mathrm{nuc})}(x)}\;\;\mathbb{1}_{4\times4}\otimes\tfrac{1}{\sqrt{2}}(T_1+\mathrm{i}T_2)\;\Psi^{(\mathrm{nuc})}(x) \;&\equiv\; \overline{\Psi^{(\mathrm{nuc})}(x)}\,\tfrac{1}{\sqrt{2}}(T_1+\mathrm{i}T_2)\;\Psi^{(\mathrm{nuc})}(x)\\ &= \overline{\Psi^{(p)}(x)}\;\psi^{(n)}(x).\end{aligned}$$

Indeed, $\psi^{(n)}(x)$ annihilates a neutron (or creates an antineutron), $\overline{\psi^{(p)}(x)}$ creates a proton (or annihilates an antiproton), cf. (1.122a) and (1.122b), respectively. A Lagrange density describing β-decay of the neutron, $n \to p + e + \bar{\nu}$, as an example, could be given by

$$\mathscr{L} = \mathscr{L}^{(\mathrm{nuc})} + \mathscr{L}^{(e)} + \mathscr{L}^{(\nu)} + \mathscr{L}_{int},$$

with $\mathscr{L}^{(\mathrm{nuc})}$ as above, $\mathscr{L}^{(e)}$ and $\mathscr{L}^{(\nu)}$ being of the form (1.126′) for the electron/positron field and for the $\nu/\bar{\nu}$ field, respectively, and the interaction being given by[41]

$$\mathscr{L}_{(\mathrm{int})} = \kappa\left(\overline{\Psi^{(\mathrm{nuc})}(x)}\gamma^{\alpha}\frac{1}{\sqrt{2}}(T_1+\mathrm{i}T_2)\Psi^{(\mathrm{nuc})}(x)\right)(\overline{\psi^{(e)}(x)}\gamma_a\psi^{(\nu)}(x)) + \mathrm{h.c.} \tag{1.213}$$

In fact, if we decided to group the neutrino and the electron in a leptonic doublet of SU(2), irrespective of their mass difference, $\Psi^{(\mathrm{lep})}(x) = (\psi^{(\nu)}(x), \psi^{(e)}(x))^T$, $\mathscr{L}_{(\mathrm{int})}$ could be written in the form

$$\begin{aligned}\mathscr{L}_{(\mathrm{int})} = \kappa &\left(\overline{\Psi^{(\mathrm{nuc})}(x)}\,\gamma^{\alpha}\,\frac{1}{\sqrt{2}}\,(T_1+\mathrm{i}T_2)\;\Psi^{(\mathrm{nuc})}(x)\right)\\ &\times\left(\overline{\Psi^{(\mathrm{lep})}(x)}\,\gamma_{\alpha}\,\frac{1}{\sqrt{2}}\,(T_1-\mathrm{i}T_2)\;\Psi^{(\mathrm{lep})}(x)\right).\end{aligned} \tag{1.213'}$$

In this example it should be clear that in the first factor the operator $(T_1+\mathrm{i}T_2)$ stands for the matrix representation $U^{(\mathrm{nuc})}\,(T_1+\mathrm{i}T_2)$ in the space of proton and neutron, while the operator $(T_1-\mathrm{i}T_2)$ in the second factor stands for $U^{(\mathrm{lep})}(T_1-\mathrm{i}T_2)$.

[41] In fact, this ansatz for L_{int} conserves parity and is not in accord with experiment. A more realistic description of β-decay contains $\gamma^{\alpha}(1-\gamma_5)$ instead of γ^{α} in both factors of (1.213), and $\kappa = G_{\mathrm{F}}/\sqrt{2}$ with G_{F} the Fermi constant, cf. Chap. 3.

In the first example the two representations had the same dimension $M = 2$. This need not always be so. For instance, let us consider a triplet of hermitean bosonic field operators $\Phi(x) = (\Phi_1(x), \Phi_2, (x), \Phi_3(x))$ such that $\Phi_{\pm 1} = (\Phi_1 \pm \mathrm{i}\Phi_2)/\sqrt{2}$ and $\Phi_0 = \Phi_3$ describe the three charge states of the pion. The triplet is the adjoint representation of SU(2), that is, it has the same dimension $M = N = 3$ as the Lie algebra. (The doublet representation, $M = 2$, is the fundamental, or defining, representation.) A realistic Lagrange density describing the emission and absorption of pions by nucleons is[42]

$$\mathscr{L}_{int}^{(\text{ß,nuc})} = \lambda \, (\overline{\Psi^{(\text{nuc})}(x)} \, \gamma_5 \, \boldsymbol{T} \, \Psi^{(\text{nuc})}) \bullet \Phi \, (x), \tag{1.214}$$

with $\bullet$ denoting the Euclidean scalar product $\boldsymbol{T} \bullet \Phi = \Sigma T_i \Phi_i$. The formal analogy to the scalar product in ordinary space $\mathbb{R}^3$ suggests at once that the expression (1.214) is invariant with respect to SU(2) transformations.

Note that in either example, (1.213') or (1.214), the specific combination of generators guarantees, among others, the conservation of electric charge. In the example (1.213') a neutron is converted into a proton (charge $+ |e|$) with simultaneous creation of an electron (charge $- |e|$) and an antineutrino (uncharged). An analogous term $\overline{\Psi^{(\text{nuc})}}(T_1 + \mathrm{i}T_2)\Psi^{(\text{nuc})}$ in the example (1.214) is accompanied by the field $\Phi_{-1} = (\Phi_1 - \mathrm{i}\Phi_2)/\sqrt{2}$. The latter annihilates a positively charged pion π^+, or creates a π^-, so that (1.214) describes the vertices $n \to p + \pi^-$ and $\pi^+ + n \to p$ (as well as analogous, charge-conserving, vertices involving $\bar{p}$ and $\bar{n}$).

1.10.2 *Charge Conjugation for Fermionic Multiplets

Let us consider the action of charge conjugation C on a multiplet of fermions described by a multicomponent spinor (1.207). The action of C on any individual Dirac field $\psi^{(i)}$ is given by (1.98) of Sect. 1.5. Note, however, that there could be an additional phase factor η_i whose square can be taken to be 1, viz.

$$\psi_C^{(i)} = \eta_i \, \mathrm{i}\gamma^2 \, \psi^{(i)*} = \eta_i \, \mathrm{i}\gamma^2\gamma^0 \overline{\psi^{(i)}}^T, \quad \text{with } \eta_i^2 = 1. \tag{1.215}$$

At the same time C takes us from the world of particles to the world of antiparticles whose additive quantum numbers are the opposite of those of their particle partners.

For example, the doublet (p, n) belongs to the eigenvalues $(+1/2, -1/2)$ of T_3 in SU(2) respectively. Their antiparticles $(\bar{p}, \bar{n})$ also fall in a doublet of SU(2) but belong to the eigenvalues $(-1/2, +1/2)$ of T_3, respectively. Hence, if in the basis $\{p, n\}$ T_3 is given by the diagonal matrix $U(T_3) = \text{diag}\,(1/2, -1/2)$, its charge

[42] The nucleonic factor contains the matrix γ_5 because the pion field is pseudoscalar, not scalar, with respect to Lorentz transformations and because the interaction is known to conserve parity which means that (1.214) must be even under P.

conjugate T_3^C in the basis $\{\bar{p}, \bar{n}\}$ is $U\left(T_3^C\right) = \text{diag}(-1/2,\ 1/2)$. At the same time, the ladder operators $(T_1 \pm \mathrm{i}T_2)$, through charge conjugation, will exchange their roles, i.e.

$$T_3^C = -T_3,\ \ (T_1^C\ \pm\ iT_2^C) = (T_1\ \mp\ \mathrm{i}T_2). \tag{1.216}$$

For a single Dirac field the phase factor η_i is arbitrary because it is unobservable. In case of a multiplet of the group G such as (1.207), however, the requirement that C take the Lie algebra spanned by $(U(T_1),\ \ldots,\ U(T_N))$ acting on the space of particles to the representation $\left(U\left(T_1^C\right), \ldots U\left(T_N^C\right)\right)$ in the space of the antiparticles may fix some or all of the relative phases η_i. For example, with $G = \text{SU}(2)$ and the doublet representation, the requirement (1.216) will be seen to imply that the product of the two phases in the charge conjugate of (1.207) be $\eta_1\,\eta_2 = -1$.

Let $\eta\ =\ \text{diag}\ (\eta_1,\ \eta_2,\ \ldots,\ \eta_M)$ denote the diagonal matrix of the charge conjugation phases in (1.215). As before, let $U(T_k)$ be the generators of $Lie(G)$ in the basis of particle states, $U\left(T_k^C\right)$ the generators in the basis of the antiparticle states. Then we have the following theorem:

Theorem *The two representations of the Lie algebra in the space of particles and of antiparticles, respectively, are related by*

$$U(T_k^C) = -\eta U^*\,(T_k)\eta. \tag{1.217}$$

Here U^* denotes the complex conjugate of U (not the hermitean conjugate).

The proof goes as follows.

Consider a finite transformation $U(g)$ with $g \in G$, acting on Ψ, (1.207), $\Psi' = U(g)\Psi$. The inverse transformation is $\Psi\ =\ U(g)^\dagger\Psi'$, or, written in components, $\psi^{(i)} = \Sigma_{k=1}^M U_{ki}^*(g)\psi'^{(k)}$. Applying charge conjugation to this equation we obtain

$$\psi_C^{(i)} = \eta_i\,\mathrm{i}\gamma^2\,\psi^{(i)*} = \mathrm{i}\gamma^2\,\eta_i\ \sum_{k=1}^{M} U_{ki}(g)\,\psi'^{(k)*}. \tag{1.218}$$

Clearly, the matrix η of charge conjugation phases must not depend on the basis of particle states, that is, it must be the same for Ψ and for Ψ'. Thus, if we multiply (1.218) by $\eta_i U_{li}^*(g)$ from the left and sum over i, its right-hand side becomes, by the unitarity of $U(g)$, $\mathrm{i}\gamma^2\psi'^{(l)*}$ which is equal to $\eta_l\psi_C'^{(l)}$. Its left-hand side is equal to $\Sigma_{i=1}^M\ \ U_{li}^*(g)\eta_i\,\psi_C^{(i)}$. Multiplying the whole equation by η_l we obtain

$$\psi_C'^{(l)} = \eta_l\ \sum_{i=1}^{M} U_{li}^*(g)\ \eta_i\ \psi_C^{(i)}\ \text{ or }\ \Psi_C' = \eta U^*\,(g)\eta\,\Psi_C. \tag{1.219}$$

This equation shows that if Ψ transforms with $U(g)$ then Ψ_C transforms with $\eta U^*(g)\eta$. In order to translate this result to the Lie algebra we return to (1.209). Taking the compex conjugate of (1.209) in the representation $U(g)$ and multiplying

with η from the left and from the right, we have

$$\eta U^*(g)\eta = \exp\left\{-\mathrm{i}\sum_{k=1}^{N}\alpha_k\,\eta U^*(T_k)\,\eta\right\}. \tag{1.220}$$

Here, the phase matrix η can be taken inside the exponential series because its square is the unit matrix, $\eta^2 = \mathbb{1}_{M\times M}$. Finally, we note that in the space of *anti*particles, finite group transformations should be written with the same sign convention as in (1.209), i.e. as the exponential series $\exp\left\{\mathrm{i}\Sigma\alpha_k U\left(T_k^C\right)\right\}$. Comparing with the result (1.220) yields the relation (1.217) of the theorem.

Remarks: The phase factos η_i need not be of square 1. It is not difficult to carry out the analysis with phase factors whose square is not unity. For instance, the left factor η in relation (1.217) will be replaced by its complex conjugate. On the other hand, there is no loss of generality in imposing $\eta_i^2 = 1$, as we did.

As an example let us apply relation (1.217) to $G = \mathrm{SU}(2)$ and the case of a doublet, $M = 2$. Without loss of generality we may take $\eta_1 = 1$, $\eta_2 \equiv \bar{\eta}$. Then

$$-\eta\sigma^{(3)*}\eta = -\begin{pmatrix}1&0\\0&\bar{\eta}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&\bar{\eta}\end{pmatrix} = -\begin{pmatrix}1&0\\0&-1\end{pmatrix} = -\sigma^{(3)},$$

independently of the value of $\bar{\eta}$. On the other hand, in order to fulfill the conditions (1.216) on the ladder operators, we must have $U\left(T_1^C\right) = U(T_1)$ and $U\left(T_2^C\right) = -U(T_2)$. Now,

$$-\eta\sigma^{(1)*}\eta = -\begin{pmatrix}0&\bar{\eta}\\\bar{\eta}&0\end{pmatrix} = -\bar{\eta}\sigma^{(1)}, -\eta\sigma^{(2)*}\eta = -\begin{pmatrix}0&\mathrm{i}\bar{\eta}\\-\mathrm{i}\bar{\eta}&0\end{pmatrix} = +\bar{\eta}\sigma^{(2)}.$$

The above conditions require $\bar{\eta} = -1$ which means that C converts the doublet (p, n) into the doublet $(\bar{p}, -\bar{n})$. It should be clear that we could have fixed η_1 differently, by a common phase for the doublet as a whole. What is independent of any such convention is the requirement $\eta_1\eta_2 = -1$ that follows from the theorem. Thus, in this example, we obtain

$$C:\begin{pmatrix}p\\n\end{pmatrix} \to \eta_1\begin{pmatrix}\bar{p}\\-\bar{n}\end{pmatrix}. \tag{1.221}$$

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Exercises

1.1. Derive the Euler-Lagrange equations of $\mathscr{L}_{\mathrm{M}}$, (1.108).
1.2. Verify that the matrices σ^{μ} are both unitary and hermitean.
1.3. Prove equation (1.45) by explicit calculation of the exponential series for the case $\kappa = \frac{1}{2}$. Show that the transformation $U \equiv D(0,\ \pi,\ 0)$ does indeed effect the transition from cogredience (i.e. transformation behaviour $D(\theta_i)$) to contragredience (i.e. transformation behaviour $D^{-1} = D^*$), and vice versa. *Hint*: First prove the relations

$$UJ_i + J_i^* U = 0 \quad \text{and} \quad Ue^{i\sigma J} U^{-1} = e^{-i\alpha J *}$$

(Note the dependence on the phase convention for the angular momentum matrices).
1.4. Study plane wave solutions of (1.69) for the case $m = 0$ and show that they are eigenstates of helicity.
1.5. Carry out the calculation that leads to (1.131). Do the same for (1.139).
1.6. Derive the Euler–Lagrange equations for the Lagrangian (1.163) and compare with (1.69).
1.7. Study the behaviour of the Lagrangian (1.163) under parity P, charge conjugation C, and time reversal T. Show that it is invariant under the combined transformation $\Theta = PCT$.
1.8. Prove the relation

$$(\sigma^{\alpha})_{aB}(\sigma_{\alpha})_{cD} = -2\varepsilon_{ac}\varepsilon_{BD}$$

1.9. Physical amplitudes are always bilinear functions of spinor fields. Show that the fact that the density matrix ρ is not hermitean does not conflict with observables being real.
1.10. For which orientation of ζ and in which basis is the density matrix (1.153) diagonal? Interpret the answer. Write down the explicit form of ρ for polarization along the momentum.
1.11. Choose a basis $(n^{(0)},\ n^{(1)},\ n^{(2)},\ n^{(3)})$ in four-dimensional momentum space such that (i) $n^{(0)\mu} = (1/m)p^{\mu}$; (ii) the vectors $n^{(i)\alpha}$ are spacelike; (iii) $(n^{(\alpha)} \cdot n^{(\beta)}) = g^{\alpha\beta}$ (orthogonality): (iv) $n_{\mu}^{(\alpha)} g_{\alpha\beta} n_{\nu}^{(\beta)} = g_{\mu\nu}$ (completeness). For example:

$$n^{(1)} = (0,\ \boldsymbol{n}^1),\ n^{(2)} = (0,\ \boldsymbol{n}^2),\ n^{(3)} = \left(\frac{|\boldsymbol{p}|}{m},\ \frac{E_p \boldsymbol{p}}{m|\boldsymbol{p}|} \right)$$

with $\boldsymbol{n}^1 \perp \boldsymbol{n}^2 \perp \boldsymbol{p}$. Calculate the polarization along the direction $\boldsymbol{n}^1$ for a state that is characterized by a point ζ on or inside the Poincaré sphere.

1.12. Show that the relationship (1.46) holds true for boosts.

1.13. Construct the Hamilton density for a free Majorana field. Show that when integrated over all space this density gives zero.

1.14. Show that reflections with respect to planes containing the momentum $\boldsymbol{p}$ exchange positive and negative helicity states.

Chapter 2
Electromagnetic Processes and Interactions

The electron, the muon, and their neutrinos are important tools in testing the structure of the fundamental electromagnetic and weak interactions. On the other hand, if these interactions are known, they serve as ideal probes for the internal structure of complex hadronic targets such as nucleons and nuclei. Although electroweak interactions should in fact be discussed as a whole and on the same footing, purely electromagnetic interactions play a distinctive role, for obvious experimental reasons: At low and intermediate energies the effective electromagnetic coupling is larger by many orders of magnitude than the weak couplings, so that electromagnetic processes are measurable to much higher accuracy than purely weak processes.

The *fundamental* aspects of unified electroweak (and strong) interactions are discussed below, in Chap. 3. The present chapter deals primarily with *applications* of charged leptons to problems of nucleon and nuclear structure, and to selected precision tests of quantum electrodynamics (QED) at low momentum transfers. In most of these applications the electromagnetic interactions effectively appear in the form of external fields in the leptonic particle's Dirac equation. This is the domain where the physics of (electromagnetically) interacting leptons can still be described in the framework of an effective, though relativistic, single particle theory. In contrast to this, the topics discussed in Chap. 3 will make use of the full intrinsic many-body nature of Dirac theory.

2.1 Electron Scattering from a Composite Target: Qualitative Considerations

Electron scattering at high and very high energies is an important tool for the investigation of the structure of various strongly interacting particles (hadrons). Among these only proton, neutron and nuclei can be prepared as targets in scattering experiments. Hence most of what we know about internal hadronic structure

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concerns protons and neutrons. Nevertheless there is also some information on long-lived hadrons such as pions from electron–positron colliding beam experiments in which pairs of such hadrons are created.[1]

If electron scattering from a nucleon or a nucleus is to give more information on the target than just its electric charge, the electron's de Broglie wavelength $\lambda\!\!\!^{-} = 1/k$ must have a magnitude comparable to the *spatial size* of the nucleon or nucleus, respectively. The radius of the proton is about $r_{\mathrm{p}} \simeq 0.86\,\mathrm{fm}$; the charge radius of nuclei is approximately

$$r_{\mathrm{c}} \simeq 1.1\ \mathrm{fm}\ A^{1/3} \tag{2.1}$$

(A being the nuclear mass number). Thus $\lambda\!\!\!^{-}$ should be of the order of, or smaller than, about 1 fm. Hence its momentum must be of the order of or greater than

$$k = \hbar c/\lambda\!\!\!^{-} \simeq 200\ \mathrm{MeV}.$$

Obviously, at these energies the electron is highly relativistic, its energy is very large as compared to its rest mass

$$E = \sqrt{\boldsymbol{k}^2 + m^2} \gg m.$$

In fact, in most cases we will neglect the mass altogether. We then deal with a massless charged fermion which behaves in a way somewhat similar to neutrinos, with the exception that the electron spin can assume any direction. The following simple estimates may serve to illustrate qualitatively what one learns from the study of *elastic scattering* and of *inelastic scattering to discrete excited states* in a hadronic target (nucleon or nucleus). A more detailed and quantitative analysis of these processes follows in the next sections.

Consider first elastic scattering of an electron by an extended object with spherically symmetric charge density $\rho(r)$ and total charge Ze. $\rho(r)$ shall be normalized to unity,

$$\int \rho(r)\mathrm{d}^3r = 4\pi \int_0^\infty \rho(r) r^2 \mathrm{d}r = 1. \tag{2.2}$$

The corresponding electrostatic potential $\phi(r)$ is related to the charge density through Poisson's equation (in natural units),

$$\Delta\phi(r) = -Ze\rho(r).$$

We calculate the differential cross section for elastic scattering in Born approximation and, for the moment, neglect the spin of the electron. In fact, in

[1] The essential difference between these two types of experiments is that in electron scattering the invariant momentum transfer is *spacelike* while in electron–positron collisions it is *timelike*.

electromagnetic scattering of electrons at very high energies, the spin of the electron is not essential.[2] We shall show below that for $E \gg m$, a spherically symmetric, parity even, potential does not lead to polarization of an initially unpolarized electron beam. This is in contrast to low energies of the order of the mass, $E \simeq m$, where there is polarization through spin–orbit coupling (Mott scattering). With this approximation the electron is then described by a Klein–Gordon equation with external electrostatic potential

$$V(r) = -e\phi(r).$$

To order $(Z\alpha)$ the scattering amplitude is given by

$$f(\boldsymbol{p}', \boldsymbol{p}) \simeq -\frac{k}{2\pi} \int e^{-\mathrm{i}\boldsymbol{p}'\boldsymbol{r}}(-e\phi(r))\mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{r}}\mathrm{d}^3r \tag{2.3}$$

($\boldsymbol{p}$, $\boldsymbol{p}'$ being the initial and final momenta of the electron, respectively, $k = |\boldsymbol{p}| = |\boldsymbol{p}'|$).

Introducing the momentum transfer $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$, for which

$$q \equiv |\boldsymbol{q}| = 2k \ \sin(\theta/2) \tag{2.4}$$

(θ being the scattering angle), the amplitude $f(\boldsymbol{q})$ can be expressed in terms of the charge density $\rho(r)$ by integrating by parts twice and making use of Poisson's equation.[3] One obtains

$$f(\boldsymbol{q}) \simeq \frac{Ze^2k}{2\pi q^2} \int \mathrm{e}^{\mathrm{i}\boldsymbol{q}\boldsymbol{r}}\rho(r)\mathrm{d}^3r = \frac{2Z\alpha k}{q^2} \int \mathrm{e}^{\mathrm{i}\boldsymbol{q}\boldsymbol{r}}\rho(r)\mathrm{d}^3r, \tag{2.5}$$

where we have replaced $e^2/4\pi$ by α (natural units).

Thus, the differential cross section is

$$\left(\frac{d\sigma}{\mathrm{d}\Omega}\right)_{\text{no spin}} = |f(\boldsymbol{q})|^2 \simeq \left(\frac{Z\alpha}{2k}\right)^2 \frac{1}{\sin^4(\theta/2)}|F(\boldsymbol{q})|^2,$$

with the *charge form factor* $F(\boldsymbol{q})$ defined as follows:

$$F(\boldsymbol{q}) = \int \rho(r)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\boldsymbol{r}}\mathrm{d}^3r. \tag{2.6}$$

[2] This is true because electromagnetic interactions are invariant under parity.

[3] In order to do this in a mathematically correct manner the $1/r$ potential must be multiplied by a convergence factor, say $\mathrm{e}^{-\alpha r}$, and the limit $\alpha \to 0$ must be taken at the end.

When the electron spin in included, the scattering cross section for a spin zero target just gets another factor $\cos^2(\theta/2)$, so that we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} |F(\boldsymbol{q})|^2, \tag{2.7}$$

with

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \left(\frac{Z\alpha}{2k}\right)^2 \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}. \tag{2.8}$$

The Mott cross section (2.8) is derived below, including the necessary kinematics if recoil of the struck target becomes important. However, already at this point, we can read off a few qualitative physical features from these formulae.

(i) In the forward direction, $\boldsymbol{q} = 0$, the form factor $F(\boldsymbol{q})$ is equal to one, by virtue of the normalization condition (2.2): $F(0) = 1$. The same result obtains for *all* momenta $\boldsymbol{q}$ if the charge distribution is concentrated in a point,

$$\rho_{\mathrm{point}}(r) = \frac{1}{4\pi}\frac{1}{r^2}\,\delta(r) \to F(\boldsymbol{q}) \equiv 1 \quad \forall \boldsymbol{q}. \tag{2.9}$$

Thus, the Mott cross section (2.8) describes the scattering from a point charge Ze placed at the origin.

(ii) If the charge is not pointlike the cross section (2.7) is modulated by the form factor $F(\boldsymbol{q})$. It is this form factor which contains information about the target beyond its charge Ze. Hence, elastic electron scattering measures, in essence, the *spatial distribution* of the charge density $\rho(r)$. In particular, if the Born approximation is applicable, then (2.6, 2.7) show that the cross section is just the square of the Fourier transform of $\rho(r)$.

(iii) If the region over which $\rho(r)$ is appreciably different from zero is characterized by a radius R, the momentum transfer must be chosen such that

$$qR \gtrsim 1. \tag{2.10}$$

Indeed, if q is chosen too small, i.e. $qR \ll 1$ the form factor does not yet deviate much from unity and little information is obtained. If q is too large, i.e. $qR \gg 1$, the exponential in (2.6) oscillates rapidly and $F(\boldsymbol{q})$ becomes unmeasurably small. Thus q must be tuned to the size of the extended object that one wants to map. The quantitative details and the nature of the information obtained by means of elastic scattering are worked out in some of the following sections (Sects. 2.4, 5).

(iv) A similar situation is encountered when we consider inelastic scattering to discrete excited states. In that case the ground state charge density (which we assumed spherically symmetric, for simplicity) is replaced by a transition density $\rho_{\mathrm{fi}}(\boldsymbol{r})$ or more complicated functionals of charge and current densities of the target particle. Equivalently, the elastic charge form factor is replaced

by inelastic form factors. As in the case of elastic scattering, the cross section depends on a leptonic part which is known and some kinematics, whilst the hadronic structure is contained in the form factors.

In the simplest case (electric charge scattering from a nucleus to a discrete excited state) the form factor for the transition from state i to state f, and with multipolarity λ, will be proportional to

$$F_{\mathrm{fi}}(q) \propto \int_0^\infty \rho_{\mathrm{fi}}(r)\, j_\lambda(qr) r^2 \mathrm{d}r. \tag{2.11}$$

This follows from expanding the exponential $\mathrm{e}^{\mathrm{i}\boldsymbol{qr}}$ in terms of spherical harmonics and from the selection rules for angular momentum imposed by the spins and parities of initial and final target state. The expression (2.11) is reminiscent of the transition amplitude for the corresponding photoexcitation of state f. In that case q is replaced by the photon energy $k_\gamma = E_\mathrm{i} - E_\mathrm{f}$. For given energy this is a fixed number. Furthermore, in many cases $k_\gamma r$ is small compared to one over the domain where $\rho_{\mathrm{fi}}(r)$ is appreciably different from zero. In this case the Bessel function in (2.11) can be replaced by its limiting form for small argument,

$$j_\lambda(k_\gamma r) \simeq (k_\gamma r)^\lambda / (2\lambda + 1)!!. \tag{2.12}$$

Thus, the γ-transition depends essentially only on one specific *moment* of the transition charge density. The power behaviour $(k_\gamma r)^\lambda$ with $k_\gamma r \ll 1$ limits the γ-transition to the lowest possible multipolarity.

In *electron* scattering, in contrast, q is a variable momentum transfer which can be chosen as large as one wishes. This means that a complete mapping of $F_{\mathrm{fi}}(q)$ and, thereby, of $\rho_{\mathrm{fi}}(r)$ can be obtained, at least in principle. Also, all multipolarities compatible with the angular momentum and parity selection rules can contribute on equal footing if q is chosen appropriately. For instance, an E5 electro-excitation can have as large a cross section as an E2 transition.

(v) Finally, if both the momentum transfer and the energy transfer to the target are chosen sufficiently large one reaches a new domain where the electron starts to probe, in a rather general sense, the *constituent structure of the target*. In the case of scattering from nuclei this is called *quasi-free scattering*, expressing the fact that the energy transfer is much larger than the binding energy of the nucleons in the nucleus so that the nucleus behaves like a cloud of almost free nucleons.

In the case of *nucleons* the analogous domain is called the *deep inelastic* region. Although there are essential differences to the case of nuclei, the main idea is the same as in quasi-free scattering from nuclei: Deep inelastic scattering, where both momentum and energy transfers are large, must be sensitive to interactions within the target at very small distances.

2.2 Elastic Scattering from a Spin Zero Target, Born Approximation

The elastic scattering of an electron from a spin zero target is the simplest case. It can be dealt with by means of standard Green function techniques of potential scattering without having to invoke covariant perturbation theory and Feynman rules. The result is the correct covariant cross section in the Born approximation and contains many of the essential features of more complicated situations such as scattering on spin-1/2 targets or similar. Because of this simplicity we consider this case first.

The "spin-zero target" may be a pion or any nucleus whose spin is zero. We start with the kinematics of Fig. 2.1. Let

$$k = (E;\boldsymbol{k}), \quad k' = (E';\boldsymbol{k}') \tag{2.13}$$

be the four-momenta of the electron in the *laboratory system* before and after the scattering. Similarly,

$$p = (M;0), \quad p' = \left(\sqrt{M^2 + \boldsymbol{p}'^2};\boldsymbol{p}'\right) \tag{2.14}$$

denote the four-momenta of the target before and after the scattering process, respectively, with M the target mass. As the electron energy is chosen large compared to its rest mass, $E \simeq |k|$ and $E' \simeq |k'|$ and the square of the four-momentum transfer is

$$q^2 := -(k - k')^2 = -(E - E')^2 + (\boldsymbol{k} - \boldsymbol{k}')^2 \simeq 4EE' \sin^2(\theta/2) \tag{2.15}$$

[q^2 is the same as $-t$, see below, (2.36)].

Setting $m_e \simeq 0$, energy–momentum conservation gives the relationship

$$E' = E \frac{1}{1 + 2(E/M)\sin^2(\theta/2)}. \tag{2.16}$$

We note that the denominator of (2.16) is a typical recoil term. Whether or not this recoil term is important depends on the mass of the target. For a nucleus, for instance, $M \simeq A \cdot 940\,\text{MeV}$ will in general be large compared to E if the electron energy is chosen to be a few hundred MeV.

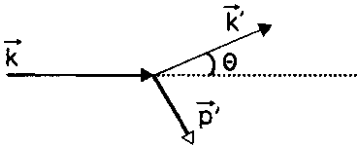

Fig. 2.1 Kinematics of electron scattering in the laboratory system

Suppose we can characterize the ground state of the target by a spherically symmetric charge distribution. In the case of a nucleus of charge number Z this is

$$Ze\rho(r) = \left\langle \psi_0 \left| e \sum_{i=1}^{Z} \delta(\boldsymbol{r} - \boldsymbol{r}_i) \right| \Psi_0 \right\rangle$$

with $\boldsymbol{\Psi}_0$ the (spherical) ground state wave function.[4] The corresponding electrostatic potential $V(r)$ is given by (1.186) and must be inserted into the Dirac equation (1.187) in which we neglect the mass term on the right-hand side.

We wish to construct solutions of (1.187) which describe the scattering of electrons whose incoming momentum is directed along the positive 3-axis. In order to avoid the complications due to the infinite range of the Coulomb potential we assume, as usual that $\lim_{r\to\infty} rV(r) = 0$. This means that we multiply the Coulomb potential by a screening factor and that we take the correct limit at the end of our calculation. As is well-known, using this mathematically incorrect procedure we miss the typical constant and logarithmic phases of the Coulomb scattering problem. However, very much like in the nonrelativistic case, these complications are irrelevant for our discussion; they may be inserted in the final results.

For $r \to \infty$ the two-component spinors $\phi(\boldsymbol{x})$ and $\chi(\boldsymbol{x})$ must be eigenstates of helicity. Thus, in the *centre-of-mass system* they have the asymptotic behaviour

$$\phi(\boldsymbol{x}) \sim u_+(0,0)\mathrm{e}^{\mathrm{i}k^* z} + f(\theta,\varphi)\frac{\mathrm{e}^{\mathrm{i}k^* r}}{r}u_+(\theta,\varphi), \tag{2.17a}$$

$$\chi(\boldsymbol{x}) \sim u_-(0,0)\mathrm{e}^{\mathrm{i}k^* z} + g(\theta,\varphi)\frac{\mathrm{e}^{\mathrm{i}k^* r}}{r}u_-(\theta,\varphi), \tag{2.17b}$$

where $k^* = |k_{\mathrm{c.m}}| = \left|k'_{\mathrm{c.m.}}\right| \simeq E^*$ and where

$$\begin{aligned} u_\pm(\theta,\varphi) &= D^{(1/2)*}(0,\theta,\varphi)u_\pm(0,0), \\ u_+(0,0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_-(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{2.18}$$

cf. (1.141). f and g denote the scattering amplitudes whose squares give the cross section in the c.m. system. $u_\pm(0,0)$ are positive and negative helicity eigenstates, respectively, of the incoming state whose momentum points in the positive 3-direction. Similarly

$$u_+(\theta,\varphi) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi/2}\cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\varphi/2}\sin(\theta/2) \end{pmatrix}, \quad u_-(\theta,\varphi) = \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\varphi/2}\sin(\theta/2) \\ \mathrm{e}^{\mathrm{i}\varphi/2}\cos(\theta/2) \end{pmatrix} \tag{2.19}$$

are the helicity eigenstates along the outgoing momentum.

[4] In the case of an elementary particle $\rho(r)$ is the Fourier transform of the form factor, see below. This leaves open the question of whether or not $\rho(r)$ may be computable from a constituent picture.

Two important properties follow from the equations (1.187) with m set equal to zero:

(i) Equations (1.187a) and (1.187b) are completely decoupled. The potential $V(r)$, which is spherically symmetric, cannot change the helicity.
(ii) From invariance under *parity* and under *rotations* one shows that $f(\theta, \varphi)$ and $g(\theta, \varphi)$ must be equal (see exercise 2.1). Thus, *a fast electron cannot be polarized by a spherically symmetric potential*. [Note that this is not so for energies comparable to the rest mass. At such energies the spin–orbit force can indeed flip the spin of the electron (Mott scattering).] Therefore, it is sufficient to solve one of the equations (1.187), for example the first one, which reads

$$(\mathrm{i}\boldsymbol{\sigma} \cdot \nabla + k^*)\phi(x) = V(r)\phi(x). \tag{2.20}$$

We solve this equation with the aid of the appropriate Green function which satisfies the equation

$$(\mathrm{i}\boldsymbol{\sigma} \cdot \nabla_x + k^*)G(\boldsymbol{x} - \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{2.21}$$

whereby

$$\phi(x) = u_+(0,0)\mathrm{e}^{\mathrm{i}k^* z} + \int G(\boldsymbol{x} - \boldsymbol{x}')V(r')\phi(\boldsymbol{x}')\mathrm{d}^3x'. \tag{2.22}$$

The Green function with the correct asymptotic behaviour is given by

$$G(\boldsymbol{x} - \boldsymbol{x}') = (\mathrm{i}\boldsymbol{\sigma} \cdot \nabla_x - k^*)\frac{\mathrm{e}^{\mathrm{i}k^*|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{2.23}$$

This follows from the equations

$$(\mathrm{i}\boldsymbol{\sigma} \cdot \nabla_x + k^*)(\mathrm{i}\boldsymbol{\sigma} \cdot \nabla_x - k^*) = -(\Delta + k^{*2}),$$

$$(\Delta + k^{*2})\frac{\mathrm{e}^{\mathrm{i}k^*|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} = -\delta(\boldsymbol{x} - \boldsymbol{x}').$$

First Born approximation means replacing $\phi(\boldsymbol{x}')$ on the r.h.s. of (2.22) by the incoming plane wave. Thus

$$\phi(\boldsymbol{x}) \simeq u_+(0,0)\mathrm{e}^{\mathrm{i}k^* z} + \frac{1}{4\pi}\int \mathrm{d}^3x'(\mathrm{i}\boldsymbol{\sigma} \cdot \nabla_x - k^*)\frac{\mathrm{e}^{\mathrm{i}k^*|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x} - \boldsymbol{x}'|}V(r')u_+(0,0)\mathrm{e}^{\mathrm{i}k^* z},$$

from which we must now extract the scattering amplitude $f(\theta, \varphi)$. For finite r', but taking the limit $r \to \infty$, we have

$$(\mathrm{i}\boldsymbol{\sigma}\cdot\nabla_x - k^*)\frac{\mathrm{e}^{\mathrm{i}k^*|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} = -(\mathrm{i}\boldsymbol{\sigma}\cdot\nabla_{x'} + k^*)\frac{\mathrm{e}^{\mathrm{i}k^*|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$\underset{r\to\infty}{\sim} -(\mathrm{i}\sigma\cdot\nabla_{x'} + k^*)\frac{\mathrm{e}^{\mathrm{i}k^* r}}{r}\mathrm{e}^{-\mathrm{i}k^*\hat{x}\cdot x'}.$$

If we set $\boldsymbol{k}' := k^*\cdot\hat{\boldsymbol{x}} = k^*\boldsymbol{x}/r$, this gives for $r\to\infty$

$$\phi(\boldsymbol{x}) \sim u_+(0,0)\mathrm{e}^{\mathrm{i}k^* z} - \frac{\mathrm{e}^{\mathrm{i}k^* r}}{r}\frac{1}{4\pi}\int \mathrm{d}^3x'\mathrm{e}^{\mathrm{i}\boldsymbol{q}\boldsymbol{x}}V(r')(\boldsymbol{\sigma}\cdot\boldsymbol{k}' + k^*)u_+(0,0) \tag{2.24}$$

with $\boldsymbol{q} := \boldsymbol{k} - \boldsymbol{k}'$. The r.h.s of (2.24) is to be identified with the general form (2.17a), giving

$$f(\theta,\varphi) = -\frac{1}{4\pi}\int \mathrm{d}^3x'\mathrm{e}^{\mathrm{i}\boldsymbol{q}\boldsymbol{x}}V(r')u_+^\dagger(\theta,\varphi)(\boldsymbol{\sigma}\cdot\boldsymbol{k}' + k^*)u_+(0,0).$$

The scalar product under the integral sign is easily worked out by making use of the equation satisfied by the spinor $u_+(\theta,\varphi)$:

$$u_+^\dagger(\theta,\varphi)\left(\boldsymbol{\sigma}\cdot\boldsymbol{k}' - k^*\right) = 0.$$

One finds

$$u_+^\dagger(\theta,\varphi)(\boldsymbol{\sigma}\cdot\boldsymbol{k}' + k^*)u_+(0,0) = 2k^*u_+^\dagger(\theta,\varphi)u_+(0,0) = 2k^*\mathrm{e}^{\mathrm{i}\varphi/2}\cos(\theta/2),$$

which then gives

$$f(\theta,\varphi) = -\frac{k^*}{2\pi}\mathrm{e}^{\mathrm{i}\varphi/2}\cos(\theta/2)\int \mathrm{d}^3r\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}V(r).$$

Alternatively, we may integrate by parts and make use of Poisson's equation $\Delta V(r) = Ze^2\rho(r)$ to obtain

$$f(\theta,\varphi) = -\frac{2Z\alpha k^*}{q^2}\mathrm{e}^{\mathrm{i}\varphi/2}\cos(\theta/2)\int \mathrm{d}^3r\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\rho(r) \tag{2.25}$$

$\left(\alpha = \mathrm{e}^2/4\pi \simeq \frac{1}{137}\right)$. The differential cross section in the centre-of-mass system is then given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{c.m.}} = |f(\theta,\varphi)|^2 = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{Mott}} F^2(\boldsymbol{q}), \tag{2.26}$$

where

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{Mott}} = \frac{(Z\alpha)^2\cos^2(\theta/2)}{4k^{*2}\sin^4(\theta/2)} \tag{2.27}$$

is the Mott cross section (2.8), θ is the scattering angle in the c.m. system and $F(\boldsymbol{q})$ is the (real) charge form factor of the charge distribution $\rho(r)$ (Mott, 1929).

2.3 A Few Properties of Form Factor and Cross Section

Before we proceed to scattering from targets with nonvanishing spin we wish to discuss a few properties of the results obtained in the previous section.

The order of magnitude of the Mott cross section is easily estimated. Take, for instance, $k^* = 200\,\mathrm{MeV} \triangleq (200/\hbar c)\,\mathrm{fm}^{-1} = 1.01\,\mathrm{fm}^{-1}$, $\theta = 90°$ and $Z = 82$. This gives

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{Mott}} = (82)^2 \times 2.6 \times 10^{-31}\,\mathrm{cm}^2 = 1.7 \times 10^{-27}\,\mathrm{cm}^2.$$

A few properties of the form factor have already been mentioned above, in Sect. 2.1. If the target is a nucleus the charge distribution can be calculated from the ground state wave function Ψ_0,

$$\rho(\boldsymbol{r}) = \frac{1}{Z}\int \mathrm{d}^3 r_1 \int \mathrm{d}^3 r_2 \ldots \int \mathrm{d}^3 r_A \sum_{\nu=1}^{z} \delta(\boldsymbol{r} - \boldsymbol{r}_\nu)|\boldsymbol{\Psi}_0(\boldsymbol{r}_1 \ldots \boldsymbol{r}_A)|^2,$$

so that

$$F(q^2) = \frac{1}{Z}\int \mathrm{d}^3 r_1 \int \mathrm{d}^3 r_2 \ldots \int \mathrm{d}^3 r_A \sum_{\nu=1}^{z} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}_\nu}|\boldsymbol{\Psi}_0(\boldsymbol{r}_1 \ldots \boldsymbol{r}_A)|^2. \tag{2.28}$$

We verify the property $F(0) = 1$ which expresses the fact that forward scattering depends only on the total charge of the target. If $\rho(\boldsymbol{r})$ is spherically symmetric, the form factor (2.6) can be written as follows. We make use of the expansion

$$\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} = 4\pi \sum_{l=0}^{\infty} \mathrm{i}^l\, j_l(qr) \sum_{m=-l}^{+l} Y^*_{lm}(\hat{\boldsymbol{q}})\, Y_{lm}(\hat{\boldsymbol{r}}) \tag{2.29}$$

and of the orthogonality property of the spherical harmonics to obtain[5]

$$\begin{aligned} F(q^2) &= 4\pi \int_0^\infty \rho(r)\, j_0(qr) r^2 \mathrm{d}r \\ &= \frac{4\pi}{q}\int_0^\infty \rho(r)\sin(qr) r\,\mathrm{d}r. \end{aligned} \tag{2.30}$$

[5]If $\rho(\boldsymbol{r})$ is not spherically symmetric the formalism developed for inelastic scattering below may be consulted.

If (qr) is small over the domain where $\rho(r)$ is appreciably different from zero, one may expand the form factor in powers of q,

$$F(q^2) = 4\pi \int_0^\infty \rho(r) r^2 \mathrm{d}r - \frac{1}{6} q^2 4\pi \int_0^\infty \rho(r) r^4 \mathrm{d}r | \mathcal{O}(q^4)$$
$$= 1 - \frac{1}{6} q^2 \langle r^2 \rangle_{\text{r.m.s}} + \mathcal{O}(q^4 \langle r^4 \rangle). \tag{2.31}$$

Here $\langle r^2 \rangle_{\text{r.m.s.}}$ denotes the root-mean-square radius

$$\langle r^2 \rangle_{\text{r.m.s.}} = -6 \frac{\partial F(q^2)}{\partial q^2}\bigg|_{q^2=0} = 4\pi \int_0^\infty [\rho(r) r^2] r^2 \mathrm{d}r. \tag{2.32}$$

As the momentum transfer increases, more and more moments $\langle r^{2n} \rangle$ come into play. Eventually, if the form factor is known for all momenta q^2, all even moments are determined. This is equivalent to saying that the charge distribution $\rho(r)$ has been mapped completely and is obtained from the form factor by

$$\rho(r) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 q \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} F(q^2), \tag{2.33}$$

In the case of *nuclei* $\rho(r)$ is given by the wave functions of the protons in the nuclear ground state. The r.m.s. radius is then the average r.m.s. radius of the protons (so-called *charge radius*),

$$\langle r^2 \rangle_{\text{r.m.s.}} = \frac{1}{Z} \sum_{i=1}^{Z} \langle r_i^2 \rangle. \tag{2.34}$$

In the case of the *nucleon* it is not clear a priori what causes its finite charge distribution. The finite extension of charge within the proton may be due to the virtual meson cloud surrounding the proton, and/or to a bound state substructure in which case the charge density reflects some properties of the ground state of the proton's constituents. In any event, the primary physical quantity is the form factor, not the charge density. It is the form factor which describes the particle's coupling to the photon (Coulomb field) and which enters in the expressions for scattering amplitudes and cross sections. Once the form factor is given, we may *define* the charge density through (2.33) and, in particular, the r.m.s. radius through the derivative of $F(q^2)$, cf. (2.32).

In Sect. 2.2 we have calculated the cross section in the c.m. system. It is not difficult to transform it to the laboratory system or any other system of reference. For that purpose it is useful to write first the cross section in a Lorentz invariant form. Let us introduce Lorentz-scalar variables (Mandelstam variables), s and t, and let us write these, in the c.m. frame, in terms of k^* and θ (neglecting the electron mass).

$$s = (k + p)^2 = M^2 + 2k^* \sqrt{M^2 + k^{*2}} + 2k^{*2},$$
$$t = (k - k')^2 = -2k^{*2}(1 - \cos\theta) \equiv -q^2.$$

Inverting these equations one obtains

$$k^* = (s - M^2)/2\sqrt{s},$$
$$\cos^2(\theta/2) = [(s - M^2)^2 + st]/(s - M^2)^2.$$

The invariant cross section $\mathrm{d}\sigma/\mathrm{d}t$ is calculated by means of

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\right)_{\text{c.m.}} \frac{\mathrm{d}\cos\theta}{\mathrm{d}t} = \int_0^{2\pi} \mathrm{d}\varphi \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{c.m.}} \frac{\mathrm{d}\cos\theta}{\mathrm{d}t},$$

yielding

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{4\pi(Z\alpha)^2}{t^2} F^2(t) \frac{(s - M^2)^2 + st}{(s - M^2)^2}. \tag{2.35}$$

It is now easy to calculate the cross section in the laboratory system, where

$$s = M^2 + 2ME,$$
$$t = -2EE'(1 - \cos\theta) = -2E^2 \frac{1 - \cos\theta}{1 + (E/M)(1 - \cos\theta)}. \tag{2.36}$$

Thus,

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = \frac{2E^2}{[1 + (E/M)(1 - \cos\theta)]^2},$$
$$\frac{(s - M^2)^2 + st}{(s - M^2)^2} = \frac{\cos^2(\theta/2)}{1 + (E/M)(1 - \cos\theta)}.$$

Finally, knowing that

$$\int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\varphi = \frac{\mathrm{d}\sigma}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}\cos\theta}$$

we find

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{lab}} = \left(\frac{Z\alpha}{2E}\right)^2 \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} F^2(q^2) \frac{1}{1 + 2(E/M)\sin^2(\theta/2)}. \tag{2.35′}$$

Thus, the cross section in the laboratory system contains the recoil factor $\{1 + 2(E/M)\sin(\theta/2)\}^{-1}$ which we encountered earlier in (2.16).

Clearly, the method of transforming the cross section from one frame of reference to another that we have developed here, is quite general. It consists in writing first the cross section in Lorentz invariant form (2.35), and then in specializing to any desired frame of reference. In this context, we also refer to the general formulae collected in App. B.

2.4 Elastic Scattering from Nucleons

2.4.1 Current Matrix Elements and Form Factors

The scattering cross section (2.26, 2.35) that we have derived in the last two sections holds for any spin zero target, a nucleus, a pion or any other elementary or composite particle with spin zero. The result is fully covariant and thus may also be derived in the framework of covariant perturbation theory. In other words the same cross section must be obtained from the Feynman rules for quantum electrodynamics which are summarized in App. C (see exercise 2.2).

If a particle has no internal structure caused by interactions other than electromagnetic, we say it is *pointlike*. For example, to the best of our knowledge, electron and muon seem to be such particles. In this case the Feynman rules apply as they are given in App. C. In particular, at any photon–fermion vertex we have to write a factor γ_α, the Lorentz index having to be contracted with the photon polarization vector $\varepsilon^\alpha(k)$ for an external photon, or with one of the indices of the photon propagator $D^{\alpha\beta}(k)$ for an internal photon line. This is a reflection of the fact that the photon couples to the electromagnetic current j_α^{em}, and that for a pointlike fermion (on its mass shell) the matrix element of j_α^{em} leading from momentum state p to momentum state p' is given by

$$\left\langle p' \left| j_\alpha^{\mathrm{em}}(0) \right| p \right\rangle = \frac{1}{(2\pi)^3} Q \overline{u(p')} \gamma_\alpha u(p). \tag{2.37}$$

If, to the contrary, the particle does have internal structure due to other interactions not described by QED,[6] then the Feynman rules are incomplete. They cannot tell us the explicit form of the particle's coupling to an (external or internal) photon line. Nevertheless, these couplings can be reduced considerably and can be expressed in terms of a few real Lorentz scalar functions (form factors) which allow one to parametrize the internal structure of the particle in a very condensed form.[7] This is achieved by making use of some general properties of the electromagnetic current, such as its behaviour under Lorentz transformations, time reversal, current conservation, hermiticity, isospin content, etc.

We exemplify these matters for the case of proton and neutron, i.e. strongly interacting particles of spin 1/2. The spin zero case is similar though somewhat simpler. It is left as an exercise for the reader. The relevant matrix element of the electromagnetic current is

[6] For example, hadrons are composite objects and, to some extent, they are also "dressed" by pion clouds.

[7] Note that also a pointlike particle of QED builds up form factors i.e. internal structure by interaction with the Maxwell field. These effects are calculable from QED in higher order perturbation theory.

$$\left\langle p',\ s' \mid j_\alpha^{\mathrm{em}}(x) \mid\ p,\ s \right\rangle, \tag{2.38}$$

where initial and final states are on-mass-shell states of given momentum and spin, and $p^2 = p'^2 = M^2$. Let us work out the restrictions on this matrix element that follow from current conservation, from the space–time structure of the electromagnetic current and hermiticity.

(i) *Current conservation*. Let $x'_\mu = x_\mu + a_\mu$ be an arbitrary translation, that transforms a given field operator according to

$$F(x) \ \rightarrow\ F'(x') = U(a)F(x)U^{-1}(a) = F(x+a). \tag{2.39}$$

Using the generalized Heisenberg equations of motion

$$-\mathrm{i}\partial^\mu F(x) = [P^\mu, F(x)], \tag{2.40}$$

where P^μ are the four energy–momentum operators, one shows that (see exercise (2.3))

$$U(a) = \exp\{\mathrm{i}a_\mu\, P^\mu\}. \tag{2.41}$$

If we consider a matrix element of $F(x)$ between specific eigenstates of energy and momentum, we can make use of (2.39) to transform $F(x)$ to any other point $x' = x + a$ of Minkowski space, i.e.

$$\begin{aligned}\langle q_\mathrm{f}|F(x)|q_i\rangle &= \langle q_\mathrm{f}|U^{-1}(a)F(x+a)\ U(a)|q_\mathrm{i}\rangle \\ &= \langle U(a)q_\mathrm{f}|F(x+a)|\ U(a)q_\mathrm{i}\rangle \\ &= \mathrm{e}^{\mathrm{i}a\cdot(q_\mathrm{i}-q_\mathrm{f})}\ \langle q_\mathrm{f}|F(x+a)|q_\mathrm{i}\rangle. \end{aligned} \tag{2.42}$$

In particular, we may take $a_\mu = -x_\mu$ to obtain

$$\langle q_\mathrm{f}|F(x)|q_\mathrm{i}\rangle \ =\ \mathrm{e}^{-\mathrm{i}x\cdot(q_\mathrm{i}\,-\,q_\mathrm{f})}\ \langle q_\mathrm{f}|F(0)|q_\mathrm{i}\rangle. \tag{2.42$'$}$$

Thus, the x-dependence of any such matrix element between eigenstates of four-momentum is a simple exponential; the remaining factor $\langle q_\mathrm{f}|F(0)|q_\mathrm{i}\rangle$ no longer depends on x. Formulae such as (2.42) or (2.42′) in which space–time arguments are shifted may be called *translation formulae*.

Suppose $F(x)$ is a current operator $J_\alpha(x)$ and suppose that the divergence of J_α is known, $\partial_\alpha J^\alpha(x) = \phi(x)$. Then from (2.42′)

$$\begin{aligned}\langle q_\mathrm{f}|\partial^\alpha\ J_\alpha(x)|q_\mathrm{i}\rangle &= (\partial^\alpha \mathrm{e}^{-i(q_\mathrm{i}-q_\mathrm{f})\cdot x})\langle q_\mathrm{f}|J_\alpha(0)|q_\mathrm{i}\rangle \\ &= \langle q_\mathrm{f}|\phi(x)|q_\mathrm{i}\rangle = e^{-\mathrm{i}(q_\mathrm{i}-q_\mathrm{f})\cdot x}\ \langle q_\mathrm{f}|\phi(0)|q_\mathrm{i}\rangle. \end{aligned}$$

Thus, one finds the relation

$$(q_\mathrm{i} - q_\mathrm{f})^\alpha \, \langle q_\mathrm{f}| \, J_\alpha \, (0) \, |q_\mathrm{i}\rangle = \mathrm{i}\langle q_\mathrm{f}| \, \phi \, (0) \, |q_\mathrm{i}\rangle \, . \tag{2.43}$$

If the current is conserved–which is the case for the electromagnetic current – we obtain the condition

$$(q_\mathrm{i} - q_\mathrm{f})^\alpha \, \langle q_\mathrm{f}| \, j_\alpha \, (0) \, |q_\mathrm{i}\rangle = 0 \tag{2.44}$$

(we drop the superscript "em" for simplicity).
(ii) *Covariance*. As j_α is a Lorentz vector, its matrix elements between nucleon states must also transform as Lorentz vectors. For the construction of such vectors we have at our disposal the vectors p'_α and p_α as well as the γ-matrices and combinations thereof. The only vectors that can be formed are[8]

$$\overline{u(p')}\,\gamma_\alpha u(p), \qquad (p'-p)^\beta \, \overline{u(p')}\,\sigma_{\alpha\beta} u(p),$$
$$(p+p')_\alpha \, \overline{u(p')}\,u(p), \qquad (p-p')_\alpha \, \overline{u(p')}\,u(p),$$

where we have defined

$$\boldsymbol{\sigma}_{\alpha\beta} := \frac{1}{2}\,\mathrm{i}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha). \tag{2.45}$$

That the first two of these are indeed *vector* operators is not difficult to show. Indeed $\overline{\boldsymbol{\Psi}(x)}\gamma_\alpha\boldsymbol{\Psi}(x)$ is a vector operator as should be clear from Chap. 1 Similarly, $\overline{\boldsymbol{\Psi}(x)}\sigma_{\alpha\beta}\boldsymbol{\Psi}(x)$ is a tensor operator. Knowing that p_α and p'_α are vectors and making use of the expansion (1.122), the assertion is proved. Among these covariants only three are independent. The external particles are on their mass shell and obey the free Dirac equation. In this case one has the Gordon identity relating the first three covariants (exercise 2.14).

Thus, the most general covariant decomposition must have the form (dropping the spin indices $s,\ s'$, for the sake of clarity),

$$\langle p'| \, j_\alpha \, (0) \, |p\rangle$$
$$= \frac{1}{(2\pi)^3} \, \overline{u(p')} \left\{ \gamma_\alpha F_1 \, (q^2) + \frac{\mathrm{i}}{2M}\sigma_{\alpha\beta}q^\beta \, F_2 \, (q^2) + \frac{1}{2M}q_\alpha F_3(q^2) \right\} u(p), \tag{2.46}$$

where q is the momentum transfer

$$q := p' - p. \tag{2.47}$$

(We have set $Q = 1$, taking out a factor $|e|$ to be inserted at each photon vertex). The functions F_i must be Lorentz scalars and, thus, can only depend on Lorentz scalar quantities such as $p^2,\ p'^2$ and q^2. Since $p^2 = p'^2 = m^2$, these are in fact constants,

[8]The same forms apply for antiparticle states, with $u(p)$ replaced by $\upsilon(p)$.

so the only true variation must be in the variable q^2. The divergence condition (2.44) implies that $F_3(q^2)$ must vanish identically for $q^2 \neq 0$. The other two terms on the r.h.s. of (2.46) fulfill this condition separately, as one easily verifies by means of the Dirac equations (1.84,1.84').

(iii) *Hermiticity of electromagnetic current*. By definition we have

$$\left\langle p \left| j_\alpha^\dagger(0) \right| p' \right\rangle = \langle p' | j_\alpha(0) | p \rangle^*. \tag{2.48a}$$

As j_α is a hermitean operator this is also equal to

$$\langle p | j_\alpha(0) | p' \rangle . \tag{2.48b}$$

The expression (2.48a), more explicitly, gives

$$\begin{aligned}
&\left(u(p')^\dagger \gamma_0 \left[\gamma_\alpha F_1 + \frac{\mathrm{i}}{2M} \sigma_{\alpha\beta} (p'-p)^\beta F_2 \right] u(p) \right)^* \\
&= u^\dagger(p) \left[(\gamma_0\gamma_\alpha)^\dagger \gamma_0 F_1^* - \frac{\mathrm{i}}{2M} (\gamma_0\sigma_{\alpha\beta})^\dagger (p'-p)^\beta F_2^* \right] u(p') \\
&= \overline{u(p)} \left\{ \gamma_0\gamma_\alpha^\dagger\gamma_0 F_1^* + \frac{\mathrm{i}}{2M} \gamma_0\sigma_{\alpha\beta}^\dagger\gamma_0 (p-p')^\beta F_2^* \right\} u(p'),
\end{aligned}$$

where we have inserted $(\gamma_0)^2 = 1$ between $u^\dagger(p)$ and the square brackets and have interchanged p and p' in the second term. We know from (1.79b) that $\gamma_0\gamma_\alpha^\dagger\gamma_0 = \gamma_\alpha$. Using the definition (2.45) one sees that also $\gamma_0\sigma_{\alpha\beta}^\dagger\gamma_0 = \sigma_{\alpha\beta}$. As (2.48a) must equal (2.48b) we conclude that the form factors F_1 and F_2 are real:

$$F_1^*(q^2) = F_1(q^2), \quad F_2^*(q^2) = F_2(q^2). \tag{2.49}$$

We add a few comments on these results: The form factor F_2 has actually been defined such as to make it real by the choice of the factor i in front of the second term of (2.46). Without this factor F_2 would have come out pure imaginary. Similarly, the factor $1/2M$ is a matter of convention, chosen so as to give $F_2(q^2)$ the same dimension as $F_1(q^2)$. In applying the divergence condition (2.44) we have used the fact that the two spinors belong to the same mass. If the external fermions have different masses the relations following from current conservation look different. Similarly, if the electromagnetic current is taken between two *different* particles, hermiticity does not suffice to derive reality properties of form factors. However, if the interactions are invariant under time reversal, the combination of hermiticity and time reversal invariance implies again reality conditions for the form factors. Examples of this will be met below in the context of weak interactions.

2.4.2 Derivation of Cross Section

The physical interpretation of the form factors F_1 and F_2 is discussed below. For the moment, we note that $F_1(q^2 = 0) = 1$ is the charge of the particle in units of $|e|$, see below. We first turn to the computation of the differential cross section,

$$\mathrm{d}\sigma = \frac{(2\pi)^2\,\delta^{(4)}(p + k - p' - k')}{[2E_k/(2\pi)^3]\;[2E_p/(2\pi)^3]\;|\boldsymbol{v}_{12}|}\;\frac{1}{4}\sum_{\text{spins}}|T_{\text{fi}}|^2\;\frac{d^3k'\mathrm{d}^3p'}{2E_{k'}\,2E_{p'}}. \tag{2.50}$$

T_{fi} is the T-matrix element, to be obtained from Feynman rules for lowest order perturbation theory. $\boldsymbol{v}_{12}$ is the relative velocity of electron and nucleon in the initial state. It is useful to calculate $\mathrm{d}\sigma$ first in the centre-of-mass system and then, in a second step, to write it in a manifestly invariant form from which the cross section in any system of reference can be obtained. In the c.m. system,

$$k = \left(\sqrt{m^2 + q^{*2}}\,;\,\boldsymbol{q}\right),\;\; p = \left(\sqrt{M^2 + q^{*2}}\,;-\boldsymbol{q}\right).$$

Introducing the invariant variables *s, t, u* one has

$$s = (p + k)^2 = (p' + k')^2 = m^2 + M^2 + 2(pk) = m^2 + M^2 + 2(p'k'), \tag{2.51a}$$

$$t = (k - k')^2 = (p' - p)^2 = 2\,m^2 - 2(kk') = 2M^2 - 2(pp'), \tag{2.51b}$$

$$\begin{aligned} u &= (k - p')^2 = (p - k')^2 = m^2 + M^2 - 2(p'k) \\ &= m^2 + M^2 - 2(pk') = 2m^2 + 2M^2 - t - s. \end{aligned} \tag{2.51c}$$

In particular, in the c.m. system

$$q^* := |\boldsymbol{q}| = \frac{1}{2\sqrt{s}}\;\sqrt{(s - M^2 - m^2)^2 - 4M^2m^2}, \tag{2.52}$$

$$t = -2q^{*2}(1 - \cos\theta). \tag{2.53}$$

The Møller factor in the denominator of (2.50) is an invariant and can be written as

$$\begin{aligned} E_k\,E_p|\boldsymbol{v}_{12}| &= \sqrt{(pk)^2 - p^2k^2} \\ &= \frac{1}{2}\;\sqrt{(s - M^2 - m^2)^2 - 4M^2m^2} = q^*\,\sqrt{s}. \end{aligned}$$

As we are interested in $\mathrm{d}\sigma/\mathrm{d}\Omega$ or, equivalently, $\mathrm{d}\sigma/\mathrm{d}t$, all other variables in the final state must be integrated over. The integration over $\boldsymbol{p}'$ may be done first, giving $\boldsymbol{p}' = -\boldsymbol{k}' \equiv \boldsymbol{q}'$:

$$\mathrm{d}\sigma = \frac{(2\pi)^{10}}{16q^*\,\sqrt{s}E'_k\,E'_p}\,\frac{1}{4}\sum|T_{\mathrm{fi}}|^2\,\delta^{(1)}\,(W-E_k-E_p)\;q^{*\prime 2}\,\mathrm{d}q^{*\prime}\,\mathrm{d}\Omega,$$

where

$$W := \sqrt{M^2+q^{*\prime 2}}+\sqrt{m^2+q^{*\prime 2}}.$$

$q^{*\prime}$ is the modulus of the three-momentum in the final state. By the remaining δ-function it becomes equal to q^*. This leaves us with the integration over $q^{*\prime}$ or, equivalently, over W, provided we make the replacement

$$\mathrm{d}q^{*\prime} = \frac{\mathrm{d}q^{*'}}{\mathrm{d}W}\mathrm{d}W = \frac{E_{p'}\,E_{k'}}{q^{*\prime}\,W}\,\mathrm{d}W.$$

This gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{16s}(2\pi)^{10}\,\frac{1}{4}\sum|T_{\mathrm{fi}}|^2. \tag{2.54}$$

The invariant quantity $\mathrm{d}\sigma/\mathrm{d}t$ is obtained from this by integrating over the azimuth ϕ and by making the replacement

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} = \frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 2q^{*2}\frac{\mathrm{d}\sigma}{\mathrm{d}t}.$$

with the expression (2.52) for q^{*2}, this gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{(2\pi)^{11}}{8[(s-M^2-m^2)^2-4M^2m^2]}\,\frac{1}{4}\sum|T_{\mathrm{fi}}|^2. \tag{2.55}$$

The next step is the construction of the T-matrix element and the calculation of the spin summation. The Feynman rules give the R-matrix R_{fi}, which is related to T_{fi} by (B2) of App. B. So we have

$$R_{\mathrm{fi}} = \frac{-\mathrm{i}e^2}{(2\pi)^2}\int \mathrm{d}^4\kappa\,\overline{u(k')}\,\gamma_\alpha\,u(k)(\kappa^2+\mathrm{i}\varepsilon)^{-1}\,g^{\alpha\beta}\,\langle p'|j_\beta(0)|p\rangle$$
$$\times(2\pi)^3\,\delta^{(4)}\,(k-\kappa-k')\,\delta^{(4)}\,(p+\kappa-p').$$

The integration over κ, the momentum of the virtual photon, yields $\kappa = k - k' = p' - p$ and leaves us with one δ-function for overall energy-momentum conservation.

Thus

$$T_{\mathrm{fi}} = \frac{-e^2}{(2\pi)^3}\,\overline{u(k')}\,\gamma^\alpha u(k)\,\frac{1}{t}\,\langle p'|j_\alpha(0)|p\rangle.$$

For the actual calculation of the cross section it is useful to rewrite the nucleonic matrix element by means of the Gordon identity

$$\langle p'|j_\alpha(0)|p\rangle = \frac{1}{(2\pi)^3}\,\overline{u(p')}\,\{(F_1+F_2)\,\gamma_\alpha - \frac{1}{2M}(p+p')_\alpha F_2\}\,u(p). \quad (2.46')$$

The spin summations are best carried out by means of the trace techniques of App. C2. Here we have to calculate the expression

$$\begin{aligned}\mathscr{M} :&= \frac{1}{4}\sum_{\text{spins}}\left|\overline{(u(k')\,\gamma^\alpha\,u(k))}\left(\overline{u(p')}\left[\gamma_\alpha(F_1+F_2) - \frac{1}{2M}\,P_\alpha F_2\right]u(p)\right)\right|^2\\ &= \frac{1}{4}\,\mathrm{tr}\{\gamma^\alpha(\not{k}+m)\gamma^\beta\,(\not{k}'+m)\}\mathrm{tr}\left\{\left[(F_1+F_2)\gamma_\alpha - \frac{P_\alpha}{2M}F_2\right](\not{p}+M)\right.\\ &\quad\left.\times\left[(F_1+F_2)\gamma_\beta - \frac{P_\beta}{2M}\,F_2\right](\not{p}'+M)\right\},\end{aligned}$$

where we have set $p+p' = P$. There are basically four expressions to be calculated, namely

$$\begin{aligned}\text{(a)}\quad &\frac{1}{4}\,\mathrm{tr}\,\{\gamma^\alpha(\not{k}+m)\gamma^\beta(\not{k}'+m)\}\,\mathrm{tr}\{\gamma_\alpha(\not{p}+M)\,\gamma_\beta\,(\not{p}'+M)\}\\ &= 8\{2m^2M^2 - M^2(kk') - m^2(pp') + (pk)\,(p'k') + (pk')(p'k)\},\end{aligned}$$

which from (2.51) is equal to

$$\begin{aligned}&= 2\{2(s-M^2-m^2)^2 + 2st + t^2\}.\\ \text{(b)}\quad &\frac{1}{4}\,\mathrm{tr}\{\gamma^\alpha(\not{k}+m)\,\not{P}\,(\not{k}'+m)\}\,\mathrm{tr}\{\gamma_\alpha(\not{p}+M)\,(\not{p}'+M)\}\\ &= 4M\{m^2P^2 + 2(Pk)\,(Pk') - P^2(kk')\}\\ &= 8M\{(s-M^2-m^2)^2 + t(s-m^2)\}.\\ \text{(c)}\quad &\frac{1}{4}\mathrm{tr}\{\not{P}(\not{k}+m)\,\not{P}\,(\not{k}'+m)\}\\ &= m^2P^2 + 2(Pk)\,(Pk') - P^2(kk')\\ &= 2\{(s-M^2-m^2)^2 + t(s-m^2)\}.\\ \text{(d)}\quad &\frac{1}{4}\,\mathrm{tr}\{(\not{p}+M)\,(\not{p}'+M)\} = M^2 + (pp') = 2M^2 - \frac{1}{2}t.\end{aligned}$$

All traces in $\mathscr{M}$ are reducible to these prototypes. We find

$$\begin{aligned}\mathscr{M} &= 4[(s-M^2-m^2)^2+t(s-m^2)]\\ &\times\left[(F_1+F_2)^2-2F_1F_2-F_2^2-\frac{t}{4M^2}F_2^2\right]\\ &+2(t^2+2m^2t)\,(F_1+F_2)^2.\end{aligned}$$

Inserting this into (2.55) and replacing $e^2/4\pi=\alpha$, we finally obtain

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{4\pi\alpha^2}{t^2}\,\frac{(s-M^2-m^2)^2+t(s-m^2)}{(s-M^2-m^2)^2-4M^2m^2}\\ &\times\left\{F_1^2(t)-\frac{t}{4M^2}F_2^2(t)+\frac{t(t+2m^2)}{2[(s-M^2-m^2)^2+t(s-m^2)]}(F_1(t)+F_2(t))^2\right\}. \quad (2.56)\end{aligned}$$

This formula was first derived by Rosenbluth (1950). It applies to the scattering of any charged lepton of mass m from a complex target with spin 1/2. The target structure is contained entirely in the Lorentz scalar functions $F_1(t)$ and $F_2(t)$, the significance of which will become clear below.

As an exercise let us use (2.56) to derive the differential cross section in the laboratory system for electron scattering on the nucleon. The electron energy shall be chosen so large that the electron mass can be neglected. Setting $m=0$, the variables s and t in the laboratory system are

$$\begin{aligned}s &\simeq M^2+2ME,\\ t &\simeq -2EE'(1-\cos\theta)=-2E^2\,\frac{1-\cos\theta}{1+(E/M)(1-\cos\theta)}\end{aligned}$$

From this one obtains

$$\begin{aligned}\frac{t^2}{2[(s-M^2)^2+ts]} &\simeq \frac{-tE^2(1-\cos\theta)}{2M^2E^2(1+\cos\theta)}=-\frac{t}{2M^2}\mathrm{tg}^2(\theta/2),\\ \frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} &= \frac{2E^2}{(1+(E/M)(1-\cos\theta))^2},\end{aligned}$$

so that

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{lab}} &= \left(\frac{\alpha}{2E}\right)^2\frac{\cos^2(\alpha/2)}{\sin^4(\theta/2)}\,\frac{1}{1+2(E/M)\,\sin^2(\theta/2)}\\ &\times\left\{F_1^2(t)-\frac{t}{4M^2}F_2^2(t)-\frac{t}{2M^2}[F_1(t)+F_2(t)]^2\,\mathrm{tg}^2(\theta/2)\right\}. \quad (2.57)\end{aligned}$$

This is the generalization of formula (2.35′) to a target with spin 1/2.

2.4.3 Properties of Form Factors

The form factors $F_1(t)$ and $F_2(t)$ are defined through the covariant decomposition (2.46) of the nucleonic one-particle matrix element of the electromagnetic current operator $j_\alpha(x)$. In this section we work out the physical interpretation of these form factors. For this purpose it is convenient to consider the matrix element (2.46) in the specific frame of reference where the sum of spatial three-momenta of initial and final nucleon states vanishes, i.e. $\boldsymbol{p} + \boldsymbol{p}' = 0$. In this frame the limit of vanishing four-momentum transfer, $q^2 \to 0$, leads us automatically into the rest frame of the particle. It is in the particle's rest frame that we are able to relate F_1 and F_2 to static properties of nucleons.

(i) *Electric form factor* . From (2.46, 2.46′) the charge density (fourth component of $j_\alpha(x)$) is given by

$$\langle \boldsymbol{p}|\, j_0(0)|\, -\boldsymbol{p}\rangle = \frac{1}{(2\pi)^3}\overline{u(\boldsymbol{p})}\left\{(F_1+F_2)\gamma_0 - \frac{E_p}{\boldsymbol{M}}F_2\right\} u(-\boldsymbol{p}).$$

Inserting the explicit form of the spinors (1.90) in the standard representation, one finds

$$\begin{aligned}\overline{u(\boldsymbol{p})}\,\gamma_0\, u(-\boldsymbol{p}) &= u^\dagger(\boldsymbol{p})\, u(-\boldsymbol{p}) = 2M,\\ \overline{u(\boldsymbol{p})}\, u(-\boldsymbol{p}) &= 2E_p,\end{aligned}$$

independently of the spin direction. Furthermore, $\boldsymbol{q} = 2\boldsymbol{p}$ and $t = (p'-p)^2 = -4\boldsymbol{p}^2 = -4E_p^2 + 4M^2$. Thus

$$(2\pi)^3\,\langle \boldsymbol{p}|\mathrm{j}_0(0)|\, -\boldsymbol{p}\rangle = 2M(F_1+F_2) - \frac{2E_p^2}{M}F_2 = 2M\left[F_1 + \frac{t}{4M^2}F_2\right].$$

On the basis of this result it is natural to define

$$G_\mathrm{E}(t) := F_1(t) + \frac{t}{4M^2}\,F_2(t) \tag{2.58}$$

as the *electric form factor* of the nucleon. It is easy to verify that for the proton

$$F_1^{(\mathrm{p})}(0) = G_\mathrm{E}^{(\mathrm{p})}(0) = 1, \tag{2.59a}$$

which expresses the fact that the proton carries one unit of the positive elementary charge. Indeed, we know that

$$\left\langle p' \,\middle|\, \int \mathrm{d}^3x\, j_0(x) \,\middle|\, p\right\rangle = 1\langle p'|p\rangle = 2E_p\delta(\boldsymbol{p}-\boldsymbol{p}').$$

On the other hand, from the decomposition (2.46), and using translation invariance as in (2.42′),

$$\begin{aligned}\int \mathrm{d}^3 x \langle p' \mid j_0(x) \mid p\rangle &= (2\pi)^3\, \delta(\boldsymbol{p}-\boldsymbol{p}')\ \langle p'|j_0(0)|p\rangle \\ &= \delta(\boldsymbol{p}-\boldsymbol{p}')\ F_1^{(p)}(0) u^\dagger(\boldsymbol{p})\, u(\boldsymbol{p}) = 2E_p F_1^{(p)} \delta(\boldsymbol{p}-\boldsymbol{p}').\end{aligned}$$

Thus $F_1^{(p)}(0) = 1$.

Similarly for the neutron

$$F_1^{(\mathrm{n})}(0) \;=\; G_\mathrm{E}^{(\mathrm{n})}(0) \;=\; 0. \tag{2.59b}$$

The r.m.s. radii of F_1 and G_E, which are defined by (2.32) are not the same, however. One finds the relationship

$$\langle r^2\rangle_{G_\mathrm{E}} = \langle r^2\rangle_{F_1} + \frac{1}{4M^2} F_2(0). \tag{2.60}$$

Here, $F_2(0)$ is found to be the anomalous magnetic moment below.

(ii) *Magnetic form factor*. We know from Sect. 1.9.2 that the magnetic properties are obtained from matrix elements of the spatial current density, i.e. from

$$\langle \boldsymbol{p}|j^i(0)|-\boldsymbol{p}\rangle = \frac{1}{(2\pi)^3}(F_1+F_2)\overline{u(\boldsymbol{p})}\gamma^i u(-\boldsymbol{p})$$

[where we have used (2.46′)]. With the explicit solutions (1.90) we have

$$\begin{aligned}\overline{u(\boldsymbol{p})}\,\gamma^i\, u(-\boldsymbol{p}) &= u^\dagger(\boldsymbol{p}) \begin{pmatrix} 0 & \sigma^{(i)} \\ \sigma^{(i)} & 0 \end{pmatrix} u(-\boldsymbol{p}) \\ &= \chi^\dagger[-\sigma^{(i)}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) + (\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\sigma^{(i)}]\chi \\ &= -2\varepsilon_{ikl}\, p_k \chi^\dagger \sigma^{(l)}\chi = -\varepsilon_{ikl} q_k \chi^\dagger \sigma^{(l)}\chi \\ \langle \boldsymbol{p}|j^i(0)|-\boldsymbol{p}\rangle &= -\frac{1}{(2\pi)^3}\,(F_1+F_2)\ \varepsilon_{ikl} q_k \chi^\dagger \sigma^{(l)}\chi.\end{aligned}$$

Comparing this to the formulae in Sect. 1.9.2 we see that the combination

$$G_\mathrm{M}(t) := F_1(t) + F_2(t) \tag{2.61}$$

may be interpreted as the *magnetic form factor*.[9] In particular, $G_\mathrm{M}(0)$ is equal to the total magnetic moment of the particle; $F_1(0)$ gives the "normal" magnetic moment; $F_2(0)$ the "anomalous" magnetic moment. In the case of proton and neutron,

[9] $G_\mathrm{E}(t)$ and $G_\mathrm{M}(t)$ are also called Sachs form factors.

$$F_1^{(\mathrm{p})}(0) = 1, \quad F_2^{(\mathrm{p})}(0) \equiv \mu_{\mathrm{an}}^{\mathrm{p}} = 1.792847351(28), \tag{2.62a}$$

$$F_1^{(\mathrm{n})}(0) = 0, \quad F_2^{(\mathrm{n})}(0) \equiv \mu_{\mathrm{an}}^{\mathrm{n}} = -1.9130427(5) \tag{2.62b}$$

When rewritten in terms of the Sachs form factors (2.58) and (2.61) the differential cross section (2.57) reads

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{lab}} = \left(\frac{\alpha}{2E}\right)^2 \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \frac{1}{1 + 2(E/M)\sin^2(\theta/2)} \times \left\{ \frac{1}{1 - t/4M^2} G_{\mathrm{E}}^2(t) - \frac{t}{4M^2} \left[\frac{1}{1 - t/4M^2} + 2\mathrm{tg}^2(\theta/2) \right] G_{\mathrm{M}}^2(t) \right\}. \tag{2.57′}$$

Therefore, if one plots the quantity in curly brackets at fixed t and multiplied by $\mathrm{ctg}^2(\theta/2)$, as a function of $\mathrm{ctg}^2(\theta/2)$, the data must fall onto a straight line with slope and intercept, respectively, as follows

$$\frac{G_{\mathrm{E}}^2 + \tau G_{\mathrm{M}}^2}{1 + \tau} \quad \text{and} \quad 2\tau G_{\mathrm{M}}^2$$

where $\tau = -t/4M^2$.

2.4.4 *Isospin Analysis of Nucleon Form Factors*

Isospin invariance is an approximate symmetry of strong interactions. It is a spectrum symmetry in the sense that strongly interacting particles can be classified in mass degenerate multiplets of the isospin group SU(2). While the strong interactions are invariant under isospin transformations, the electromagnetic interactions are not. That is, the strong interactions transform like scalars under isospin transformations, the electromagnetic interaction $j^\alpha A_\alpha$ does not. Nevertheless, it may be expanded in terms of multipole operators in isospin space, viz.

$$j_\alpha(x) = j_\alpha^{(0)}(x) + j_\alpha^{(1)}(x) + \cdots, \tag{2.63}$$

where $j_\alpha^{(0)}$ denotes an isoscalar operator, $j_\alpha^{(1)}$ denotes the third component of an isovector operator. There are good indications that the electromagnetic current operator contains only isoscalar and isovector operators, i.e. that the expansion (2.63) ends with the second term, but a priori this is not known. The two terms on the right-hand side carry the quantum numbers of the vector mesons ω, ϕ and ρ, respectively, as summarized in Table 2.1.

In the case of nucleons it is easy to isolate the isoscalar and isovector parts of the nucleon form factors. Proton and neutron form a doublet of isospin, the proton is assigned $I_3 = +1/2$, the neutron $I_3 = -1/2$. Let $O_{\mu=0}^{(\kappa)}$ be any tensor

Table 2.1 Properties of electromagnetic current.

	Isospin		Spin Parity J^π	Charge conjugation C	G-Parity G	Analogue vector meson states
	I	I_3				
$j_\alpha^{(0)}(x)$	0	0	$\left\{\begin{matrix}0^+\\1^-\end{matrix}\right\}$	$-$	$-$	$\omega(782)$
						$\phi(1020)$
$J_\alpha^{(1)}(x)$	1	0	$\left\{\begin{matrix}0^+\\1^-\end{matrix}\right\}$	$-$	$+$	$\rho(770)$

operator with isospin κ and three component $\mu = 0$ in isospin, having nonvanishing matrix elements between one nucleon states. Then from the Wigner–Eckart theorem, we have

$$\langle \frac{1}{2} I_3 | O_\mu^{(\kappa)} | \frac{1}{2} I_3 \rangle = (-)^{1/2-I_3} \begin{pmatrix} \frac{1}{2} & \kappa & \frac{1}{2} \\ -I_3 & 0 & I_3 \end{pmatrix} \left(\frac{1}{2} \parallel O^{(\kappa)} \parallel \frac{1}{2} \right), \tag{2.64}$$

where $\left(\frac{1}{2} \left\| O^{(\kappa)} \right\| \frac{1}{2}\right)$ denotes the reduced matrix element. The first two $3j$-symbols with $\mu = 0$ are given by [EDM57, DST63]

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -I_3 & 0 & I_3 \end{pmatrix} = \frac{(-)^{1/2-I_3}}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -I_3 & 0 & I_3 \end{pmatrix} = \sqrt{1/6}.$$

All other such symbols for $\kappa \geq 2$ vanish because the triangle rule is not fulfilled. Thus

$$\left\langle \frac{1}{2} I_3 \right| \sum_\kappa O_0^{(\kappa)} \left| \frac{1}{2} I_3 \right\rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{2} || O^{(0)} || \frac{1}{2} \right) + (-)^{1/2-I_3} \sqrt{\frac{1}{6}} \left(\frac{1}{2} || O^{(1)} || \frac{1}{2} \right). \tag{2.65}$$

As a consequence, we see that the isoscalar may be isolated by taking the sum of matrix elements (2.65) over I_3, while the isovector is isolated by taking the difference. This leads to the following definitions of isoscalar and isovector nucleon form factors

$$F_i^{(s)} = \frac{1}{2}(F_i^{(\mathrm{p})} + F_i^{(\mathrm{n})}) \tag{2.66a}$$

$$(i = 1, 2).$$

$$F_i^{(\upsilon)} = \frac{1}{2}(F_i^{(\mathrm{p})} - F_i^{(\mathrm{n})}) \tag{2.66b}$$

Analogous definitions may be introduced for the electric and magnetic form factors (2.58) and (2.61).

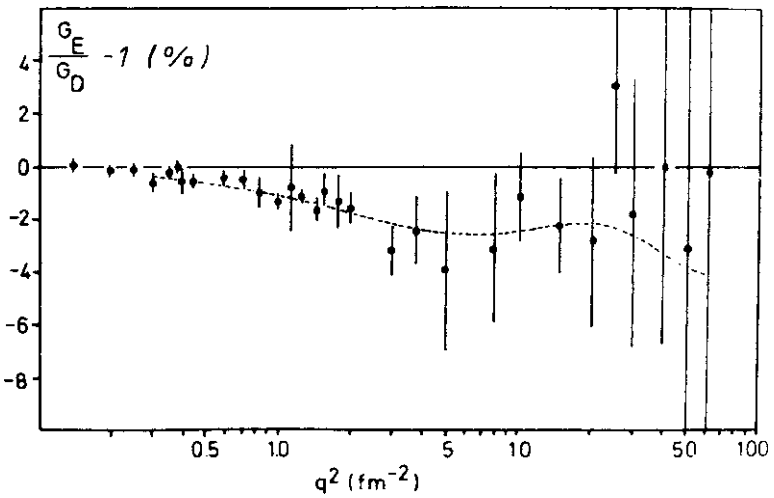

Fig. 2.2 The ratio $G_{\mathrm{E}}^{\mathrm{p}}/G_{\mathrm{D}}$ in percentage deviation from 1, versus q^2. Figure taken from Simon et al. (1980)

It is customary to show the measured electric form factor $G_{\mathrm{E}}^{\mathrm{p}}$ of the proton for low momentum transfers, divided by the so-called *dipole* fit:

$$G_{\mathrm{D}}(q^2) := \frac{1}{(1 + q^2/q_0^2)^2} \text{ with } \quad q_0^2 = 18.23\,\mathrm{fm}^{-2}. \tag{2.67}$$

This dipole dependence of the form factors $G_{\mathrm{E}}^{\mathrm{p}}(q^2)$ and $G_{\mathrm{M}}^{\mathrm{p}}(q^2)/\mu^{\mathrm{p}}$ on q^2 is only of historical interest insofar as early data seemed to be in good agreement with this simple ansatz. Better, empirical fits, useful for theoretical analyses, are found in the literature (Borkowski et al. 1976). Nevertheless, it is customary to plot the data in terms of this formula. From the data one deduces the following value for the r.m.s radius of the proton (Simon et al. 1980). As an example we quote:

$$\left\langle r_E^2 \right\rangle_P^{1/2} = (0.862 \pm 0.012)\,\mathrm{fm}.$$

Figures 2.2, 2.3 show the electric and magnetic form factors $G_{\mathrm{E}}^{\mathrm{p}}$ and $G_{\mathrm{E}}^{\mathrm{p}}/\mu^{\mathrm{p}}$ of the proton, in units of G_{D}, (2.67), and in the form of percent deviations from that ansatz. Figure 2.4, finally, shows data for the electron-proton elastic cross section, again in units of the dipole fit, i.e. in units of the cross section (2.57′) with

$$G_{\mathrm{E}}^{\mathrm{p}} = G_{\mathrm{M}}^{\mathrm{p}}/\mu^{\mathrm{p}} \equiv G_{\mathrm{D}}(q^2).$$

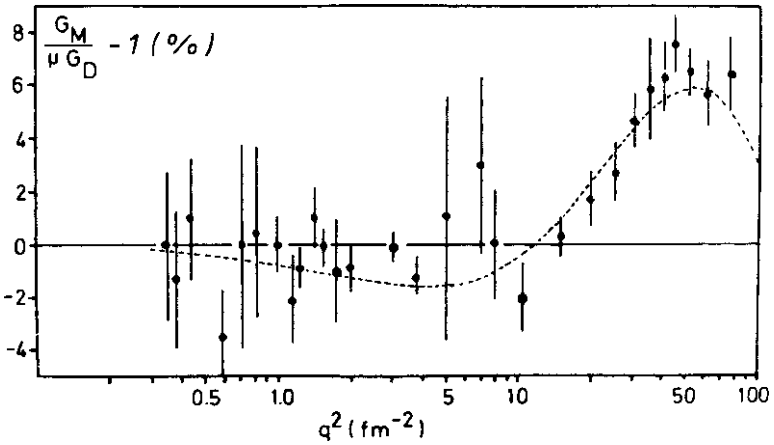

Fig. 2.3 The ratio $G_M^P/\mu^P G_D$ versus q^2. Figure taken from Simon et al. (1980)

2.5 *Elastic and Inelastic Electron Scattering from Nuclei

In Sects. 2.2, 2.4 we have derived the cross section for elastic scattering of electrons from spin zero and spin 1/2 targets. These formulae hold for any kind of target, composite or "elementary". The internal structure of the target is hidden in the Lorentz invariant form factors whose definition is based only on Lorentz covariance and current conservation.

We now extend these results to elastic scattering from nuclei of arbitrary spin, as well as to inelastic scattering to discrete nuclear excited states. As we have seen, *elastic* scattering from a target with spin zero depends on one single form factor, the electric form factor. In case of a target with spin 1/2 there are two form factors: the electric and the magnetic dipole form factor. For a target with spin $J = 1$ or higher there are, in addition, electric quadrupole form factors or more generally, form factors of multipolarity λ up to $\lambda_{\max} = 2J$. This is a consequence of conservation of angular momentum which requires that nuclear initial and final state spins form a triangle with λ, the multipolarity of the form factor, viz. $\boldsymbol{J}_i + \boldsymbol{\lambda} + \boldsymbol{J}_f = 0$. In addition, conservation of parity selects the kind of multipoles that can contribute to elastic scattering, e.g. electric monopole, magnetic dipole, electric quadrupole.

In studying inelastic scattering to discrete excited states of the nucleus we encounter a very similar situation. The initial state with spin and parity $J_i^{\pi_i}$ (this is generally the nuclear ground state), goes over into a final state with spin and parity $J_f^{\pi_f}$ through excitation by means of multipole fields with multi-polarity λ and π_λ such that

$$\boldsymbol{J}_i + \boldsymbol{\lambda} + \boldsymbol{J}_f = 0, \qquad \pi_i \pi_\lambda = \pi_f.$$

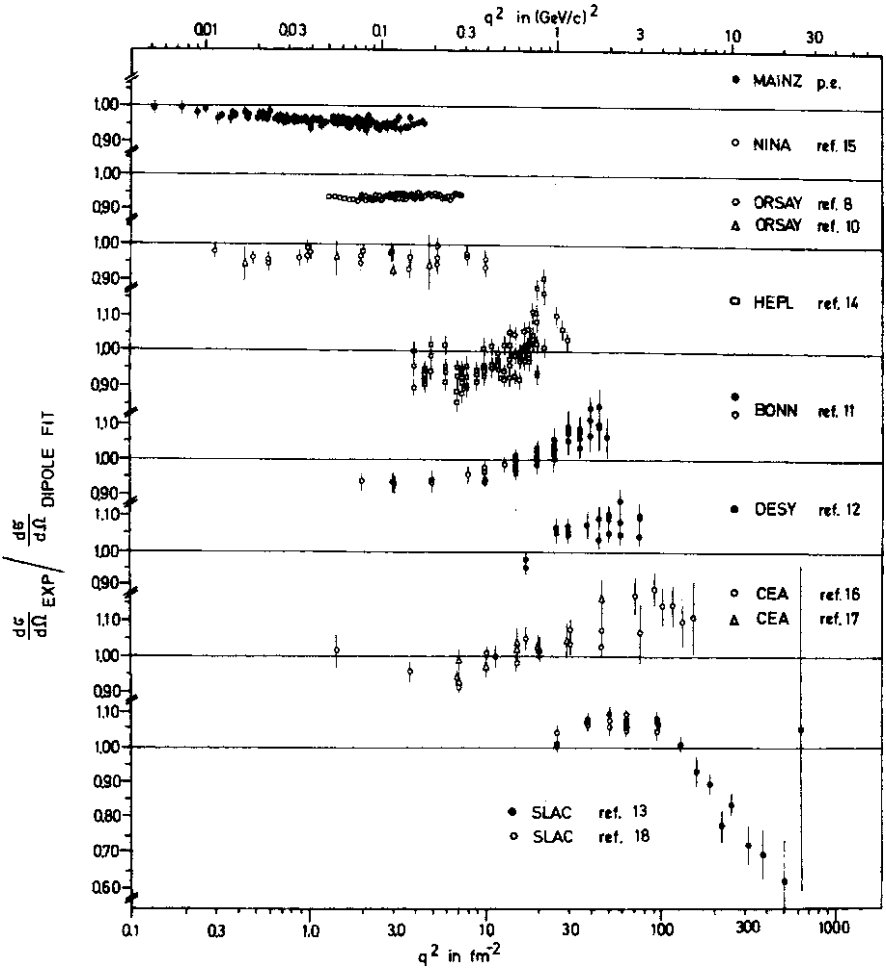

Fig. 2.4 Ratio of electron–proton scattering cross sections to the dipole formula, as measured in various laboratories, versus q^2. Compilation taken from Borkowski et al. (1975)

Because of the close similarity of these two situations we treat them within the same formalism. Everything that follows for inelastic scattering below can always be specialized to elastic scattering by taking $J_\mathrm{f} = J_\mathrm{i}$ and $\pi_\mathrm{f} = \pi_\mathrm{i}$.

In a first step we calculate the cross sections as before, using the Born approximation in describing initial and final states of the electron. This approximation is very good for the proton, $Z = 1$. Depending on the accuracy of experiments one wishes to analyze, the Born approximation may still be acceptable for light nuclei, up to about $Z \cong 10$. With increasing charge number of the nucleus it becomes less reliable; for heavy nuclei such as lead ($Z = 82$) it fails badly, as it neglects the strong distortion of initial and final electron waves due to the nuclear

Coulomb field. In this case a complete partial wave analysis on the basis of exact eigenstates in the nuclear electric field must be worked out. Nevertheless, it can be seen that the cross sections calculated in the Born approximation contain all relevant qualitative physical features of the scattering process. The exact cross sections, obtained from partial wave analysis, *differ quantitatively* from the Born cross sections but their structure carries the same (qualitative) physical information. Thus, Coulomb distortion is a problem of purely technical nature, and we leave its discussion to a later section.

2.5.1 Multipole Fields

The theory of electron scattering from nuclei leading to final states with definite spin and parity is based on an expansion of the virtual photon interaction in terms of multipole fields. In this subsection we collect a few results and formulae which are needed in the sequel [ROS55, JAC75]: We start from Helmholtz' differential equation for vector fields on $\mathbb{R}^3$,

$$(\nabla^2 + k^2)\boldsymbol{B}(\boldsymbol{r}) = 0. \tag{2.68}$$

This equation follows from the wave equation for any field $\boldsymbol{B}(\boldsymbol{r},\ t)$ with harmonic time dependence,

$$\boldsymbol{B}(\boldsymbol{r}, t) = \mathrm{e}^{\mathrm{i}\omega t}\boldsymbol{B}(r).$$

$\hbar\omega$ is the energy, $\hbar k$ the momentum, k the wave number of this field, and $\hbar\omega = \hbar ck$ or, in natural units, $\omega = k$. $\boldsymbol{B}(\boldsymbol{r})$ can be expanded in terms of a complete set of solutions of the Helmholtz equation with definite angular momentum and definite parity. These basic solutions are called *multipole fields*. They form a complete and orthogonal set of vector functions in the radial variable r and on the unit sphere.

Let us begin with the definition of *vector spherical harmonics*. Let ζ_m ($m = +1,\ 0,\ -1$) be spherical unit vectors, defined in terms of Cartesian unit vectors by

$$\begin{aligned}\zeta_0 &= e_3,\\ \zeta_1 &= -\frac{1}{\sqrt{2}}(e_1 + \mathrm{i}e_2),\\ \zeta_{-1} &= \frac{1}{\sqrt{2}}(e_1 - \mathrm{i}e_2).\end{aligned} \tag{2.69}$$

These vectors obey the symmetry relation

$$\zeta_m^* = (-)^m\zeta_{-m} \tag{2.70a}$$

and the orthogonality relation

$$\zeta_m^* \zeta_m = \delta_{mm}. \tag{2.70b}$$

As defined in (2.69) the ζ_m transform as a spherical tensor of rank one under rotations. The vector spherical harmonics are then defined as follows:

$$\boldsymbol{T}_{Jl M} := \sum_{m_l m_s} (l m_l, 1 m_s | JM) Y_{l m_l} \zeta_{m_s}, \tag{2.71}$$

where $(lm_l, 1m_s|JM)$ denotes the Clebsch–Gordan coefficients that couple the angular momenta l and 1 to $J = l + 1,\ l, l - 1$. By construction, T_{JlM} transform under rotations with the unitary rotation matrices $D^{(J)*}_{MM'}(\phi,\ \theta,\ \psi)$, i.e. they are spherical tensors of rank J. In addition they have vector character due to the fact that they contain the spherical unit vectors ζ. The m-th spherical component is given by

$$\begin{aligned}(\boldsymbol{T}_{Jl M})_m &\equiv (\boldsymbol{T}_{Jl M} \cdot \zeta_m) \\ &= (-)^m (l M + m, 1 - m | JM) Y_{l M+m}.\end{aligned}$$

The index l indicates the behaviour of the $\boldsymbol{T}_{JlM}$ under the parity operation. From their definition (2.71) one sees that under space reflection

$$\boldsymbol{T}_{Jl M}(\pi - \theta, \varphi + \pi) = (-)^l \boldsymbol{T}_{Jl M}(\theta, \varphi). \tag{2.72}$$

It is easy to verify the orthogonality property

$$\int \mathrm{d}\Omega (\boldsymbol{T}^*_{J'l'M'} \boldsymbol{T}_{Jl M}) = \delta_{JJ'} \delta_{ll'} \delta_{MM'}, \tag{2.73}$$

which follows from the orthogonality of the ordinary spherical harmonics and of the vectors ζ, as well as from some known properties of Clebsch-Gordan coefficients. Finally, we note that the vector harmonics also form a *complete* set of vector-like functions on the unit sphere. (The completeness follows from completeness of spherical harmonics). Some special cases are

$$\begin{aligned}\boldsymbol{T}_{10M} &= \frac{1}{\sqrt{4\pi}} \zeta_M, \\ \boldsymbol{T}_{10M} &= -\frac{1}{\sqrt{4\pi}} \frac{\boldsymbol{r}}{|\boldsymbol{r}|}.\end{aligned}$$

Returning to the Helmholtz equation (2.68), we now construct solutions with definite angular momentum and definite parity. These have the form

$$\boldsymbol{B}_{Jl M}(\boldsymbol{r}) = f_l(r) \boldsymbol{T}_{Jl M}(\theta, \varphi).$$

Inserting this ansatz into (2.68) we are led to a differential equation for the function $f_l(r)$ alone which reads

$$\left\{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)+k^2-\frac{l(l+1)}{r^2}\right\}f_l(r)=0,$$

or, with $z=kr$,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{2}{z}\frac{\mathrm{d}}{\mathrm{d}z}+1-\frac{l(l+1)}{z^2}\right)f_l(z)=0. \tag{2.74}$$

This equation is well-known in the theory of Bessel functions. We choose the following fundamental system of solutions,

$$\begin{aligned} f_l^{\mathrm{I}}(r) &= j_l(kr) && \text{spherical Bessel function,}\\ f_l^{\mathrm{II}}(r) &= h_l^{(1)}(kr) && \text{spherical Hankel function of first kind}\\ &= j_l(kr)+\mathrm{i}n_l(kr) \end{aligned} \tag{2.75}$$

[$n_l(kr)$ is a spherical Neumann function]. This specific set is chosen in view of the physical boundary conditions that we require for the multipole fields: f_l^{I} is the solution regular at the origin $r=0$, whilst f_l^{II} describes asymptotically outgoing spherical waves,[10]

$$h_l^{(1)}(x)\underset{x\to\infty}{\sim}\frac{1}{x}\exp\left\{\mathrm{i}(x-(l+1))\frac{\pi}{2}\right\}. \tag{2.76}$$

Finally, the functions f_l satisfy the completeness relation

$$\int_0^\infty f_l^*(k'r)\,f_l(kr)r^2\mathrm{d}r=\frac{\pi}{2k^2}\delta(k-k'). \tag{2.77}$$

Equipped with this knowledge we can now define a set of *multipole fields*

(i) *magnetic multipole fields*

$$\boldsymbol{A}_{lm}(\mathrm{M}):=f_l(kr)\boldsymbol{T}_{j=llm}; \tag{2.78}$$

(ii) *electric multipole fields*

[10]These and further properties of spherical Bessel and Hankel functions can be found in [ABS65] and in other monographs on special functions.

$$\boldsymbol{A}_{lm}(\mathrm{E}) = -\sqrt{\frac{l}{2l+1}}\, f_{l+1}(kr)\boldsymbol{T}_{ll+1m} + \sqrt{\frac{l+1}{2l+1}}\, f_{l-1}(kr)\boldsymbol{T}_{ll-1m}; \quad (2.79)$$

(iii) *longitudinal multipole fields*

$$\boldsymbol{A}_{lm}(\mathrm{L}) := \sqrt{\frac{l+1}{2l+1}}\, f_{l+1}(kr)\boldsymbol{T}_{ll+1m} + \sqrt{\frac{l}{2l+1}}\, f_{l-1}(kr)\boldsymbol{T}_{ll-1m}. \quad (2.80)$$

The properties of these fields, as well as the reason for the nomenclature, are discussed extensively in the literature on multipole fields [ROS55, JAC75]. In particular, differential properties of them can best be discussed by means of the techniques of angular momentum algebra for which we refer to standard monographs [FAR59, EDM57, ROS57]. Especially useful is the "gradient formula" which reads

$$\nabla f(r)Y_{lm} = -\sqrt{\frac{l+1}{2l+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} - l\frac{f}{r}\right)\boldsymbol{T}_{ll+1m} + \sqrt{\frac{l}{2l+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{l+1}{r}f\right)\boldsymbol{T}_{ll-1m}. \quad (2.81)$$

By means of these techniques one shows that *magnetic* and *electric* multipole fields are divergence free

$$\nabla \cdot \boldsymbol{A}_{lm}(\tau) = 0, \ \tau = \mathrm{E}, M. \quad (2.82)$$

Both potentials vanish for $l = 0$. This is evident in the case of (2.79); in the case of (2.78) it follows from (2.71) since $(00,\ 1\,\mathrm{m}|00)$ vanishes. Thus, electric and magnetic multipole fields are transverse, i.e. fulfill (2.82), and may be used to describe photon states of definite angular momentum and parity. According to (2.72) their parities are $(-)^l$ and $(-)^{l+1}$, respectively. However, as the interaction with matter always involves the product of such vector fields with the current operator which is itself odd under parity, the rules for parity change in electromagnetic transitions are

$(-)^{l+1}$ in magnetic (Ml) transitions,
$(-)^l$ in electric (El) transitions.

The longitudinal fields (2.80) are not divergenceless (hence their name), and they exist also for $l = 0$. Thus, while there are no transverse monopole fields, a longitudinal field can carry total angular momentum zero.

On the basis of the orthogonality and completeness relations (2.73) and (2.77) one shows easily that the multipole fields fulfill the orthogonality relations

$$\int_0^\infty r^2\mathrm{d}r \int \mathrm{d}\Omega \boldsymbol{A}^*_{l'm'}(\boldsymbol{r},\tau')\boldsymbol{A}_{lm}(\boldsymbol{r},\tau) = \frac{\pi}{2k^2}\delta(k-k')\delta_{\tau\tau'}\delta_{ll'}\delta_{mm'}. \quad (2.83)$$

The importance of these results lies in the fact that a given vector field $\boldsymbol{F}(\boldsymbol{r})$ which is sufficiently regular, can be expanded in terms of orthogonal multipole fields whose

parity and angular momentum properties are simple. Whenever matrix elements of $\boldsymbol{F}$ between states of definite angular momentum and parity are to be calculated, only one or a few terms of the multipole expansion give nonvanishing contributions.

Finally, using the techniques of angular momentum algebra, one can show that the multipole fields can also be written in terms of the orbital angular momentum operator applied to spherical harmonics, in terms of curl and gradient of such functions. This provides equivalent representations that are sometimes useful.

$$\boldsymbol{A}_{lm}(\mathrm{M}) = f_l(kr)\frac{1}{\sqrt{l(l+1)}}\boldsymbol{l}Y_{lm}, \tag{2.78′}$$

$$\begin{aligned}\boldsymbol{A}_{lm}(\mathrm{E}) &= -\frac{\mathrm{i}}{k}\nabla\times(f_l(kr)\boldsymbol{T}_{llm}) \\ &= -\frac{\mathrm{i}}{k\sqrt{l(l+1)}}\nabla\times\boldsymbol{l}(f_l(kr)Y_{lm}),\end{aligned} \tag{2.79′}$$

$$\boldsymbol{A}_{lm}(\mathrm{L}) = \frac{1}{k}\nabla(f_l(kr)Y_{lm}). \tag{2.80′}$$

The notation (2.78–2.80) is more useful in calculating matrix elements between states of good angular momentum and parity because one can then make use of the Wigner–Eckart theorem and all the tricks of angular momentum algebra. The representation (2.78′–2.80′) on the other hand, is very useful if one wants to use identities of vector calculus in order to transform interaction terms to a more convenient form.

2.5.2 Theory of Electron Scattering

There are several ways of deriving the Hamiltonian that describes the interaction of (arbitrarily relativistic) electrons with a static target.

(i) One may analyze the scattering process in a semi-classical treatment, starting from a retarded interaction between two given charge and current densities.

$$H_{\mathrm{ret}} = \int \mathrm{d}^3 r_{\mathrm{n}} \int \mathrm{d}^3 r_{\mathrm{e}} \frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}_{\mathrm{n}}-\boldsymbol{r}_{\mathrm{e}}|}}{|\boldsymbol{r}_{\mathrm{n}}-\boldsymbol{r}_{\mathrm{e}}|}\left\{\rho_{\mathrm{n}}(\boldsymbol{r}_{\mathrm{n}})\rho_{\mathrm{e}}(\boldsymbol{r}_{\mathrm{e}}) - \boldsymbol{j}_{\mathrm{n}}(\boldsymbol{r}_{\mathrm{n}})\boldsymbol{j}_{\mathrm{e}}(\boldsymbol{r}_{\mathrm{e}})\right\}. \tag{2.84}$$

One then expands the retarded Green function that appears in (2.84) in terms of transverse and longitudinal multipole fields. The procedure is conceptually simple but technically somewhat involved and we refer to the literature for details [ROS61] (Scheck 1966).

(ii) Alternatively, one can calculate the interaction in the framework of quantum electrodynamics, formulated in the Coulomb gauge. The interaction is then given by the *instantaneous* electrostatic Coulomb interaction plus the terms

arising from the exchange of virtual but still transverse photons between the electron and the target [HEI63, SAK84]. In this case, instead of using plane waves it is appropriate to expand and quantize the transverse photon field in terms of the magnetic and electric multipole fields (2.78, 2.79).

In either case the scattering matrix element is found to be

$$\begin{aligned}
&\langle \mathrm{f}; k' | H_{\mathrm{int}} | \mathrm{i}; k \rangle \\
&= \left\langle \mathrm{f}, k' \left| \sum_{l=1}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi \mathrm{i}}{l(l+1)} \int \mathrm{d}^3 r_{\mathrm{n}} \int \mathrm{d}^3 r_{\mathrm{e}} \right.\right. \\
&\quad \times \left\{ \frac{1}{k} \left[\boldsymbol{j}_n \cdot \nabla \times \boldsymbol{l} \begin{pmatrix} j_l(kr_<) \\ h_l^{(1)}(kr_>) \end{pmatrix} Y_{lm}^*(\hat{\boldsymbol{r}}_{\mathrm{n}}) \right) \right] \left[\boldsymbol{j}_{\mathrm{e}} \cdot \nabla \times \boldsymbol{l} \begin{pmatrix} h_l^{(1)}(kr_>) \\ j_l(kr_<) \end{pmatrix} Y_{lm}(\hat{\boldsymbol{r}}_{\mathrm{e}}) \right) \right] \\
&\quad + k \left[\boldsymbol{j}_n \cdot \boldsymbol{l} \begin{pmatrix} j_l(kr_<) \\ h_l^{(1)}(kr_>) \end{pmatrix} Y_{lm}^*(\hat{\boldsymbol{r}}_n) \right) \left[\boldsymbol{j}_e \cdot \boldsymbol{l} \begin{pmatrix} h_l^{(1)}(kr_>) \\ j_l(kr_<) \end{pmatrix} Y_{lm}(\hat{\boldsymbol{r}}_{\mathrm{e}}) \right) \right] \\
&\quad \left. + \int \mathrm{d}^3 r_n \int \mathrm{d}^3 r_e \frac{1}{r_>} \rho_n(\boldsymbol{r}_n) \rho_{\mathrm{e}}(\boldsymbol{r}_{\mathrm{e}}) | \mathrm{i}, k \right\rangle . \qquad (2.85)
\end{aligned}$$

In this expression i and f denote initial and final state of the target, respectively; for example, i stands for initial total momentum p of the target and for all other target quantum numbers such as angular momentum, parity and any other internal quantum numbers as there may be. The notation $r_<$ and $r_>$ serves as a shorthand for the requirement that

$$\begin{aligned}
&\text{if } r_{\mathrm{e}} > r_{\mathrm{n}}, \quad \text{the combination } j_l(kr_{\mathrm{n}}) h_l^{(1)}(kr_{\mathrm{e}}), \quad \text{and} \\
&\text{if } r_{\mathrm{e}} > r_{\mathrm{n}}, \quad \text{the combination } h_l^{(1)}(kr_{\mathrm{n}}) j_l(kr_{\mathrm{e}})
\end{aligned}$$

must be taken in the first two terms on the r.h.s. of (2.85). This reflects the correct boundary conditions which are built into (2.85): It is always the smaller of the two radial variables that is to be inserted into j_l, the function regular at the origin, whilst the larger of the two is to be taken in $h_l^{(1)}$, the function that describes outgoing spherical waves.

Evidently, if we wish to explore the internal structure of the target, we have to choose the momentum transfer and, therefore, the electron energy high enough so that the electron penetrates sizeably into the target. In this case the integrations in (2.85) are entangled in a nontrivial way. The matrix element does not factor into a target structure function and a leptonic factor. This is unlike Coulomb excitation where penetration is unimportant (and, in fact, often unwanted) and where the lowest order cross section does factor into target and projectile properties.

The expression (2.85) is fairly general. It applies equally well to elastic and inelastic scattering. It holds independently of what basis we choose for the initial and final electron states. If we take plane waves we shall obtain the transition

matrix element in the Born approximation. In this case k is the magnitude of the three-momentum in the centre-of-mass system. If we wish to include the Coulomb distortion in the electronic states, we have to evaluate (2.85) with eigenstates of the electron in the static Coulomb field created by the target. In this case k is the asymptotic wave number determined by the electron energy.

The charge and current densities which appear in (2.85) are still very general. For the electron, we clearly have to set

$$\rho_{\mathrm{e}}(\boldsymbol{r}) = -e\gamma^0\delta(\boldsymbol{r}-\boldsymbol{r}_{\mathrm{e}}) \equiv -e\beta\delta(\boldsymbol{r}-\boldsymbol{r}_{\mathrm{e}}), \tag{2.86a}$$

$$\boldsymbol{j}_{\mathrm{e}}(\boldsymbol{r}) = -e\gamma^0\boldsymbol{\gamma}\delta(\boldsymbol{r}-\boldsymbol{r}_{\mathrm{e}}) \equiv -e\boldsymbol{\alpha}\delta(\boldsymbol{r}-\boldsymbol{r}_{\mathrm{e}}) \tag{2.86b}$$

Concerning the target, however, there are many options. For example, let the target be a nucleus treated in a nonrelativistic scheme. If we know the nuclear wave function in terms of states of individual nucleons then

$$\rho_{\mathrm{n}}(\boldsymbol{r}) = \sum_{i-1}^{A} e_i\delta(\boldsymbol{r}-\boldsymbol{r}_i), \tag{2.87a}$$

$$\boldsymbol{j}_{\mathrm{n}}(\boldsymbol{r}) = \sum_{i=1}^{A}\left[e_i\boldsymbol{v}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i) + \frac{e}{2m}g_s^i(\nabla\times\boldsymbol{s}^i)\delta(\boldsymbol{r}-\boldsymbol{r}_i)\right]. \tag{2.87b}$$

These operators are taken between nonrelativistic nuclear states (e.g. shell model states). If the nucleus is to be described by some set of effective collective coordinates (such as vibrator coordinates in case of collective matter oscillations, or the set of Euler angles in case of rigid rotator motion, etc.) then the densities are semi-classical functions containing the collective coordinates in a way determined by the underlying model.

In the following two sections we present the essential steps of the derivation of cross sections in the Born approximation (Sect. 2.5.3) and including Coulomb distortion (Sect. 2.5.4).

2.5.3 Born Approximation

Before proceeding to the calculation of the matrix element (2.85) in the first Born approximation it is useful to transform the interaction to a more convenient form. The electric part of the interaction [first term on the right-hand side of (2.85)] is transformed by means of a well-known relation for spherical Bessel and Hankel functions [WAT58],

$$j_l(kr_<)h_l^{(1)}(kr_>) = \frac{2k}{\mathrm{i}\pi}\int_0^\infty \mathrm{d}q\,\frac{j_l(qr)\,j_l(qr')}{q^2-k^2} + \frac{1}{\mathrm{i}k}\frac{r_<^l r_>^{-l-1}}{2l+1}. \tag{2.88}$$

Likewise, the magnetic term (second term of eq. (2.85)) is transformed by means of the relation [WAT58]

$$j_l(kr_<)h_l^{(1)}(kr_>) = \frac{2}{\mathrm{i}\pi k}\int_0^\infty \frac{j_l(qr)j_l(qr')}{q^2-k^2}q^2\mathrm{d}q. \tag{2.89}$$

Finally, the electric term that comes from the second term on the right-hand side of (2.88), i.e.

$$\left\langle \mathrm{f};k'\right| \int \mathrm{d}^3r_\mathrm{n}\int \mathrm{d}^3r_\mathrm{e}\ \boldsymbol{j}_\mathrm{n}\cdot\nabla\times\boldsymbol{l}\left(\begin{matrix} r_<^l \\ r_>^{-l-1}\end{matrix} Y_{lm}^*\right)\boldsymbol{j}_\mathrm{e}\cdot\nabla\times\boldsymbol{l}\left(\begin{matrix} r_>^{-l-1} \\ r_<^l\end{matrix} Y_{lm}\right)\left|\mathrm{i};\ k\right\rangle$$

can be further transformed by means of the relation [ROS57]

$$\nabla\times\boldsymbol{l}(r^\alpha Y_{lm}) = \mathrm{i}(l+1)\nabla(r^\alpha Y_{lm}), \qquad \alpha = l, -l-1. \tag{2.90}$$

Partial integration allows to shift the nabla operators onto the current densities so that, eventually, the continuity equations[11]

$$\begin{aligned}\langle \mathrm{f}|\nabla\cdot\boldsymbol{j}_\mathrm{n}|\mathrm{i}\rangle &= -\mathrm{i}k\langle \mathrm{f}|\rho_\mathrm{n}|\mathrm{i}\rangle,\\ \langle k'|\nabla\cdot\boldsymbol{j}_\mathrm{e}|k\rangle &= +\mathrm{i}k\langle k'|\rho_\mathrm{e}|k\rangle\end{aligned}$$

may by used. In this manner we obtain the equivalent expression for the scattering matrix element (2.85):

$$\begin{aligned}&\langle \mathrm{f};k'|\boldsymbol{H}_\mathrm{int}|\mathrm{i};k\rangle\\ &= \Bigg\langle \mathrm{f};k'\Big|\sum_{l=1}^\infty\sum_{m=-l}^{+l}\frac{8}{l(l+1)}\int\mathrm{d}^3r_\mathrm{n}\int\mathrm{d}^3r_\mathrm{e}\int_0^\infty\frac{\mathrm{d}q'}{q'^2-k^2}\\ &\quad\times\{\boldsymbol{j}_\mathrm{n}\cdot\nabla\times\boldsymbol{l}(j_l(q'r_\mathrm{n})Y_{lm}^*(\boldsymbol{r}))\ \boldsymbol{j}_\mathrm{e}\cdot\nabla\times\boldsymbol{l}(j_l(q'r_\mathrm{e})Y_{lm}(\hat{\boldsymbol{r}}_\mathrm{e}))\\ &\quad+q'^2\boldsymbol{j}_\mathrm{n}\cdot\boldsymbol{l}(j_l(q'r_\mathrm{n})Y_{lm}^*(\hat{\boldsymbol{r}}_\mathrm{n}))\ \boldsymbol{j}_\mathrm{e}\cdot\boldsymbol{l}(j_l(q'r_\mathrm{e})Y_{lm}(\hat{\boldsymbol{r}}_\mathrm{e}))\}\\ &\quad+\sum_{l=0}^\infty\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\int\mathrm{d}^3r_\mathrm{n}\int\mathrm{d}^3r_\mathrm{e}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\hat{\boldsymbol{r}}_\mathrm{n})Y_{lm}(\hat{\boldsymbol{r}}_\mathrm{e})\rho_\mathrm{n}(\boldsymbol{r}_\mathrm{n})\rho_\mathrm{e}(\boldsymbol{r}_\mathrm{e})\Big|\mathrm{i};k\Bigg\rangle.\end{aligned} \tag{2.91}$$

In the last term of (2.91) the sum over l runs from zero to infinity; the terms from one to infinity stem from the transformed electric term (see above), whilst the term with $l = 0$ is the monopole term of (2.85). (Note that $Y_{00} = 1/\sqrt{4\pi}$). This last term, summed over *all* l, is nothing but the instantaneous Coulomb interaction

[11] The factor $k = \omega$, the energy variable, should not be confused with the initial electron's four-momentum, see also (2.68).

$$\int \mathrm{d}^3 r_\mathrm{n} \int \mathrm{d}^3 r_\mathrm{e} \frac{1}{|\boldsymbol{r}_\mathrm{n} - \boldsymbol{r}_\mathrm{e}|} \rho_\mathrm{n}(\boldsymbol{r}_\mathrm{n}) \rho_\mathrm{e}(\boldsymbol{r}_\mathrm{e}) \tag{2.92}$$

expanded in a multipole series. Therefore, the first two terms on the right-hand side of (2.91) must represent the contributions stemming from the exchange of transverse virtual photons with momentum q'.

To calculate (2.91) in the Born approximation means taking

$$|k\rangle = u(k)\mathrm{e}^{\mathrm{i}k\cdot r_\mathrm{e}} \text{ and } \quad |k'\rangle = u(k')\mathrm{e}^{\mathrm{i}k'\cdot r_\mathrm{e}}.$$

Inserting this into (2.91) one can then perform the integration over the electron coordinates $\boldsymbol{r}_\mathrm{e}$ and over the (virtual photon) momentum q. These calculations are somewhat lengthy and tedious. We therefore give only one example that shows the technique but do not work out all the details. Take for example the instantaneous interaction term (2.92) and use the relation (cf. exercise 2.4)

$$\frac{1}{|\boldsymbol{r}_\mathrm{n} - \boldsymbol{r}_\mathrm{e}|} = \frac{4\pi}{(2\pi)^3} \int \mathrm{d}^3 q' \frac{\mathrm{e}^{\mathrm{i}q'\cdot(\boldsymbol{r}_\mathrm{n} - \boldsymbol{r}_\mathrm{e})}}{\boldsymbol{q}'^2}. \tag{2.93}$$

Integration over $\boldsymbol{r}_\mathrm{e}$ gives, together with (2.86a),

$$\begin{aligned}
&\langle \mathrm{f}; k' | \int \mathrm{d}^3 r_\mathrm{n} \int \mathrm{d}^3 r_\mathrm{e} \frac{1}{|\boldsymbol{r}_\mathrm{n} - \boldsymbol{r}_\mathrm{e}|} \rho_\mathrm{n}(\boldsymbol{r}_\mathrm{n}) \rho_\mathrm{e}(\boldsymbol{r}_\mathrm{e}) | \mathrm{i}; k \rangle \\
&\quad = -4\pi e u^\dagger(k') u(k) \sum_{l,m} (-)^m \mathrm{i}^l \frac{4\pi}{q^2} Y_{lm}(\hat{\boldsymbol{q}}) \times \int \mathrm{d}^3 r_\mathrm{n} \langle \mathrm{f} | \rho_\mathrm{n}(\boldsymbol{r}_\mathrm{n}) | \mathrm{i} \rangle j_l(q r_\mathrm{n}) Y_{l-m}(\hat{\boldsymbol{r}}_\mathrm{n}),
\end{aligned}$$

where $\boldsymbol{q}$ is the three-momentum transfer, $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$ and $q = |\boldsymbol{q}|$. This last equation suggests that one define a multipole form factor of the target

$$M(\mathrm{C}l, m; q) := \frac{(2l+1)!!}{q^l} \int \mathrm{d}^3 r \rho_\mathrm{n}(\boldsymbol{r}) j_l(qr) Y_{lm}(\hat{\boldsymbol{r}}). \tag{2.94}$$

The factor in front of the integral has been chosen such that in the limit $q \to 0$, $M(Cl, m; q)$ goes over into the static l-pole moment of the target charge density.

The transverse electric and magnetic terms in (2.91) are worked out in a similar way. In analogy to (2.94) we are led to define

electric multipole form factors

$$M(\mathrm{E}l, m; q) := \frac{(2l+1)!!}{q^{l+1}(l+1)} \int \mathrm{d}^3 r \boldsymbol{j}_\mathrm{n}(\boldsymbol{r}) \cdot \nabla \times \boldsymbol{l}(j_l(qr) Y_{lm}(\hat{\boldsymbol{r}})), \tag{2.95}$$

and *magnetic multipole form factors*

$$M(\mathrm{M}l, m; q) := -\mathrm{i} \frac{(2l+1)!!}{q^l (l+1)} \int \mathrm{d}^3 r \boldsymbol{j}_\mathrm{n}(\boldsymbol{r}) \cdot \boldsymbol{l}(j_l(qr) Y_{lm}(\hat{\boldsymbol{r}})). \tag{2.96}$$

Here again, the factors have been chosen so that for $qr \ll 1$ the form factors go over into the corresponding transverse electric and magnetic multipole terms which describe the corresponding transition induced by photons [DST63] (in the approximation of long wavelengths).

We are concerned here only with the scattering cross section for unpolarized electrons and we do not discriminate the spin orientation of the electron in the final state. Therefore, we have to calculate the incoherent sum of squared matrix elements over all initial and final spin projections and have to divide by $2(2J+1)$, J being the nuclear spin, in order to account for the average over spin orientations in the initial state. The spin summation in the electronic part is done by means of the trace techniques (see Appendix). This is a straightforward but somewhat lengthy calculation that we do not wish to carry out here as the details can be found in the literature. The result is, in the unpolarized case [UEB71] (De Forest et al. 1966),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sum_{l=0}^{\infty} \frac{\mathrm{d}\sigma_{\mathrm{E}l}}{\mathrm{d}\Omega} + \sum_{l=1}^{\infty} \frac{\mathrm{d}\sigma_{\mathrm{M}l}}{d\,\Omega}, \tag{2.97a}$$

where

$$\frac{\mathrm{d}\sigma_{\mathrm{E}l}}{\mathrm{d}\Omega} = \alpha^2 \frac{4\pi(l+1)}{l[(2l+1)!!]^2} \frac{q^{2l}}{E^2} \left\{ \frac{l}{l+1} B(\mathrm{C}l;q) V_{\mathrm{L}}(\theta) + B(\mathrm{E}l;q) V_{\mathrm{T}}(\theta) \right\} \tag{2.97b}$$

$$\frac{\mathrm{d}\sigma_{\mathrm{M}l}}{\mathrm{d}\Omega} = \alpha^2 \frac{4\pi(l+1)}{l[(2l+1)!!]^2} \frac{q^{2l}}{E^2} B(\mathrm{M}l;q) V_{\mathrm{T}}(\theta). \tag{2.97c}$$

Here the functions $V_{\mathrm{L}}(\theta)$ and $V_{\mathrm{T}}(\theta)$ stem from the summation over electron spins. In the high-energy limit ($E \gg m_{\mathrm{e}}$) they are

$$V_{\mathrm{L}}(\theta) \simeq \frac{\cos^2(\theta/2)}{4\sin^4(\theta/2)}, \quad V_{\mathrm{T}}(\theta) \simeq \frac{1+\sin^2(\theta/2)}{8\sin^4(\theta/2)}. \tag{2.98}$$

The functions $B(\tau l;q)$ are the spin-averaged, squared nuclear matrix elements of the operators (2.94)–(2.96), viz.

$$B(\tau l;q) = \sum_{M_{\mathrm{f}},m} |\langle J_{\mathrm{f}} M_{\mathrm{f}}|M(\tau l,m;q)|J_{\mathrm{i}} M_{\mathrm{i}}\rangle|^2, \tag{2.99}$$

with $\tau \equiv \mathrm{C}$ (longitudinal Coulomb multipoles), $\tau \equiv \mathrm{E}$ (electric multipoles), or $\tau \equiv \mathrm{M}$ (magnetic multipoles). As before, q is the magnitude of the three-momentum transfer, while E is the initial electron energy. Strictly speaking, the expressions (2.97) hold in the centre-of-mass frame. However, they may equally well be applied in the laboratory system provided the typical recoil terms of order of E/M, with M the target mass, can be neglected.

We note that there are no interference terms in the cross sections (2.97). The magnetic multipoles do not interfere with the electric and longitudinal multipoles

because they have different parity selection rules. The interference terms between electric and longitudinal multipoles disappear when the spin average is taken. Equations (2.97b,c), which hold for *elastic* as well as for *inelastic* scattering, demonstrate quite clearly the important new feature of electron scattering on an extended target: The scattering cross section depends on the momentum transfer q which can be chosen arbitrarily large. Thus, looking back at (2.94–2.96), the electron probes the *spatial structure* of charge and current densities within the target. In the case of elastic scattering, these are the charge, the electric and magnetic current densities of the ground state. In the case of inelastic scattering, we are probing off-diagonal matrix elements of these operators between the (initial) ground state and some (final) excited states of the target. Because of the close analogy to the elastic case one often calls these matrix elements *transition charge* and *transition current densities*.

The *selection rules* that apply to elastic and inelastic electron scattering derive from angular momentum conservation and from the behaviour of the transition operators and of the nuclear states under parity and time reversal.

Let

$$(J_\mathrm{i},\ \pi_\mathrm{i}) \quad \text{and} \quad (J_\mathrm{f},\ \pi_\mathrm{f})$$

be the spins and parities of initial and final nuclear states, respectively, and let us consider a given multipole operator $M(\tau l)$, (2.94–2.96), taken between these states. Angular momentum conservation implies that $(J_\mathrm{i},\ J_\mathrm{f},\ l)$ form a triangle $|J_\mathrm{i} - J_\mathrm{f}| \leq l \leq J_\mathrm{i} + J_\mathrm{f}$ with $l \geq 1$ for transverse electric and magnetic multipoles. Parity conservation implies that $\pi_\mathrm{i} \cdot \pi_\mathrm{f} = (-)^l$ for longitudinal and transverse electric multipoles, $\pi_\mathrm{i} \cdot \pi_\mathrm{f} = (-)^{l+1}$ for transverse magnetic multipoles. Finally, hermiticity of the electromagnetic current and invariance under time reversal give the additional relation for the nuclear reduced matrix element

$$(J_\mathrm{i}||M(\tau l)||J_\mathrm{f}) = (-)^{J_\mathrm{i}-J_\mathrm{f}+l+\eta}(J_\mathrm{f}||M(\tau l)||J_\mathrm{i}), \tag{2.100}$$

with $\eta = 0$ for $\tau = \mathrm{C}$; $\eta = 1$ for $\tau = \mathrm{E}$ or M (Donnelly et al. 1975). If the nuclear states are also eigenstates of isospin with eigenvalues $I_\mathrm{i},\ I_\mathrm{f}$, respectively, there is an additional phase factor $(-)^{I_\mathrm{i}-I_\mathrm{f}}$ in (2.100). In this case the reduced matrix element implies reduction with respect to both angular momentum and isospin (Donnelly et al. 1975).

For elastic scattering, in particular, J_i and J_f are identical. From (2.100) we then must have $(-)^{l+\eta} = +1$, $l + \eta$ must be even. Thus, only *even longitudinal* and only *odd transverse magnetic multipoles* contribute to elastic scattering.

As an illustration let us consider some examples:

(i) Elastic scattering on ^{3}He $\left(J^\pi = \frac{1}{2}^+\right)$ and ^{209}Bi $\left(\mathrm{J}^\pi = \frac{9}{2}^-\right)$. Only the following multipoles give nonvanishing contributions

^{3}He : C0,M1,
^{209}Bi : C0,C2,C4,C6,C8;M1,M3,M5,M7,M9.

(ii) Inelastic scattering from the ground state to the electric dipole giant resonance in $^{16}\mathrm{O}\left(J_{\mathrm{i}}^{\pi_{\mathrm{i}}} = 0^{+} \rightarrow J_{\mathrm{f}}^{\pi_{\mathrm{f}}} = 1^{-}\right)$. In this case only the multipoles E1 and C1 can contribute.

These selection rules for El and Ml transitions are the same as for the corresponding photonic electric and magnetic multipole transitions. (There are no Cl transitions in the case of real photons.) There is, however, one essential difference betweeen photo- and electroexcitation: In the case of photonic processes, the momentum transfer q is replaced by

$$k = E_{\mathrm{f}} - E_{\mathrm{i}} \stackrel{\Delta}{=} (E_{\mathrm{f}} - E_{\mathrm{i}})/\hbar c,$$

the photon momentum (or energy). In most practical cases $k \cdot r$ is smaller than 1 for r of the order of, or smaller than, the typical size of the nucleus. Thus, as a consequence of the behaviour of the spherical Bessel function for small argument [see (2.12)], a transition of high multipolarity l_{H} is suppressed relative to a transition of low multipolarity l_{L} by a typical factor

$$\frac{(kr)^{l_{\mathrm{H}}}}{(2l_{\mathrm{H}}+1)!!} \Big/ \frac{(kr)^{l_{\mathrm{L}}}}{(2l_{\mathrm{L}}+1)!!}. \tag{2.101}$$

For example, an E3 γ-transition amplitude is suppressed relative to an E1 transition by a factor of the order of $\frac{1}{35}(kr)^2$. Thus, the lowest multipolarity which is compatible with the selection rules will also be the dominant one.

No such ordering of successive multipoles occurs in electron scattering. Indeed, the modulus of the momentum transfer q can become arbitrarily large and the quantity $q \cdot r$ can assume any value, greater or smaller than one. So, in general, high multipoles can be equally important as low multipoles. In the limit of small momentum transfer only, $q \rightarrow 0$, we recover the ordering of photonic multipole transitions. Actually, apart from the electonic kinematic factors, El and Ml transition probabilities in electron scattering must go over, in the limit $q \rightarrow 0$, into the corresponding γ-transition probabilities. This is called the "photon point."

Examples for the use of the Born approximation in the description of electron scattering as well as an analysis of the information carried by elastic and inelastic form factors are postponed until Sects. 2.5.5,6, where we have completed the discussion of Coulomb distortion, so that we can compare the two methods of analysis, at the same time. For the moment, it may suffice to stress that, whilst a γ-transition gives us just one *moment* of the transition charge density or current density, the matrix elements of the multipole operators (2.94–2.96) which are relevant for the cross sections for electron scattering (2.97) yield continuous information on these quantities. As the momentum transfer is varied, these matrix elements probe the spatial structure of nuclear charge and current density. If it were possible to measure the cross sections up to very large momentum transfers we would eventually obtain a complete mapping of $\rho_{\mathrm{n}}(\boldsymbol{r})$ and $\boldsymbol{j}_{\mathrm{n}}(\boldsymbol{r})$.

In practice, this is not possible, however. The scattering cross section to any specific excited state falls off faster than q^{-4} with increasing momentum transfer and, at some point, becomes unmeasurably small. Furthermore, as the energy transfer and the momentum transfer increase (into what is called the "deep inelastic" region), the number of possible final states increases so much that one may not be able to follow up one particular excitation. One then measures instead fully *inclusive scattering* (i.e. summing over all final states), or *semi-inclusive scattering* (where some property of the final state is recorded). The cross section for inclusive scattering quickly starts to dominate over all exclusive reaction channels.

In summary, only limited information on the spatial structure of the target densities is obtained in practice. At low momentum transfer, one starts probing the nuclear periphery, i.e. the charge density in the neighbourhood of the nuclear radius (distance at which the density has dropped to about half its central value). As the momentum transfer increases, more and more information on the nuclear interior appears in the elastic scattering cross sections. The densities at the origin always remain the least well known. Examples of practical analysis are given below.

2.5.4 The Problem of Coulomb Distortion

The Born approximation in the calculation of electron scattering has the great advantage of being simple and transparent. The form factors are the Fourier transforms of the nuclear matrix elements of charge and current densities and thus provide us with a direct mapping of these important quantities. On the other hand, the Born approximation also has serious deficiencies: It does not take into account the distortion of the electron waves in the static Coulomb field of the nucleus. Depending on the accuracy of the available data, the neglect of Coulomb distortion may be tolerable for light nuclei, $Z = 1$ to ~ 10. However, for larger values of the nuclear charge these effects quickly become large and must be taken into account. What are the most prominent effects of Coulomb distortion?

(i) It is not difficult to see that a form factor can have zeroes at physical values of the momentum transfer q. These diffraction zeroes which reflect specific properties of the charge and current densities in coordinate space will also appear in the cross section[12]. The cross section will then exhibit a typical diffraction pattern. The diffraction zeroes, strictly speaking, are not realistic and do not appear in the exact expression for the cross section. This is easy to understand in a qualitative manner. Suppose we describe the scattering amplitude in terms of partial waves. Partial waves carry definite angular

[12]This is true if the cross section depends only on one form factor and, in the case of inelastic scattering, if retardation effects are neglected. If it contains several form factors which have their zeroes at different q^2 then cross sections do not go to zero. Obviously, this does not invalidate our discussion.

momenta. Classically, the angular momentum, with respect to the center of the nucleus, is proportional to the impact parameter b times the momentum transfer q. Thus low partial waves penetrate into the nucleus, high partial waves pass by far outside, whilst some intermediate partial waves graze the nuclear edge. In the Born approximation the diffraction zeroes come about as a result of destructive interference of partial wave amplitudes. On the other hand, if the effect of the static Coulomb field is taken into account, low partial waves are more distorted than intermediate partial waves, while very high partial waves will be affected only very little. As a consequence, the interference of the partial waves of the Born approximation is perturbed. The diffraction zeroes disappear and are replaced with *diffraction minima* of the scattering amplitude of nonvanishing value. Furthermore, the position of these minima will be displaced from the positions of the Born zeroes.

(ii) A second important effect of Coulomb distortion may also be understood qualitatively. The static field is attractive. Therefore, the exact partial waves are attracted towards the nuclear interior. In terms of the Born approximation, this means, effectively, that at least the low and intermediate partial waves are scattered at a higher effective energy k. As the previous zero occurs at a fixed value of the product

$$q \cdot R \simeq 2kR \sin(\theta/2) = \text{const.}$$

(R being the nuclear size parameter), the diffraction minimum is expected at a somewhat lower value of the scattering angle. As a simple example, consider elastic scattering from a spinless nucleus whose charge density is taken to be

$$\rho(r) = \frac{3}{4\pi R_0^3}\Theta(R_0 - r) \quad \text{(homogeneous density)}. \tag{2.102}$$

It is not difficult to calculate the charge form factor from (2.6,2.30), for this density (see exercise 2.5). One finds

$$F(q^2) = \frac{3}{z} j_1(z) = \frac{3}{z^3}(\sin z - z\cos z), \tag{2.103}$$

where $z = q \cdot R_0$. The cross section in the Born approximation is given by (2.7), and depends only on the variable

$$qR_0 \simeq 2kR_0 \sin(\theta/2).$$

Thus if we choose the energy such that $k_1 R_0^1 = k_2 R_0^2$ for two different nuclei with charges Z_1, Z_2 and radii R_0^1, R_0^2, the quantity $(k/Z)^2\, \mathrm{d}\sigma/\mathrm{d}\Omega$ will be the same

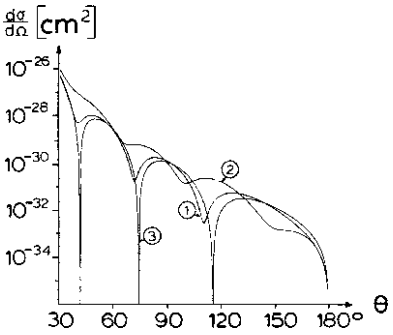

Fig. 2.5 Differential cross section for elastic electron scattering on calcium ($Z = 20$, $A = 40$) and lead ($Z = 82$, $A = 208$). Curve 1: exact result for Ca; curve 2: exact result for Pb; curve 3: cross section in the first Born approximation for Ca and Pb. The two cross sections for Ca are multiplied by the factor $(82\ R_0(\mathrm{Pb})/20 R_0(\mathrm{Ca}))^2$, so that the cross sections in the Born approximation of Ca and Pb coincide. In all three cases the nuclear charge density is assumed to be homogeneous, cf. e.g. (2.102). The radius for lead is $R_0(\mathrm{Pb}) = 4.26\,\mathrm{fm}$, the energy is $E = 300\,\mathrm{MeV}$, so that $kR_0 = 6.48$

for the two cases. This is illustrated by Fig. 2.5 which shows the scaled Born cross sections for $Z = 20$ (calcium) and $Z = 82$ (lead), with $k\ R_0 = 6.48$ (curve marked 3). The spherical Bessel function of order 1 has zeroes at $z_1 = 4.493$, $z_2 = 7.725$, $z_3 = 10.904$, etc.,[13] that is, the form factor and the Born cross section vanish at $\theta_1 = 40.6°$, $\theta_2 = 73.2°$, $\theta_3 = 114.7°$, ... For comparison, the figure also-shows the cross sections calculated by means of a full partial wave analysis. (The cross section for calcium is multiplied by the same scale factor as in the Born approximation.) We note that, even with the cross section on calcium being rescaled, the exact cross sections do not coincide.

The zeroes are replaced with minima whose position is shifted towards lower scattering angles. The shift is larger for $Z = 82$ than for $Z = 20$.

One may ask whether these strong distortion effects render useless the method of Born approximation in electron scattering. Fortunately, this is not so. As will be seen below, the physical information contained in specific features of the cross section is the same, independently of whether the Born approximation or the more exact partial wave analysis is used. Coulomb distortion is a technical complication which does not obscure the connection between properties of the nuclear densities and the cross sections. The technical and conceptual simplicity of the Born approximation can be made use of in many systematic investigations. Only when comparison with the data is made must the distortion be taken into account.

[13] See Table 10.6 of [ABS65].

2.5.5 *Partial Wave Analysis for Elastic Scattering*

On our way to constructing the partial wave decomposition of electron scattering amplitudes we need central field solutions of the Dirac equation carrying definite angular momentum. According to the formulate of App. E these have the form

$$\psi_{\kappa m}(r,\theta,\varphi) = \begin{pmatrix} g_\kappa(r)\varphi_{\kappa m} \\ \mathrm{i} f_\kappa(r)\varphi_{-\kappa m} \end{pmatrix}, \tag{2.104}$$

where κ is Dirac's quantum number. The radial wave functions g_κ and f_κ obey the system of differential equations

$$\begin{aligned} \frac{\mathrm{d}f_\kappa}{\mathrm{d}r} &= \frac{\kappa-1}{r} f_\kappa - (E - V(r) - m) g_\kappa, \\ \frac{\mathrm{d}g_\kappa}{\mathrm{d}r} &= -\frac{\kappa+1}{r} g_\kappa + (E - V(r) + m) f_\kappa. \end{aligned} \tag{2.105}$$

In the case of electron scattering at high energies we can neglect the mass term in (2.105). In this case we have the following symmetry relations

$$g_{-\kappa}(r) \simeq f_\kappa(r), \qquad f_{-\kappa}(r) \simeq -g_\kappa(r), \tag{2.106}$$

which can be read off from (2.105). Furthermore, it is appropriate to use the high-energy representation (1.187) of the Dirac equation. Using the transformation matrix S, p. 25, the central field solutions (2.104) now appear in the form

$$\psi_{\kappa m} = \begin{pmatrix} \phi_{\kappa m} \\ \chi_{\kappa m} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} g_\kappa \varphi_{\kappa m} + \mathrm{i} f_\kappa \varphi_{-\kappa m} \\ g_\kappa \varphi_{\kappa m} - \mathrm{i} f_\kappa \varphi_{-\kappa m} \end{pmatrix}. \tag{2.107}$$

For vanishing mass the symmetry relations (2.106) apply and thus

$$\psi_{-\kappa m} = -\frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} g_\kappa \varphi_{\kappa m} + \mathrm{i} f_\kappa \varphi_{-\kappa m} \\ -g_\kappa \varphi_{\kappa m} + \mathrm{i} f_\kappa \varphi_{-\kappa m} \end{pmatrix}.$$

Furthermore, we know from (2.17,2.20) in Sect. 2.2, that the two asymptotic helicity states have the same scattering amplitudes. Thus, it is sufficient to study two-component spinors $\phi_{\kappa m}$ [e.g. upper two components of (2.107)] for *positive* κ. Replacing $\kappa > 0$ by $j = \kappa - 1/2$ and introducing $F_j(r) = r f_\kappa(r)$, $G_j(r) = r g_\kappa(r)$, equations (2.105) (in the limit $m = 0$) transform into

$$\begin{aligned} \frac{\mathrm{d}F_j}{\mathrm{d}r} &= \frac{j+1/2}{r} F_j - (E - V(r)) G_j, \\ \frac{\mathrm{d}G_j}{\mathrm{d}r} &= -\frac{j+1/2}{r} G_j + (E - V(r)) F_j. \end{aligned} \tag{2.105$'$}$$

As an example consider the case $V(r) \equiv 0$. From (2.105′) one derives

$$\frac{\mathrm{d}^2 G_j}{\mathrm{d}r^2} + \left(E^2 - \frac{\kappa(\kappa+1)}{r^2}\right) G_j = 0 \qquad (\kappa = j + 1/2),$$

whose solutions can be expressed in terms of the spherical Bessel functions,

$$G_j = Nrj_\kappa(kr) = Nrj_{j+1/2}(kr)$$

F_j is obtained from the second equation (2.105′) and the well-known relation

$$j_l'(z) = \frac{l}{z} j_l - j_{l+1} = -\frac{l+1}{z} j_l + j_{l-1}.$$

One finds

$$F_j(r) = Nrj_{\kappa-1}(kr) = Nrj_{j-1/2}(kr),$$

so that, with $\kappa = j + 1/2$

$$\phi_{\kappa m}^{(V\equiv 0)} = \frac{N}{\sqrt{2}} \{ j_{j+1/2}(kr)\varphi_{\kappa m} + \mathrm{i} j_{j-1/2}(kr)\varphi_{-\kappa m} \}. \tag{2.108}$$

In deriving the partial wave decomposition of the scattering amplitude we assume, at first, that the potential $V(r)$ decreases at infinity faster than $1/r$, i.e. $\lim_{r\to\infty} rV(r) = 0$. (See discussion in Sect. 2.2) The modifications due to the long range of the Coulomb potential are considered at the end of this section.

Following (2.17) we require the solutions to have the following asymptotic form:

$$\phi_{m=1/2} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}kz} + \frac{f(\theta,\varphi)}{r} \begin{pmatrix} \mathrm{e}^{-i\varphi/2} \cos(\theta/2) \\ \mathrm{e}^{i\varphi/2} \sin(\theta/2) \end{pmatrix} \mathrm{e}^{\mathrm{i}kr}. \tag{2.109}$$

For the solution at all values of $\boldsymbol{x}$ we write the series expansion

$$\phi_{m=1/2} = \sum_{j=1/2}^{\infty} a_{j1/2} \phi_{j1/2}$$

in terms of the angular momentum eigenstates (2.107).

The incoming part of ϕ, i.e. the first term of (2.109), is readily expanded in terms of partial waves

$$\phi_{\text{in}} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}kz} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sum_{l=0}^{\infty} \mathrm{i}^l \, j_l(kr) \, \sqrt{(2l+1)4\pi} Y_{l0}$$

$$= \sqrt{4\pi} \sum_{j=1/2}^{\infty} \sum_{l=j-1/2}^{j+1/2} \mathrm{i}^l \, \sqrt{2l+1} \, j_l(kr) \left(l0, \tfrac{1}{2}\tfrac{1}{2} | j \tfrac{1}{2}\right) \varphi_{jl1/2}. \quad (2.110)$$

Here we have expressed the product of the spin up state $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and of the eigenstate Y_{l0} of orbital angular momentum in terms of states in which l and spin 1/2 are coupled to total angular momentum $j = l \pm 1/2$. Remembering the definition of the Dirac quantum number κ, we have for positive κ

$$\begin{aligned} \varphi_{\kappa m} &\equiv \varphi_{j=l-1/2, l=\kappa, m} \\ & \qquad\qquad (\kappa > 0) \\ \varphi_{-\kappa m} &\equiv \varphi_{j=\bar{l}+1/2, \bar{l}=\kappa-1, m} \end{aligned}$$

This allows us to rewrite (2.110) by inserting the explicit values of the Clebsch–Gordan coefficients, as given in Table 2.2.

One finds

$$\phi_{\text{in}} = \sum_{j=1/2}^{\infty} \mathrm{i}^{j-3/2} \, \sqrt{4\pi \left(j + \tfrac{1}{2}\right)} \{ j_{j+1/2}(kr) \, \varphi_{\kappa \, 1/2} + \mathrm{i} j_{j-1/2}(kr) \, \varphi_{-\kappa \, 1/2} \}. \quad (2.110')$$

[The reader may check that this is the same as the free solution (2.108).] For the purpose of reference we note here the well-known asymptotic behaviour of spherical Bessel functions

$$j_l(z) \underset{z \to \infty}{\sim} \frac{1}{z} \sin(z - l\pi/2). \quad (2.111)$$

Indeed, if we make the following ansatz for the full solution:

Table 2.2 Explicit expressions for Clebsch–Gordan coefficients $\left(lm_l = m - m_s; \tfrac{1}{2} m_s | jm\right)$

j	$m_s = +\frac{1}{2}$	$-\frac{1}{2}$
$l + \frac{1}{2}$	$\left(\dfrac{l + \frac{1}{2} + m}{2l+1}\right)^{1/2}$	$\left(\dfrac{l + \frac{1}{2} - m}{2l+l}\right)^{1/2}$
$l - \frac{1}{2}$	$-\left(\dfrac{l + \frac{1}{2} - m}{2l+1}\right)^{1/2}$	$\left(\dfrac{l + \frac{1}{2} + m}{2l+1}\right)^{1/2}$

$$\phi_{m=1/2} = \frac{1}{kr} \sum_{j=1/2}^{\infty} \sqrt{4\pi(j+1/2)}\, \mathrm{i}^{j-3/2}\, \mathrm{e}^{\mathrm{i}\eta_j} \{ G_j\, \varphi_{\kappa\, 1/2} + \mathrm{i} F_j\, \varphi_{-\kappa 1/2} \}. \tag{2.112}$$

the asymptotic behaviour of the radial functions can be taken to be

$$G_j \sim \sin\left(kr - \frac{j+1/2}{2}\pi + \eta_j\right), \tag{2.113a}$$

$$F_j \sim \cos\left(kr - \frac{j+1/2}{2}\pi + \eta_j\right). \tag{2.113b}$$

The asymptotic ansatz (2.113) makes sure that the *in*coming wave of ϕ, (2.112), i.e. the piece proportional to $\mathrm{e}^{-\mathrm{i}kr}/r$, is indeed equal to the incoming part of ϕ_{in}, (2.110′). η_j is the phase shift caused by the potential $V(r)$ as compared to the force-free situation where $V \equiv 0$. In order to identify the scattering amplitude $f(\theta, \varphi)$ we must isolate the *out*going spherical wave $\mathrm{e}^{\mathrm{i}kr}/r$ in the asymptotic expansion of ϕ, (2.112). We find

$$\phi|_{\mathrm{out}} \sim -\frac{\mathrm{e}^{\mathrm{i}kr}}{2\mathrm{i}kr} \sqrt{4\pi} \sum_j \sqrt{j+1/2}\, \mathrm{e}^{2\mathrm{i}\eta_j} \{\varphi_{\kappa 1/2} - \varphi_{-\kappa 1/2}\}.$$

The asymptotic piece of ϕ_{in}, (2.110′), which contains the *out*going spherical wave is determined in exactly the same way. Obviously, it is of the same form, but with $\eta_j = 0$. Upon comparison of the difference $(\phi - \phi_{\mathrm{in}})_{\mathrm{outg.spher.wave}}$ to the asymptotic form (2.109) we have

$$f(\theta, \varphi) \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi/2} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\varphi/2} \sin(\theta/2) \end{pmatrix} = -\frac{1}{2\mathrm{i}k} \sum \sqrt{4\pi(j+1/2)}\, (\mathrm{e}^{2\mathrm{i}\eta_j} - 1)\, \{\varphi_{\kappa 1/2} - \varphi_{-\kappa 1/2}\}. \tag{2.114}$$

By means of the Clebsch–Gordan coefficients of Table 2.2 we have, with $l = \kappa = j + \frac{1}{2}$,

$$\{\varphi_{\kappa 1/2} - \varphi_{-\kappa 1/2}\} = -\left(\sqrt{\frac{l}{2l+1}}\, Y_{l0} + \sqrt{\frac{l}{2l-1}}\, Y_{l-1,0} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\sqrt{\frac{l+1}{2l+1}}\, Y_{l1} - \sqrt{\frac{l-1}{2l-1}}\, Y_{l-1,1} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using the definition

$$Y_{lm} = (-)^m \sqrt{\frac{(2l+1)(l-1)!}{4\pi(l+1)!}}\, p_l^m\, \mathrm{e}^{\mathrm{i}\varphi} \tag{2.115}$$

and two recurrence relations for the associated Legendre functions

$$P_l^m - xP_{l-1}^m - (l+m-1)\ \sqrt{1-x^2}\ P_{l-1}^{m-1} = 0,$$
$$xP_l^m - (l-m+1)\ \sqrt{1-x^2}\ P_l^{m-1} - P_{l-1}^m = 0,$$
$$(x = \cos\theta)$$

from which we derive

$$P_l^1 - P_{l-1}^1 = l\frac{\sqrt{1-x^2}}{1+x}\ (P_l + P_{l-1}) = l\,\mathrm{tg}(\theta/2)\,(P_l + P_{l-1}),$$

one finds

$$\{\varphi_{\kappa 1/2} - \varphi_{-\kappa 1/2}\} = -\sqrt{\frac{l}{4\pi}}\,(P_l + P_{l-1})\ \left\{\begin{pmatrix}1\\0\end{pmatrix} + \mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{tg}(\theta/2)\begin{pmatrix}0\\1\end{pmatrix}\right\}$$
$$= -\sqrt{\frac{1}{4\pi}}\ \frac{\mathrm{e}^{\mathrm{i}\varphi/2}}{\cos\theta/2}\ (P_l + P_{l-1})\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\varphi/2}\cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\varphi/2}\ \sin(\theta/2)\end{pmatrix}.$$

Comparing this to (2.114) and setting $l = j + 1/2$ we obtain at once the final result

$$f(\theta,\ \varphi) = \frac{1}{2\mathrm{i}k}\ \frac{\mathrm{e}^{\mathrm{i}\varphi/2}}{\cos(\theta/2)}\ \sum_{j=1/2}^{\infty}(j+1/2)\ (\mathrm{e}^{2\mathrm{i}\eta_j} - 1)\ (P_{j+1/2} + P_{j-1/2}). \tag{2.116}$$

The phase factor $\mathrm{e}^{\mathrm{i}\varphi/2}$ is the same as the one obtained in the Born approximation, (2.25). As the potential $V(r)$ and hence the scattering process are axially symmetric about the 3-axis we may set $\varphi = 0$ without loss of generality.

Extension to the Coulomb potential. For potentials which do not decrease faster than $1/r$, the asymptotic form (2.109) is not correct. Very much like in the analogous nonrelativistic situation the phase factor $\mathrm{e}^{\mathrm{i}kr}$ is modified by an additional phase factor which depends on $\ln(2kr)$, so that in (2.109) we should make the replacement

$$\mathrm{e}^{\mathrm{i}kr} \ \rightarrow\ \mathrm{e}^{\mathrm{i}(kr + Z\alpha\ln(2kr))}.$$

In the expression (2.116) the phases η_j, so far, were the scattering phases due to the potential relative to the *force-free case*. In the case of a potential decreasing like $1/r$ a different procedure is indicated: The electrostatic potential created by a nucleus with spherically symmetric charge density $\rho(r)$ is, cf. (1.186),

$$V(r) = -4\pi Ze^2\left\{\frac{1}{r}\int_0^r \rho(r')r'^2\mathrm{d}r' + \int_r^\infty \rho(r')r'\mathrm{d}r'\right\}. \tag{2.117}$$

As $\rho(r)$ vanishes (or becomes negligibly small) beyond some distance R of the order of the nuclear radius, $V(r)$ approaches the pure $1/r$ potential

$$V_C(r) = -Ze^2/r \tag{2.118}$$

for $r > R$. Like in the nonrelativistic case, the radial Dirac equations (2.105) can be solved analytically for the case of the potential (2.118) of point-like charges. In particular, the scattering phases η_j^{C} of this potential can be given explicitly and the scattering amplitude f_{C} be computed from (2.116). Therefore, the scattering problem for the true potential (2.117) is solved most economically by computing the *additional* phase shift due to $V(r)$

$$\delta_j := \eta_j - \eta_j^{\mathrm{C}}, \tag{2.119}$$

where η_j is the full phase shift of the potential $V(r)$. The construction of the continuum solutions g_κ^{C} and $\mathrm{f}_k^{\mathrm{C}}$ for the potential V_{C} is straightforward but tedious and we do not work them out here[14]. The solutions which are regular at the origin have the asymptotic behaviour (in the mass zero limit),

$$\begin{aligned} rg_\kappa^{\mathrm{C}} &\sim \sin\left(kr + y\ \ln(2kr) + \delta_\kappa^0\right), \\ rf_\kappa^{\mathrm{C}} &\sim \cos\left(kr + y\ \ln(2kr) + \delta_\kappa^0\right), \end{aligned} \tag{2.120}$$

where

$$\delta_\kappa^0 \equiv \bar{\eta}_\kappa - \sigma_\kappa - (\gamma_\kappa - 1)\pi/2, \quad y = Z\alpha, \tag{2.121}$$

with

$$\begin{aligned} &\gamma_\kappa = \sqrt{\kappa^2 - (Z\alpha)^2}, \\ &\bar{\eta}_\kappa\ (m=0) = -\frac{1}{2}\ \mathrm{arctg}\ \frac{y}{\gamma_\kappa} - \frac{\pi}{2}\ \frac{1+\mathrm{sign}\kappa}{2}, \\ &\sigma_\kappa = \arg\Gamma(\gamma_\kappa + \mathrm{i}y); \end{aligned}$$

comparing this to the general form (2.113) we see that the Coulomb phase is given by

$$\eta_k^{\mathrm{C}} = \bar{\eta}_\kappa - \sigma_\kappa + (l - \gamma_\kappa + 1)\,\pi/2. \tag{2.122}$$

(up to the logarithmic term $Z\alpha$ In $2kr$), or

$$\eta_j^{\mathrm{C}} = \bar{\eta}_j - \sigma_j + \left(j + \frac{3}{2} - \gamma_j\right)\pi/2. \tag{2.122'}$$

In (2.122′) we have written the index j, not κ since we need to consider only positive κ, here; equation (2.122) holds for all κ, positive and negative. For the sake of simplicity, we have neglected the mass of the electron. If one wishes to retain the

[14] See e.g. [ROS 61].

mass terms then $y = Z\alpha\ E/\kappa$ and $\bar{\eta}_\kappa$ of (2.121) is replaced with[15]

$$\bar{\eta}_\kappa(m) = -\frac{1}{2}\text{arctg}\ \frac{y\left(1+\frac{\gamma_\kappa}{\kappa}\frac{m}{E}\right)}{\gamma_\kappa - \frac{1}{\kappa}y^2\frac{m}{E}} - \frac{\pi}{2}\ \frac{1+\text{sign}\,\kappa}{2}. \tag{2.123}$$

All other formulae (2.120–2.122) remain unchanged provided k is now understood to be the wave number $k = \sqrt{E^2 - m^2} \cdot \eta_j^{\text{C}}$ can therefore be obtained analytically form (2.122). The additional phase shifts δ_j, (2.119), which are caused by the *difference* of the true potential (2.117) and the pointlike potential (2.118) may be obtained as follows: One calculates solutions of the Dirac equations (2.105′), for a given energy and for $V(r)$ as obtained from (2.117), which are regular at the origin. As $V(r)$ goes over into $V_{\text{C}}(r)$ at some finite radius R outside the nuclear charge radius, we need the full solution (f, g) only in the inner region $0 \le r \le R$. In the outer region $r \ge R$ the exact solution is a superposition of regular (R) and irregular (I) solutions of (2.105′) with $V_{\text{C}}(r)$, viz.

$$\begin{aligned} f_\kappa &= a f_\kappa^{\text{C,R}} + b f_\kappa^{\text{C,I}}, \\ g_\kappa &= a g_\kappa^{\text{C,R}} + b g_\kappa^{\text{C,I}}. \end{aligned} \tag{2.124}$$

The phase differences δ_j may be obtained by direct comparison of the exact solution to the regular solutions $\left(f^{\text{C,R}}, g^{\text{C,R}}\right)$ at some $r > R$ at which an asymptotic expansion of these functions is meaningful. In this case, however, $rf^{\text{C.R}}$ and $rg^{\text{C.R}}$ must be expanded beyond the form (2.120), up to, say, terms of order $1/r^2$. Alternatively, the phases may be obtained from the asymptotic form of (2.124), expressing them as functions of b/a and η_j^{C} (Ravenhall et al. 1954).

As an example, Table 2.3 shows the Coulomb phases η_j^{C} for V_{C} with $Z = 82$, as well as the phase shifts δ_j for the Fermi charge density (2.125) for lead, $Z = 82$, $c = 6.6475$, $t = 2.30$. The electron energy is $k = 300\,\text{MeV}$. At this energy the nine lowest partial waves are modified appreciably by the deviation of the actual charge density from the point-like charge.

2.5.6 *Practical Analysis of Scattering Data and Information Content of Partial Waves*

In the early stages of this kind of nuclear physics it was customary to analyze elastic scattering data in terms of specific functional forms for the nuclear charge density. An ansatz that was particularly popular is the so-called *Fermi distribution*,

$$\rho(r) = N\frac{1}{1+\exp\left[(r-c)/z\right]}, \tag{2.125}$$

[15] Arctg is defined such that it goes to zero as the argument goes to zero.

Table 2.3 Coulomb phases and phase shifts, equation (2.119) for electron scattering on $^{208}_{82}$Pb. The charge density is described by (2.125) with $c = 6.6475\,\text{fm}$, $t = 2.30\,\text{fm}$

j	η_j^C	δ_j
$\frac{1}{2}$	0.4441	−1.4779
$\frac{3}{2}$	−0.2401	−0.8046
$\frac{5}{2}$	−0.5491	−0.5137
$\frac{7}{2}$	−0.7512	−0.3367
$\frac{9}{2}$	−0.9017	−0.2181
$\frac{11}{2}$	−1.0218	−0.1361
$\frac{13}{2}$	−1.1217	−0.0801
$\frac{15}{2}$	−1.2072	−0.0434
$\frac{17}{2}$	−1.2820	−0.0214
$\frac{19}{2}$	−1.3485	−0.0095
$\frac{21}{2}$	−1.4083	−0.0039

where the normalization factor

$$N = \frac{3}{4\pi c^3}\left[1 + \left(\frac{\pi z}{c}\right)^2 - 6\left(\frac{z}{c}\right)^3 \mathrm{e}^{-c/z} \sum_{n=1}^{\infty} \frac{(-)^n}{n^3} \mathrm{e}^{-nc/z}\right]^{-1}$$
$$\simeq \frac{3}{4\pi c^3} \frac{1}{1 + (\pi z/c)^2} \tag{2.126}$$

is chosen such as to normalize $\rho(r)$ to one, cf. (2.2). The parameter c is called the *radius of half-density* because $\rho(r = c) = 0.5\ \rho(r = 0)$. The parameter a is a measure for the rate at which the density falls off in the nuclear surface. A good measure for this fall-off is the so-called *surface thickness* t which is defined as follows: Let $\rho(r)$ be a function which decreases monotonically for increasing r. Let $r_{(90)}$ and $r_{(10)}$ be the radii at which the density is 90% and 10% of its value at $r = 0$, respectively. Then

$$t := r_{(10)} - r_{(90)}.$$

In the case of the Fermi distribution (2.125) one has the relation $t = 4z \ln 3 \simeq 4.394z$. The shape of the Fermi density is illustrated by Fig. 6.7, curve marked ρ_0.

As a matter of fact, the Fermi density (2.125) gives a surprisingly good description of charge and matter densities in practically all spherical nuclei except

the very lightest ones. It depends on two parameters, c and t, which have the typical values

$$c \simeq 1.1 \times Z^{1/3}\,\text{fm}, \; t \simeq 2.2\text{–}2.5\,\text{fm}. \tag{2.127}$$

There are, of course, deviations from this simple pattern, especially at, and close to, magic shells but these deviations are never very large. (We shall come back to this, in connection with muonic atoms, below.) Even though a functional form such as the Fermi function (2.125) may be quite useful as a rough parametrization, it is not adequate for the analysis of precise measurements of the elastic cross section extending over many orders of magnitude, for several reasons: Assuming a specific functional form implies a prejudice about possible shapes of the charge density and, therefore, cannot be the basis of a model-free analysis of the data. Furthermore, such an ansatz contains a finite number of parameters [two in the case of the Fermi distribution (2.125)] which are then determined by a best fit to the data. What if the data are so numerous and so precise that they contain more information on the charge distribution than what can be described by these parameters? Finally, a specific function such as (2.125) does not reflect the fact that different parts of $\rho(r)$ have different weights in the angular dependence of cross sections. In summary, one would prefer a model independent way of analysis whose final result would be an empirical density function $\rho(r)$, along with an error band that depends on r and reflects the type of experimental input as well as its error bars.

Such methods of analysis which do not rely on specific model densities have been proposed and have been applied successfully to high-precision experiments (Lenz 1969, Friedrich et al. 1972, Sick 1974, Friar et al. 1973, 1975). Figures 2.6 and 2.7 show typical examples of nuclear charge densities as determined from experiment. Instead of describing these specific methods here we prefer to discuss, in a qualitative manner, the physical information carried by the various low, intermediate and high partial waves.

Sensitivity of scattering phase to details of charge density. Obviously, the nature of charge density moments and the number of them that can be obtained from elastic scattering, depend on:

(i) the primary energy of the electron,
(ii) the range of momentum transfers, and
(iii) the experimental error bars of the differential cross section.

The essential features of such an analysis can be made transparent by studying integral representations for the scattering phases. To this end, let us consider two different charge densities $\rho^{(1)}(r)$, $\rho^{(2)}(r)$ which are both normalized to one, as before, but which differ from each other over the domain of the nucleus. Both are assumed to go quickly to zero for $r \gtrsim R$. The corresponding potentials $V^{(1)}(r)$ and $V^{(2)}(r)$, calculated from (2.117), then differ only for $r \lesssim R$ and both go over into $V_C(r)$ outside the nucleus.

Let $F_j^{(1)}$, $G_j^{(1)}$ and $F_j^{(2)}$, $G_j^{(2)}$ be the solutions of (2.105′) for $V^{(1)}$ and $V^{(2)}$, respectively, and let $\delta_j^{(1)}$ and $\delta_j^{(2)}$ be the corresponding phase shifts as defined in (2.119). From (2.105′) and from the properties of the generalized Wronskian (see

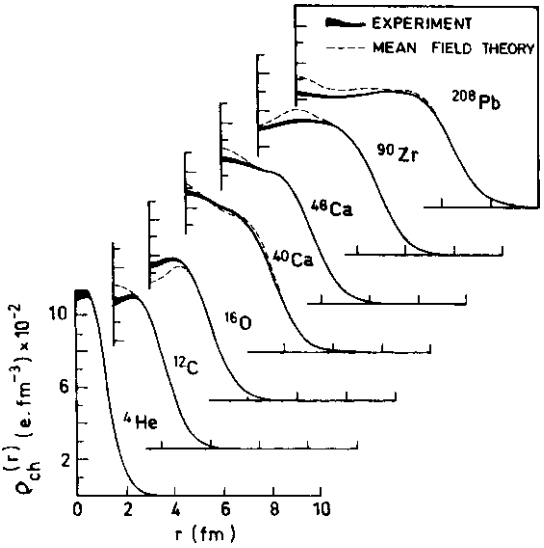

Fig. 2.6 Charge density distributions of doubly closed-shell nuclei. Thickness of solid lines corresponds to experimental uncertainties. Dashed lines are calculated densities. Taken from Frois and Papanicolas (1987)

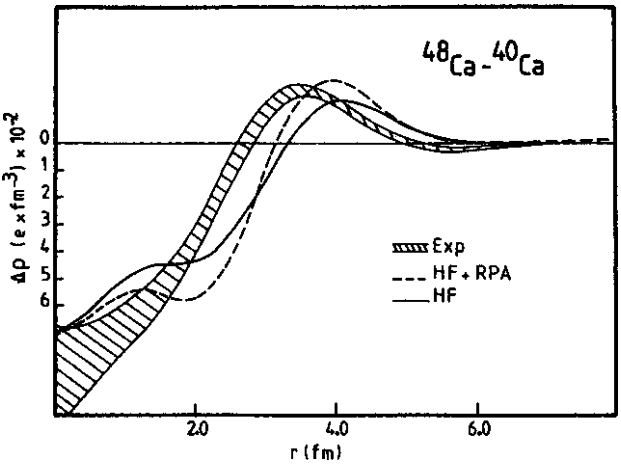

Fig. 2.7 Difference of charges densities of ^{48}Ca and ^{40}Ca. The shaded band is the result of a quasi model-independent determination from elastic electron scattering; its width reflects experimental errors of the cross sections as well as the lack of knowledge of their behaviour at large momentum transfers. Figure taken from Lect. Notes in Phys. 108 (1979), p. 58 (Proceedings of Conf. on Nuclear Physics with Electromagnetic Interactions, Mainz 1979)

exercise 2.6),

$$W(r) = F_j^{(1)}(r)G_j^{(2)}(r) - G_j^{(1)}(r)F_j^{(2)}(r), \tag{2.128}$$

one derives the relation

$$\sin\left(\delta_j^{(1)} - \delta_j^{(2)}\right) = -\int_0^\infty \mathrm{d}r (V^{(1)} - V^{(2)}) \left(F_j^{(1)} F_j^{(2)} + G_j^{(1)} G_j^{(2)}\right). \tag{2.129}$$

The potentials are related to the charge densities through Poisson's equation. Inserting this into (2.129) and integrating by parts twice yields the final result

$$\sin\left(\delta_j^{(1)} - \delta_j^{(2)}\right) = 4\pi \int_0^\infty r^2 \mathrm{d}r (\rho^{(1)}(r) - \rho^{(2)}(r)) \chi_j(r), \tag{2.130}$$

where χ_j stands for

$$\chi_j(r) = \int_0^r \frac{\mathrm{d}r'}{r'^2} \int_0^{r'} \mathrm{d}r'' \left(F_j^{(1)} F_j^{(2)} + G_j^{(1)} G_j^{(2)}\right). \tag{2.131}$$

What does (2.130) tell us about the sensitivity of the phase shifts to the charge density? To answer this question it is useful to consider high, intermediate, and low partial waves separately:

(a) *High partial waves*. Classically speaking, these partial waves correspond to electron trajectories which pass by, far outside, and do not penetrate the nucleus. The phases $\eta_j^{(i)}$ coincide practically with the phases η_j^{C} of a point-like source. As $\rho^{(1)} \simeq \rho^{(2)}$ these phases do not contain information on the nuclear charge density other than its total charge. (In the example of Table 2.3: $j \gtrsim 19/2$).

(b) *Intermediate partial waves*. These partial waves start penetrating the nucleus somewhat. However, the centrifugal potential terms in (2.105′) are still predominant so that the radial functions can be approximated, over the entire nuclear domain, by their power behaviour at the origin

$$F_j^{(i)} \sim r^{j+1/2}, \quad G_j^{(i)} \sim r^{j+3/2}, \tag{2.132}$$

Inserting this into (2.131) we find that $\chi_j(r)$ may be approximated by a simple power behaviour too, $\chi_j \sim r^{2j+1}$. Equation (2.130) then becomes

$$\sin\left(\delta_j^{(1)} - \delta_j^{(2)}\right) \sim 4\pi \int_0^\infty r^2 \mathrm{d}r (\rho^{(1)} - \rho^{(2)}) r^{2j+1}. \tag{2.130′}$$

Thus, intermediate partial waves are determined by even moments of the charge density. Obviously, these depend primarily on the density at the nuclear surface.

(c) *Low partial waves*. The low partial waves, finally, for which the approximation (2.132) becomes invalid, contain information about the nuclear interior. If these partial waves occur at all, i.e. if they are really distinct from the class b (in fact,

this is only the case if the energy is high enough), then they cannot be expressed in terms of simple moments

$$\int_0^\infty r^2 \mathrm{d}r\ \rho(r) r^{2j+1}$$

and a full partial wave analysis must be carried out.

Summarizing, we may say that the type and quality of information that is obtained from a partial wave analysis of electron scattering is very similar to the information content of the form factor in the Born approximation. Therefore, Coulomb distortion, even though important on a quantitative level, is not much more than a technical complication that does not alter the physics of the process.

2.5.7 *Miscellaneous Comments*

We close this somewhat lengthy discussion of electron scattering with a few comments and supplementary remarks, as well as some suggestions for further reading. The first three supplements concern elastic scattering, the last one concerns inelastic scattering.

Dispersion corrections to elastic scattering. In our treatment of elastic scattering we have assumed the nucleus to be an inert system of stationary charge and current densities. In other terms, we have calculated the fields created by the nuclear ground state at the site of the electron and we have calculated the cross sections from a one-particle wave equation for the electron, in the *external field approximation*. In reality, the nucleus is a dynamical system and has its own internal degrees of freedom which manifest themselves in its rich excitation spectra. These internal degrees of freedom may play a role in elastic scattering if second-order processes of the type sketched symbolically in Fig. 2.8 become important: In a first step the electron excites the nuclear ground state A to some excited intermediate state A* which deexcites again to the ground state in a second scattering process.

These dispersion corrections are notoriously difficult to calculate in a reliable manner. Fortunately, they are not large and can be neglected in most practical situations.

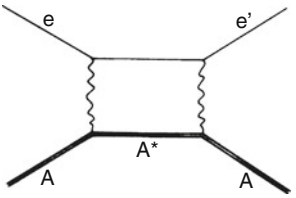

Fig. 2.8 So-called disperision corrections to electron scattering: a two-step process in which the nucleus is excited to a intermediate state

Elastic scattering on strongly deformed nuclei. So far we have considered only spherically symmetric nuclear charge densities $\rho(r)$. This situation allowed us to separate angular and radial motion of the electron. Obviously, this situation applies only for nuclei with spin zero, $J = 0$. For nuclei with $J = 1/2$ we have formally the same situation as for the nucleon: Here there is charge scattering from a spherically symmetric charge density as before, plus scattering on a magnetic moment density (M1 scattering). For nuclei with $J \geq 1$ the nuclear charge density also has nonvanishing higher (even) moments, viz.

$$\rho(\boldsymbol{r}) = \left\langle JM = J \left| \sum_{i=1}^{Z} \delta(\boldsymbol{r} - \boldsymbol{r}_i) \right| J\ M = J \right\rangle$$
$$= \rho_0(r) + \sum_{\kappa=1}^{[J]} \sqrt{\frac{4\kappa+1}{16\pi}} \rho_{2\kappa}(r) Y_{2\kappa,0}(\theta). \tag{2.133}$$

In the case of strongly deformed nuclei the quadrupole term will, in general, be predominant, i.e.

$$\rho(\boldsymbol{r}) \simeq \rho_0(r) + \sqrt{\frac{5}{16\pi}} \rho_2(r) Y_{20}(\theta), \tag{2.134}$$

where $\rho_2(r)$ denotes the radial quadrupole density. The factors in (2.133) have been chosen such that the nuclear spectroscopic quadrupole moment, by its traditional definition, is given by the second moment of $\rho_2(r)$

$$Q_s = \int_0^\infty [\rho_0(r) r^2] r^2 \mathrm{d}r. \tag{2.135}$$

While the calculation of the scattering cross section in the Born approximation is straightforward (see above), the corresponding partial wave analysis is technically more complicated, but still feasible. There may be a difficulty on the experimental side; Such strongly deformed nuclei have rotational excitations of rather low energy, 50–200 keV. It then depends on the energy resolution of a given experimental set up whether or not the truly elastic scattering from the nuclear ground state can be distinguished from the excitation cross sections for these low-lying rotator states. Thus, elastic scattering of electrons may not be the ideal tool for investigating the charge density of deformed nuclei. We will see below that in this case muonic atoms offer a more direct approach to this quantity.

Summation of partial wave amplitudes. There is a technical difficulty in summing the partial wave series (2.116): As it stands this series converges very slowly. The origin of this problem is not difficult to understand. For this purpose let us consider first the scattering on a point charge in the nonrelativistic case. The potential $V_C(r)$ of a point charge introduces a $1/r$ singularity into the Schrödinger equation. The corresponding scattering amplitude, which can be derived in closed form, is proportional to $1/\sin^2(\theta/2)$ and hence becomes singular at $\theta = 0$. This singularity

can be "smoothed" and the convergence of the partial wave series accelerated by means of the following procedure. Suppose we wish to sum an expression of the form (Ravenhall et al. 1954).

$$F(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta).$$

If this series converges only very slowly in practice, one may replace it by a so-called reduced series which is defined by

$$(1-\cos\theta)^m F(\theta) = \sum_{l=0}^{\infty} c_l^{(m)} P_l(\cos\theta),$$

and where m is some positive integer. Using standard recurrence formulae for Legendre polynomials one derives the recurrence relations for the coefficients $c_l^{(m)}$

$$c_l^{(m+1)} = c_l^{(m)} - \frac{l}{2l-1} c_{l-1}^{(m)} - \frac{l+1}{2l+3} c_{l+1}^{(m)},$$
$$c_l^{(0)} = a_l.$$

The modified series which is obtained after a few iterations converges more rapidly than the original series.

Inelastic scattering and partial wave analysis. The problem of Coulomb distortion in inelastic electron scattering is essentially the same as in elastic scattering. For light nuclei the conceptually simple method of Born approximation may still be adequate (depending, again, on the accuracy of the data). For medium and heavy nuclei it is not. The inelastic form factors of the Born approximation, in general, have zeroes in the physical domain of momentum transfers. Therefore, the cross section, if it is dominated by one form factor, again exhibits the typical diffraction pattern of the Born approximation. The diffraction zeroes are replaced with minima and are shifted from their initial position if Coulomb distortion is taken into account. Also the absolute values of the cross section can be widely different from what they are in the Born approximation, depending on the value of the scattering angle.

The partial wave analysis of inelastic scattering is conceptually similar to the case of elastic scattering but technically much more involved. The first such analyses were carried out in the mid-nineteen-sixties[16]. (Griffy et al. 1963, Scheck 1966). A good starting point is the interaction term in the form of (2.85). It is appropriate to expand the electron wave functions in terms of the central field solutions (2.104). The initial state $|k\rangle$ is determined by the requirement that in the asymptotic domain it contain the plane wave and incoming spherical waves. Similarly, the

[16]The much simpler case of monopole excitations was treated in Alder et al. (1963).

final state is the analogous superposition of plane wave and outgoing spherical wave. Inserting these (properly normalized) functions into the matrix element (2.85) yields a multiple sum of matrix elements of the interaction between states of good angular momentum and parity. This fact allows one to make use of the selection rules due to angular momentum and parity conservation and to perform all angular and spin integrals by means of standard angular momentum algebra. This leaves one with a sum over radial integrals involving the radial functions $f_k(r)$, $g_k(r)$ and some nuclear radial quantity. We know from the analysis of elastic scattering that for $E \simeq 300\,\mathrm{MeV}$ about the nine lowest partial waves penetrate into the nucleus. Obviously, these are the ones which are sensitive to details of the transition charge and current densities. Here, the radial integrals entering the expansion of the scattering amplitude are obtained by numerical integration. For the higher partial waves the eigenfunctions of the point charge may be used and, if one is lucky, the radial integrals can be done analytically (Reynolds et al. 1964). The complexity of such calculations lies in the high number of partial waves which contribute and in the complicated pattern of terms allowed by the selection rules of angular momentum.

2.6 Muonic Atoms – Introduction

Like any other long-lived negatively charged particle the muon can be captured in the static Coulomb field of a nucleus and thus can form a hydrogen-like exotic atom. This system has peculiar and unique properties, regarding both its *spatial dimensions* and its dynamical *time structure*, which make it an important tool both in exploring electroweak interactions and in probing properties of the nucleus.

This section summarizes first the properties of the muon which are relevant for the subsequent sections of this chapter. We then give a first qualitative picture of the properties of the muonic atoms and their applications. The section closes with a derivation of bound central field solutions in static Coulomb potentials.

Specific and quantitative applications of muonic atoms to quantum electrodynamics and to the investigation of nuclear properties are treated in Sects. 2.7 and 2.8, respectively.

2.6.1 Properties of Free Muons

The muon has all properties of a "heavy electron". It appears in two charge states μ^- and μ^+, which are antiparticles of each other. Its charge is equal to the charge

of the electron[17]; its spin is 1/2; it carries *lepton number* (see Chap. 3). Its *mass* is about 207 times larger than the electron mass. More precisely

$$m_\mu/m_e = 206.768259(62). \tag{2.136}$$

This number is obtained by combining the measured values of

(i) the ratio of magnetic moments μ_μ/μ_p of muon and proton;
(ii) the hyperfine splitting $\Delta E = E(F = 1) - E(F = 0)$ of muonium;
(iii) the anomaly of the g-factor of the muon, $a_\mu = \frac{1}{2}(|g_\mu| - 2)$.[18]

[The same combination of data gives very precise information on the equality of the charges of the muon and the electron].

The mass value is known to better than 1 ppm [RPP94]:

$$m_\mu = 105.658367(4)\text{MeV}/c^2. \tag{2.137}$$

The fact that μ^- carries the charge $Q = -|e|$ means that it has exactly the same coupling to the electromagnetic field as the electron. In Chap. 3 we shall see that also the weak interactions of muons are exactly the same as those of electrons. Actually, the same statements seem to apply also to the τ-lepton with mass

$$m_\tau = 1776.82(16)\text{MeV}/c^2. \tag{2.138}$$

In this sense the interactions of leptons are *universal*: The structure of the coupling terms to the Maxwell field and to the bosons of weak interactions are identically the same for the three kinds of "electrons" e^-, μ^-, τ^-. Their coupling strengths (i.e. their electric and weak "charges", respectively) to photons and to weak bosons respectively are the same.

All *quantitative* differences in physical properties of electrons, muons and τ-leptons (such as anomaly of magnetic dipole moments, scattering cross sections, decay amplitudes) will be due solely to the difference in their *masses*. This can be understood in a qualitative manner as follows. A specific physical situation is always characterized by a typical spatial dimension (examples: Bohr radius $a_B = \hbar^2/e^2m$, Compton wavelength $\lambda\!\!\!^{-} = \hbar/mc$) and typical momenta (examples: momentum in a bound atomic state, momentum transfer in a scattering amplitude) which yield the bulk of the quantity that one wishes to calculate. However, the scale of these characteristic dimensions is set by the mass of the particle or, in some cases, by the ratio or difference of masses of different leptons (examples: vacuum polarization due to virtual electron-positron pairs in electronic and in muonic atoms, g-factor anomaly for electrons and for muons). We shall encounter many examples below, in this chapter and in Chap. 3.

[17] This is known to at least 2 ppm. Cf. the summary by H. Primakoff in "Muon physics" [MUP77].
[18] See e.g. Scheck (1978).

The *magnetic moment* of the muon relative to the magnetic moment of the proton is known very precisely from measurements in muonium:

$$|\mu_\mu|/\mu_\mathrm{p} = 3.18334547(47) \tag{2.139}$$

It is practically a normal Dirac moment,

$$\mu_\mu = \frac{1}{2} g_\mu \frac{Q}{2m_\mu} (Q = -|e|), \tag{2.140}$$

with $|g_\mu| \simeq 2$, the deviation of $|g_\mu|$ from 2, the so-called anomaly a_μ, being predictable on the basis of higher order radiative corrections. This anomaly is defined as $a_\mu = \frac{1}{2}\left(|g_\mu| - 2\right)$. The measured value of this anomaly is (Bailey et al. 1979).

$$a_\mu = 116592089(54) \times 10^{-11} \tag{2.141}$$

In the same experiment the equality of $g_{\mu-}$ and $-g_{\mu+}$ which is predicted by the invariance of the muon's interactions under the combined operation *CPT* (charge conjugation C, space reflection P, and time reversal T) has been tested with the result

$$\frac{g_+ - g_-}{g_\mathrm{average}} = -0.11 \pm 0.12 \tag{2.141'}$$

In contrast to the electron, the muon is unstable. Its primary decay mode is

$$\mu^- \to e^- \bar{\nu}_e \nu_\mu \quad (\mu^+ \to e^+ \nu_e \bar{\nu}_\mu);$$

its lifetime is

$$\tau_\mu = 2.197034(21) \times 10^{-6}\mathrm{s} \tag{2.142}$$

(see Chap. 3). This time is very long compared to typical time scales of electromagnetic processes of muons in a target.

2.6.2 Muonic Atoms, Qualitative Discussion

As the Pauli exclusion principle is not effective between muons and electrons, the trapped muon runs through its Bohr cascade towards the 1s-state, irrespective of the presence of the electronic shells of the host atom. Before we turn to the quantitative analysis of muonic atoms let us first discuss their characteristic spatial dimensions, energy scales and time scales.

Energy scales and spatial dimensions. Since the Bohr radius $a_\mathrm{B} = 1/Z\alpha m$ is inversely proportional to the mass of the charged lepton (more precisely: the *reduced* mass of the lepton-nucleus system) the orbits of a muonic atom are smaller by a

factor of about 207 [see(2.136)] than the orbits of the electrons of the host atom. If the principal quantum number is smaller than n_0, where

$$n_0^2 a_{\mathrm{B}}(m_\mu) \lesssim a_B(m_e),$$

the muonic orbits $(n, l = n-1)$ lie inside the electronic 1s-orbit. This happens for

$$n \lesssim n_0 \simeq \sqrt{m_\mu/m_e} \simeq 14.$$

Therefore, the states with $n \lesssim n_0$ of the muonic atom are essentially hydrogen-like up to screening effects by the electronic shells of the host atom. Screening will be the smaller, the lower the orbit. Thus, for a first qualitative orientation we may use the equations for the nonrelativistic hydrogen atom. The binding energy of a bound state with principal quantum number n is

$$E_n^{(\mathrm{n.r.})} = -m_\mu (Z\alpha)^2/2n^2. \tag{2.143}$$

The first relativistic correction to this is of order $(Z\alpha)^4$ but its magnitude relative to (2.143) is independent of the mass. Thus, the relative importance of relativistic effects in muonic atoms is approximately the same as in electronic atoms. The relevant parameter is $Z\alpha$. For light nuclei these effects will be small, for heavy nuclei such as lead $(Z = 82)$ where $Z\alpha \simeq 0.6$ they will be important. For most estimates and qualitative considerations the formulae of the hydrogen atom will be adequate. The wave functions are [see (6.52)]:

$$\psi_{nlm}(\boldsymbol{r}) = \frac{1}{r} y_{nl}(r) Y_{lm}(\theta, \varphi), \tag{2.144}$$

$$y_{nl}(\boldsymbol{r}) = \sqrt{\frac{(l+n)!}{a_B(n-l-1)!}} \frac{1}{n(2l+1)!} z^{l+1} e^{-z/2} \times {}_1F_1(-n+l-1; 2l+2; z), \tag{2.145}$$

where

$$a_B = 1/Z\alpha\mu, \tag{2.145a}$$

μ being the reduced mass, and z is the dimensionless variable

$$z = \frac{2}{n a_{\mathrm{B}}} r. \tag{2.145b}$$

The symbol ${}_1F_1$ denotes the confluent hypergeometric function [ABS65].

Let us calculate the Bohr radius for a light, two medium weight and a heavy nucleus, and let us compare it to the nuclear r.m.s. radius:

Nucleus	$\langle r^2\rangle^{1/2}$[fm]	$a_B(m)$[fm]
^{4}He($Z=2$)	3.1	128
^{16}O($Z=8$)	2.6	32
^{40}Ca($Z=20$)	3.5	12.8
^{208}Pb($Z=82$)	5.5	3.12

We see from this comparison that from about $Z = 20$ upwards the low muonic orbits start penetrating into the nucleus more and more. Accordingly, the low muonic states must become more and more sensitive, as Z increases, to the finite size of the nucleus and, in particular, to the deviation of the nuclear charge distribution from a point charge. Of course, for large Z the estimates for the binding energies and the radii of low-lying orbits given above become unrealistic. Let us illustrate this by comparing the 2p–1s transition energy in *lead*, for a point charge and for a realistic charge distribution[19]:

$$(E_{2p1/2} - E_{1s})_{\substack{\text{point charge}\\(Z=82)}} = (-5.38 + 20.99)\,\text{MeV} = 15.61\,\text{MeV},$$

$$(E_{2p1/2} - E_{1s})_{\text{finite size}} = (-4.78 + 10.52)\,\text{MeV} = 5.74\,\text{MeV}.$$

We note that the 1s state of this heavy atom shows a much weaker binding than in the pure $1/r$ potential. The 2p state is also shifted upwards, but by a much smaller proportion than the 1s state. At the same time the radial wave functions are also affected by the finite extension of the nuclear charge density. As the states are less bound than for a point-like charge, the radial functions are driven towards larger values of r. Nevertheless, it is still true that the muon in a 1s state of a heavy atom penetrates strongly into the nuclear interior.

Time scales in muonic atoms. The fate of the muon between the moment of its creation, say, from pion decay,

$$\pi^- \to \mu^- + \bar{\nu}_\mu \tag{2.146}$$

in a continuum state until it is trapped in some high-lying Bohr orbit of a target atom, i.e. the moderation of the muon from its initial positive energy down to zero kinetic energy through ionization and inelastic scattering processes in the target, is complicated and, in fact, not too well known. For our purposes it will be sufficient to know that these early stages take a relatively short time, of the order of 10^{-10}–10^{-12} s. The muon eventually lands in some bound state with quantum numbers (n, l). The question as to what the initial distribution in n and l is, can be (and has been) studied experimentally by looking at the intensities of γ-transitions

[19]The point charge values are calculated from (2.162) below. The finite size values are taken from Engfer et al. (1974).

between these highest states of the cascade. There is, as yet, no satisfactory theory of these initial distributions–a fact that may not seem so surprising if one realizes that the initial (n, l) distribution is a complicated function of the structure and chemical composition of the target material.

The cascade proceeds predominantly through

Auger transitions, i.e. through emission of electrons in the host atom, and *electric dipole* γ-radiation.

Auger transitions are important mainly in the upper part of the cascade. They are relatively more important for the lighter elements.[20]

In the lower part of the cascade the transition energies become large, and E1 γ-transitions quickly take over. The selection rules of Auger and of γ(E1)-transitions are

$$\text{Auger}: \quad l_\mathrm{f} = l_\mathrm{i} \pm 1, \quad \Delta n = \text{minimal}, \tag{2.147}$$

$$\text{E1}: \quad l_\mathrm{f} = l_i \pm 1, \quad \Delta n = \text{maximal}. \tag{2.148}$$

The selection rule for Δl is a strict one, the rules for Δn are somewhat empirical. They arise from the energy dependence of the rates and from the n-dependence of the transition matrix elements.

The selection rules have the effect of favouring *circular orbits* $(n, l = n-1)$; the lower n, the higher are the relative intensities for transitions between circular orbits. Transitions between inner, non-circular states have comparatively low intensities.

It is not difficult to estimate the time scale of these γ-transitions. The transition probability for an electric dipole transition is given by

$$T(\mathrm{E1}) = 8\pi c \frac{2\alpha}{9} \left(\frac{\Delta E}{\hbar c}\right)^3 \frac{1}{2l_\mathrm{i}+1} (l_\mathrm{f} \,\|Y_1\|\, l_\mathrm{i})^2 \,\langle n_\mathrm{f} l_\mathrm{f}|r|n_\mathrm{i} l_\mathrm{i}\rangle^2 . \tag{2.149}$$

Let us calculate T(E1) for a transition between circular orbits $(n_\mathrm{i} \equiv n, l_\mathrm{i} = n-1)$ $(n_\mathrm{f} = n-1,\ l_\mathrm{f} = n-2)$ with the wave functions (2.144). We have

$$\begin{aligned}
(l_\mathrm{f} \,\|Y_1\|\, l_\mathrm{i}) \ &= (-)^{l_\mathrm{i}+1} \sqrt{\frac{3}{4\pi}} \sqrt{(2l_\mathrm{i}+1)(2l_\mathrm{i}-1)} \begin{pmatrix} l_\mathrm{i} & 1 & l_\mathrm{i}-1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= -\sqrt{\frac{3(n-1)}{4\pi}}, \\
\Delta E \equiv E_n \ - E_{n-1} &= \frac{2n-1}{2n^2(n-1)^2} (Z\alpha)^2 \mu c^2, \\
\langle n, n-1|r \ |n-1, n-2\rangle &= a_\mathrm{B} \frac{2^{2n+1} n^{n+1} (n-1)^{n+2}}{(2n-1)^{2n} \sqrt{2(2n-1)(n-1)}}.
\end{aligned}$$

[20] See Vol. I, Chap. III of [MUP77].

Putting these results together, we find

$$\overline{h}T(\mathrm{E1}; n \to n-1) = \frac{2^{4n} n^{2n-4} (n-1)^{2n-2}}{3(2n-1)^{4n-1}} \alpha^5 \mu c^2 Z^4,$$

or, expressing μ in terms of the electron mass,

$$T(\mathrm{E1}; n \to n-1) \simeq 5.355 \times 10^9 \frac{2^{4n} n^{2n-4} (n-1)^{2n-2}}{(2n-1)^{4n-1}} \times \frac{\mu}{m_\mathrm{e}} Z^4 [\mathrm{s}^{-1}]. \quad (2.150)$$

The transition probability is proportional to the reduced mass and to the fourth power of the nuclear charge. For the sake of illustration let us calculate the transition time

$$\tau(\mathrm{E1}; n \to n') = 1/T(\mathrm{E1}; n \to n')$$

for the transitions $\{(n = 14,\ l = 13) \to (n = 13,\ l = 12)\}$ and $\{2\mathrm{p} \to 1\mathrm{s}\}$:

$$\tau(\mathrm{E1}; 14 \to 13) \simeq 2.26(1 + 0.113/A)10^{-7}/Z^4 \mathrm{s}, \quad (2.150a)$$

$$\tau(\mathrm{E1}; 2 \to 1) \simeq 7.72(1 + 0.113/A)10^{-12}/Z^4 \mathrm{s}. \quad (2.150b)$$

[The correction factor stems from the reduced mass $1/\mu \simeq (1/m_\mu)(1 + m_\mu/Am_\mathrm{N})$]. These times must be compared with the lifetime (2.142) of the muon, $\tau \simeq 2.2 \times 10^{-6}$ S. In a very light system such as hydrogen ($Z = 1$) the upper part of the cascade is relatively slow, so that many muons will decay during the cascade before they reach the 1s state. In a heavy atom such as lead ($Z = 82$), however, the cascade times are scaled down by the factor Z^4. Rough interpolation between $n = 14$ and $n = 2$ shows that the whole cascade (assuming E1 transitions only) will take about 10^{-14} s. This time is extremely short as compared to τ_μ.

We conclude: For medium and heavy nuclei the trapping time and the cascade are very short as compared to the muon lifetime. In these atoms the muon behaves exactly like a stable heavy electron. In very light atoms such as muonic hydrogen, however, the upper part of the cascade is affected appreciably by muon decay, at the expense of the intensities of the cascade γ-transitions. These qualitative considerations are confirmed by detailed cascade calculations on computers. A detailed knowledge of the cascade is important in many experimental situations. For instance, if one sets out to study properties of inner states such as the metastable 2s state, these calculations are essential in predicting the relative populations of these states.

After having reached the 1s state the muon either decays, or is captured by the nucleus through the weak interaction process

$$\mu^- + (Z, A) \to (Z-1, A) + \nu_\mu.$$

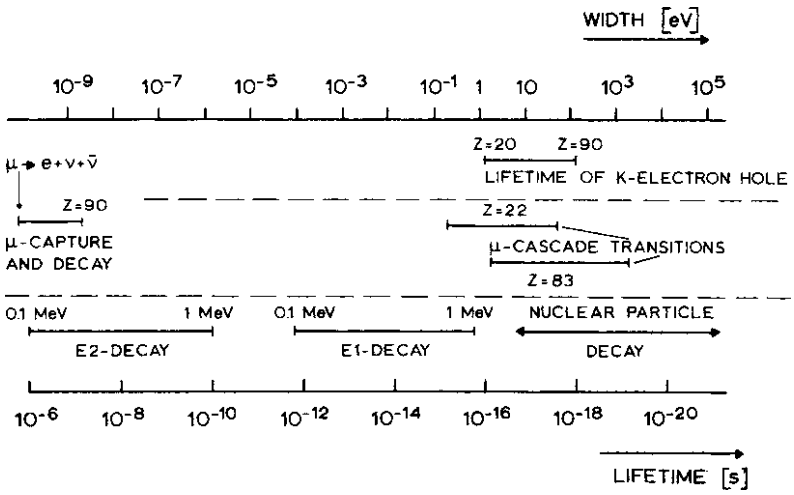

Fig. 2.9 Comparison of various lifetimes which are relevant for the dynamics in a muonic atom. The lower part of the figure shows typical times for nuclear γ-and particle decays. Taken from [MUP77] Vol. I, Chap. III

The capture rate Γ_{cap}, roughly, increases like Z^4. In light elements decay width and capture width are of similar magnitude. However, as Z increases, the capture reaction becomes predominant. In the heavy atoms, the lifetime of the muon is reduced, due to capture, to about 10^{-7} s.

These time scales of the muonic atom are illustrated in Fig. 2.9. They should be compared to typical time scales in the electronic shell as well as in the nuclear excitation spectrum. As to the former, we show, as an example, the lifetime of a hole state in the K-shell. As may be seen from the figure this atomic lifetime is comparable to cascade times of the muon. One consequence is this: If the muon, in the upper part of the cascade, has created a hole in e.g. the K-shell, it will make the rest of its cascade in presence of an incomplete electronic shell. This may be relevant in precision measurements of muonic transitions if the screening of the nuclear charge by the electrons has to be taken into account.

Regarding the nuclear excitation spectrum it can happen that due to accidental (near) degeneracy between muonic transition energies and the energies of certain nuclear excited states, the nucleus remains in an excited state when the muon has reached the 1s orbit. Comparison of typical lifetimes of such nuclear states with the lifetime $\tau(1s)$ of the muon in its 1s state (due to free decay and capture) shows a remarkable fact: The nuclear lifetimes are generally much shorter than $\tau(1s)$. Therefore, the nucleus can return to its ground state, through emission of a γ-ray, *while the muon remains in the 1s orbit*. This offers the unique possibility of observing a nuclear transition in the presence of the muon in the 1s orbit which,

as we know, penetrates strongly into the interior of the nucleus. The additional charge – e leads to *isomer shifts*, the interaction of the nuclear magnetic moment with the muon's magnetic moment to *magnetic hyperfine structure*.

On the basis of these qualitative considerations we may group the information obtainable from muonic atoms according to the spatial extension of the orbits in question.

(a) For *intermediate and heavy elements* we distinguish

(a.1) *High-lying orbits*: These are the ones which overlap strongly with the electronic cloud. These states are affected in an essential way by the state of the host atom and by the chemical composition and physical structure of the target.

(a.2) *Very low orbits*: As the muon moves well inside the lowest electronic shell, screening effects are very small and often negligible. These orbits penetrate into the nucleus and, therefore, are sensitive to the spatial structure of nuclear charge, magnetization and current densities. As the transition energies are comparable to nuclear excitation energies and as the overlap with nuclear states is large, dynamical mixing effects between muonic and nuclear states can occur. Also, instead of emitting γ-rays the muon can transfer its energy to the nucleus which then decays via fission or via emission of neutrons. These radiationless transitions are especially important in heavy elements. Muon induced fission is an important tool for the study of fission in transuranium elements (fission barriers, fission isomers).

(a.3) *Intermediate orbits* ($3 \lesssim n \lesssim 6$): These are the ones which are the most hydrogen-like. Indeed, effects due to the finite size of the nucleus are small and can be calculated to a high degree of accuracy. Likewise, the screening effects due to the electronic cloud are small and under good control. Energies and wave functions of these states can be calculated to very high precision. These orbits are ideally suited for model-independent measurements of *static nuclear moments* (especially electric quadrupole and hexadecapole moments), and of *radiative corrections*.

Figure 2.10 shows the densities $R_{nl}^2(r)\cdot r^2$ of selected muonic circular orbits in a heavy element (bismuth, $Z = 83$), in comparison to the nuclear charge density and to the density of K and L electrons. The figure illustrates quite clearly the three groups of orbits discussed above.

(b) In *very light elements* (such as hydrogen and helium) we have essentially only orbits of type (a.3). The electrons of the host atom are completely stripped off by the Auger effect so that the muon sees the bare charge of the nucleus. On the nuclear side, the muonic orbit radii are such that penetration effects are small. Thus, the study of energies of very light muonic atoms concerns primarily tests of radiative corrections as predicted by quantum electrodynamics.

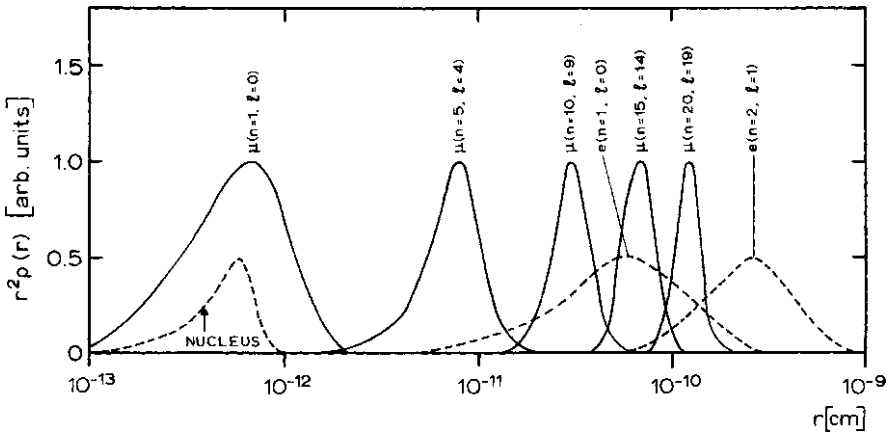

Fig. 2.10 Spatial structure of a heavy muonic atom. The scale in r (abscissa) is logarithmic; the densities are not normalized. Take from [MUP77] Vol. I, Chap. III

2.6.3 Dirac Bound States in a Central Field

In a quantitative analysis of data on muonic atoms (i.e. transition energies and X-ray intensities) we need the exact bound state solutions of the Dirac equation. In the case of a spherically symmetric potential (central field) the total angular momentum $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$ commutes with the energy and can be chosen diagonal. The problem is then reduced to the calculation of the radial wave functions $f_\kappa(r)$ and $g_\kappa(r)$ of the central field solutions [QP07]

$$\phi_{km}(\boldsymbol{r}) = \begin{pmatrix} g_\kappa(r)\ \varphi_{km} \\ \mathrm{i} f_\kappa(r)\ \varphi_{-\kappa m} \end{pmatrix}. \tag{2.151}$$

We recall the meaning of the symbols in (2.151):

$$\kappa = \pm 1,\ \pm 2,\ \pm 3,\ \ldots \text{ (Dirac quantum number)},$$

$$j = |\boldsymbol{\kappa}| - \tfrac{1}{2} \text{ and } \begin{cases} l = \kappa & \text{for } \kappa > 0, \\ l = -\kappa - 1 & \text{for } \kappa < 0, \end{cases}$$

$$\varphi_{\kappa m} \equiv \varphi_{jlm} = \sum_{m_l, m_s} \left(l m_l \frac{1}{2}\ m_s | j m \right) Y_{lm} \chi_{m_s}.$$

$f(r)$ and $g(r)$ satisfy a system of first-order differential equations, (2.105), which contains the spherically symmetric potential $V(r)$, viz.

$$\frac{\mathrm{d}f_\kappa}{\mathrm{d}r} = \frac{\kappa-1}{r} f_\kappa - \{E - V(r) - m\} g_\kappa,$$
$$\frac{\mathrm{d}g_\kappa}{\mathrm{d}r} = -\frac{\kappa+1}{r} g_\kappa + \{E - V(r) + m\} f_\kappa. \tag{2.152}$$

g_κ is called the "large" component, f_κ the "small" component: in the nonrelativistic limit $m - |E| \ll 1$, f_κ vanishes, whilst g_κ goes over into the corresponding Schrödinger wave function (see below).

Case (a): Potential of a point-like charge
As nuclei are well confined systems the electrostatic potential (2.117) around any nucleus approaches rapidly the potential of a point charge, $V_\mathrm{c} = -Z\alpha/r$, outside the nuclear radius. Therefore, this potential is the reference case to which the actual potential should be compared. The bound states of the true potential are shifted with respect to the case of the point charge. The corresponding wave functions are more or less distorted Coulombic wave functions. Information about the true potential is primarily contained in the shifts of the energy levels, to some extent also in the wave functions and in observables derived from them.

Technically, the bound state problem is different for the two cases and the potential of the point charge must be considered separately. The reason for this technical difference is easy to understand: The true potential of an extended charge distribution is regular at the region, $r = 0$. (Consider, for instance, the homogeneous density, for which $V(r) \sim C_0 + C_1 r^2$ is parabolic.) $V_\mathrm{c}(r)$, on the contrary, is singular at $r = 0$. In any relativistic wave equation, both V_c and its square enter. This is obvious in the Klein–Gordon equation (6.36) where the term $(E - V_\mathrm{c}(r))^2$ is added to the kinetic energy. In the case of Dirac spinors we know that each component also satisfies the Klein–Gordon equation. Alternately, this may be recovered directly from (2.152) by deriving from them uncoupled second-order equations for $f(r)$ and $g(r)$ separately (so-called "iterated form" of the Dirac equation).

In either case, the term ${V_\mathrm{c}}^2(r)$, being proportional to $1/r^2$, has the same type of singularity at the origin as the centrifugal potential $l(l+1)/r^2$. This singularity, as is well-known, determines the behaviour of the radial wave functions at $r = 0$. Thus, for a regular potential we expect the standard behaviour r^l or r^{-l-1}, while for the $1/r$ potential the characteristic exponent is modified by terms of order $Z\alpha$. At $r \to \infty$, on the other hand, the two cases are obviously the same. So only the solutions inside a suitably defined matching radius R, where the two potentials differ substantially, must be derived separately. After this long introduction we now turn to the derivation of the bound Coulomb states.

For very large r the differential equations (2.152) give the approximate equation

$$g''(r) - (m^2 - E^2)\, g(r) \simeq 0$$
$$f'(r) \simeq -(E-m) g(r)\,, \quad (\text{large}\, r)$$

For bound states $E < m$ and $\lambda := \sqrt{m^2 - E^2}$ is real positive. Thus $g(r)$ behaves like $\mathrm{e}^{-\lambda r}$, $f(r)$ like $\sqrt{(m-E)/(E+m)}g(r)$. For convenience we take out a factor $1/r$, in f and g, and we make the ansatz

$$g(r) = \frac{1}{r}\mathrm{e}^{-\lambda r}\ \sqrt{E+m}\ \{y_1(r) + y_2(r)\},$$
$$f(r) = \frac{1}{r}\mathrm{e}^{-\lambda r}\ \sqrt{m-E}\ \{y_1(r) - y_2(r)\}. \tag{2.153}$$

The asymptotic behaviour of $y_1(r)$ and $y_2(r)$ must be such that the exponential factor $\mathrm{e}^{-\lambda r}$ is not compensated. Let $x := 2\lambda r$, then y_1 and y_2 satisfy the system

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = \left(1 - \frac{Z\alpha E}{\lambda x}\right) y_1 - \left(\frac{\kappa}{x} + \frac{Z\alpha m}{\lambda x}\right) y_2,$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = \left(-\frac{\kappa}{x} + \frac{Z\alpha m}{\lambda x}\right) y_1 + \frac{Z\alpha E}{\lambda x} y_2. \tag{2.154}$$

The behaviour at $r = 0$ remains to be determined, keeping in mind the remarks made before. For that purpose, we write

$$y_{1,2}(x) = x^{\gamma}\phi_{1,2}(x), \tag{2.155}$$

with $\phi_{1,2}(0) \neq 0$ but finite and determine γ from (2.154): For $x = 0$ we obtain the linear system

$$\gamma\phi_1(0) = -\frac{Z\alpha E}{\lambda}\ \phi_1(0) - \left(\kappa + \frac{Z\alpha m}{\lambda}\right)\phi_2(0),$$
$$\gamma\phi_2(0) = \left(-\kappa + \frac{Z\alpha m}{\lambda}\right)\phi_1(0) + \frac{Z\alpha E}{\lambda}\ \phi_2(0)\,.$$

This homogeneous system has a nontrivial solution only if the determinant of the coefficient matrix

$$\begin{pmatrix} \gamma + Z\alpha E/\lambda & \kappa + Z\alpha m/\lambda \\ \kappa - Z\alpha m/\lambda & \gamma - Z\alpha E/\lambda \end{pmatrix}$$

vanishes. This gives $\gamma^2 = \kappa^2 - (Z\alpha)^2$. The solutions regular at the origin require the positive square root,

$$\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}. \tag{2.156}$$

With the ansatz (2.155), where γ is given by (2.156), one derives easily the differential equations satisfied by $\phi_1(x)$ and $\phi_2(x)$

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}x} = \left\{1 - \left(\gamma + \frac{Z\alpha E}{\lambda}\right)\frac{1}{x}\right\}\phi_1 - \left(\kappa + \frac{Z\alpha m}{\lambda}\right)\frac{1}{x}\phi_2,$$
$$\frac{\mathrm{d}\phi_2}{\mathrm{d}x} = -\left(\kappa - \frac{Z\alpha m}{\lambda}\right)\frac{1}{x}\phi_1 - \left(\gamma - \frac{Z\alpha E}{\lambda}\right)\frac{1}{x}\phi_2. \tag{2.157}$$

The functions $\phi_i(x)$ represent what remains of the radial functions f and g after we have taken out the characteristic exponent (x^γ) near the origin and the exponential factor ($\mathrm{e}^{-x/2}$) at infinity. Our experience with analogous problems in nonrelativistic quantum mechanics suggests that $\phi_i(x)$ are simple polynomials (they must be orthogonal for fixed κ). In fact, one can show that they can be written in terms of the well-known confluent hypergeometric function. One way of seeing this is the following: Starting from (2.157) derive a second-order differential equation for ϕ_2 (or ϕ_1) alone. One finds

$$x\frac{\mathrm{d}^2\phi_2}{\mathrm{d}x^2} + (2\gamma + 1 - x)\frac{\mathrm{d}\phi_2}{\mathrm{d}x} - \left(\gamma - \frac{Z\alpha E}{\lambda}\right)\phi_2 = 0. \tag{2.158}$$

This is, indeed, Kummer's equation with $a \equiv \gamma - Z\alpha E/\lambda$, $b \equiv 2\gamma + 1$. The solution with the required properties is [ABS65, ART31]

$$\phi_2(x) = {}_1F_1\left(\gamma - \frac{Z\alpha E}{\lambda};\ 2\gamma + 1;\ x\right). \tag{2.159a}$$

ϕ_1 is found from the second equation (2.157) and the recurrence relation

$$x\,{}_1F_1'(a;\ b;\ x) + a\,{}_1F_1\,(a;\ b;\ x) = a\,{}_1F_1\,(a + 1;\ b;\ x)$$

for the confluent hypergeometric function. Thus

$$\phi_1(x) = \frac{Z\alpha E/\lambda - \gamma}{\kappa - Z\alpha m/\lambda}\,{}_1F_1\,(1 + \gamma - Z\alpha E/\lambda;\ 2\gamma + 1;\ x). \tag{2.159b}$$

The asymptotic behaviour of ${}_1F_1$ is, to leading order in $1/x$,

$${}_1F_1\,(a;\ b;\ x) \sim \frac{\Gamma(b)}{\Gamma(b-a)}\,(-x)^{-a} + \frac{\Gamma(b)}{\Gamma(a)}\mathrm{e}^x x^{a-b}. \tag{2.160}$$

Obviously, the second term of this must vanish if we do not wish to destroy the exponential decrease of the bound state solutions, i.e. the factor $\mathrm{e}^{-x/2}$ in (2.153). This can only be achieved if the parameter a is zero or a negative integer because in that case $1/\Gamma(a)$ vanishes. When applied to ϕ_2, this gives the condition

$$Z\alpha E/\lambda - \gamma = n', \tag{2.161}$$

with

$$\lambda = \sqrt{m^2 - E^2},\ n' = 0,\ 1, 2, \ldots$$

Does the same condition also make $\phi_1(x)$ of (2.159b) remain regular at infinity? For $n' \geq 1$ this is obvious. For $n' = 0$, a little more care is necessary: The function ${}_1F_1\ (1;\ 2\gamma + 1; x)$ is not regular but the factor $Z\alpha E/\lambda - \gamma = 0$ in front of it makes ϕ_1 vanish, provided the denominator $(\kappa - Z\alpha m/\lambda)$ does not vanish. This is what has to be checked. With $\gamma = Z\alpha E/\lambda$ we have

$$\kappa^2 = \gamma^2 + (Z\alpha)^2 = (Z\alpha)^2\ \frac{\lambda^2 + E^2}{\lambda^2} = \left(Z\alpha \frac{m}{\lambda}\right)^2$$

and therefore $\kappa = \pm Z\alpha m/\lambda$. The solution $\kappa = +Z\alpha m/\lambda$ must indeed be excluded, whilst $\kappa = -Z\alpha m/\lambda$ is acceptable. Therefore, for $n' = 0$ only negative κ is allowed.

For convenience, we set $n' = n - |\kappa|$ with $n = 1, 2, \ldots$ Our results can then be summarized as follows. From (2.161) we find

$$E_{n|\kappa|} = m \left\{ 1 + \left(\frac{Z\alpha}{n - |\kappa| + \sqrt{\kappa^2 - (Z\alpha)^2}} \right)^2 \right\}^{-1/2}, \tag{2.162}$$

where the quantum numbers n, κ and $|\kappa| - 1/2$ assume the following values:

$$\begin{aligned} &n = 1,\ 2, \ldots, \\ &\kappa = \pm\ 1,\ \ \pm 2,\ \ \ldots,\ \ \pm(n-1), -n, \\ &j = |\kappa| - \frac{1}{2} = \frac{1}{2},\ \frac{3}{2},\ \ \ldots,\ n - \frac{1}{2}. \end{aligned} \tag{2.163}$$

$\kappa = +n$, as we said above, is excluded. The integer n is the familiar principal quantum number of the nonrelativistic hydrogen atom. This can be seen, for instance, by expanding the energy eigenvalues (2.162) in terms of $Z\alpha$,

$$E_{n|\kappa|} \simeq m \left\{ 1 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^4} \left(\frac{n}{|\kappa|} - \frac{3}{4} \right) \right\}, \tag{2.164}$$

the first term of which is the rest mass, whilst the second gives the binding energy (2.143) of the nonrelativistic hydrogen atom. The third term is the first relativistic correction. This term is independent of the mass (relative to the others) but depends on the angular momentum $j = |\kappa| - \frac{1}{2}$. For constant n it is relatively more important for small values of j than for large values.

As may be seen from the exact formula (2.162) the dynamical l-degeneracy of the nonrelativistic hydrogen atom is almost completely lifted. Only energy eigenvalues of equal n and $|\kappa|$ are degenerate. As κ can be positive and negative, except for the

largest value of κ where $\kappa = -n$, all j-values except for the highest $j = n - \frac{1}{2}$ have a *two*fold dynamical degeneracy in addition to the usual directional degeneracy in m_j, the magnetic quantum number. Thus, the $2s_{1/2}$ and $2p_{1/2}$ states are degenerate, the $3s_{1/2}$ and $3p_{1/2}$ states, the $3p_{3/2}$ and $3d_{3/2}$ states, and so on.

This remaining degeneracy of the relativistic atom is lifted eventually by radiative corrections (Lamb shift).

The eigenfunctions are given by our equations (2.153, 2.155, 2.159). The normalization to 1 is best performed by making use of well-known integrals involving confluent hypergeometric functions, exponentials and powers. The result is

$$\begin{aligned} g_{n\kappa}(r) &= 2\lambda N(n,\kappa)\sqrt{m+E}\, x^{\gamma-1}\mathrm{e}^{-x/2} \\ &\quad \times \Bigg\{ -(n-|\kappa|)_1F_1(-n+|\kappa|+1;2\gamma+1;x) \\ &\quad + \left(\frac{Z\alpha m}{\lambda} - \kappa\right)_1F_1(-n+|\kappa|;2\gamma+1;x)\Bigg\} \qquad (2.165a) \\ f_{n\kappa}(r) &= -2\lambda N(n,\kappa)\sqrt{m-E}\, x^{\gamma-1}\mathrm{e}^{-x/2} \\ &\quad \times \Bigg\{ (n-|\kappa|)_1F_1(-n+|\kappa|+1;2\gamma+1;x) \\ &\quad + \left(\frac{Z\alpha m}{\lambda} - \kappa\right)_1F_1(-n+|\kappa|;2\gamma+1;x)\Bigg\}. \qquad (2.165b) \end{aligned}$$

The normalization constant $N(n,\ \kappa)$ is given by

$$N(n,\kappa) = \frac{\lambda}{m}\frac{1}{\Gamma(2\gamma+1)}\left\{\frac{\Gamma(2\gamma+n-|\kappa|+1)}{2Z\alpha(Z\alpha m/\lambda-\kappa)\Gamma(n-|\kappa|+1)}\right\}^{1/2}. \qquad (2.166a)$$

As before,

$$x = 2\lambda r, \qquad (2.166b)$$

$$\lambda = \sqrt{m^2 - E^2_{n|\kappa|}} = \frac{Z\alpha m}{\sqrt{n^2 - 2(n-|\kappa|)(|\kappa|-\gamma)}}, \qquad (2.166c)$$

$$\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}. \qquad (2.166d)$$

Strictly speaking, the orbital angular momentum l is not a good quantum number. However, in the limit of weakly relativistic motion, i.e. for $Z\alpha \ll 1$, $\sqrt{m-E} = \mathcal{O}(Z\alpha)$ is small compared to $\sqrt{m+E} \simeq \sqrt{2m}$. This shows that the wave function $g_{n\kappa}(r)$ is large compared to the wave function $f_{n\kappa}(r)$. Thus, the upper component in $\phi_{n\kappa m}$, (2.151), is large compared to the lower one, and its angular momentum l can be used to label the state, even though the lower component carries a different

angular momentum $\bar{l} = l \pm 1$. The nomenclature is approximate and refers to the corresponding nonrelativistic situation. As an example let us consider all states with $n = 2$. Here we have

$$n = 2, \kappa = -1, j = \frac{1}{2} : \begin{pmatrix} l = 0 \\ \bar{l} = 1 \end{pmatrix} \text{“}2s_{1/2}\text{-state”},$$
$$n = 2, \kappa = +1, j = \frac{1}{2} : \begin{pmatrix} l = 1 \\ \bar{l} = 0 \end{pmatrix} \text{“}2p_{1/2}\text{-state”},$$
$$n = 2, \kappa = -2, j = \frac{3}{2} : \begin{pmatrix} l = 1 \\ \bar{l} = 2 \end{pmatrix} \text{“}2p_{3/2}\text{-state”}.$$

The example shows that in the relativistic case (i.e. $Z\alpha$ not small compared to 1), the $2p_{1/2}$-state is a closer parent to the $2s_{1/2}$ than to the $2p_{3/2}$. We expect relativistic effects in the $2p_{1/2}$ to be more important than in the $2p_{3/2}$.

The limit of the purely nonrelativistic case can be verified on the expressions (2.165) for the wave functions. As

$$\lambda \simeq \frac{Z\alpha m}{n}\left(1 + \frac{(Z\alpha)^2}{2n^2}\frac{n - |\kappa|}{|\kappa|}\right),$$

this means calculating the wave functions to lowest nonvanishing order in $Z\alpha$. In this limit $m - E \simeq 0$ and $f_{n\kappa}(r) \simeq 0$. For $\kappa > 0$ we have $l = \kappa$,

$$x \simeq \frac{2Z\alpha m}{n} r = \frac{2r}{n a_B} =: z,$$

and

$$g_{n\kappa=l} \simeq 2\frac{Z\alpha m}{n}\frac{Z\alpha}{n}\frac{1}{\Gamma(2l+1)}\sqrt{\frac{\Gamma(n+l+1)}{2Z\alpha(n-l)\Gamma(n-l+1)}}\sqrt{2mz}^{l-1}\mathrm{e}^{-z/2}$$
$$\times(n-l)\left\{{}_1F_1(-n+l;2l+1;z) - {}_1F_1(-n+l+1;2l+1;z)\right\},$$

which equation, by means of the recurrence relation

$$b\left\{{}_1F_1(a;b;z) - {}_1F_1(a-1;b;z)\right\} = z\ {}_1F_1(a;b+1;z) \tag{2.167}$$

goes over into the nonrelativistic wave function $(1/r)y_{nl}(r)$ of (2.145). For negative k, $\bar{l} = -\kappa - 1 = |\kappa| - 1$, $g_{n\kappa}(r)$ must go over into $(1/r)y_{n,\bar{l}}(r)$, since in the nonrelativistic limit the states $\left(n, j = i + \frac{1}{2}\right)$ and $\left(n, j = i - \frac{1}{2}\right)$ have the same radial wave function. That this is indeed so may be verified from (2.165), and the recurrence relation

$$(1 + a - b)\ {}_1F_1(a;b;z) - a\ {}_1F_1(a+1;b;z) + (b-1)\ {}_1F_1(a;b-1;z) = 0 \tag{2.168}$$

(see exercise 2.8).

Case (b): Potential of a spherically symmetric charge distribution of finite size
For simplicity we consider a *spherically symmetric* charge distribution $\rho(r)$ of finite size. (The more general case of an arbitrary density $\rho(r)$ is dealt with in Sect. 2.8) Unlike the case of a point charge, $\rho(r)$ is assumed to be regular at the origin and to admit a Taylor expansion around $r = 0$,

$$\rho(r) = \rho_0 + \rho_1 r + \frac{1}{2!}\rho_2 r^2 + \mathcal{O}(r^3). \tag{2.169}$$

Inserting this series into the formula (2.117) for the potential this yields a similar expansion for $V(r)$:

$$V(r) = -4\pi Z\alpha^2 \left\{ \int_0^\infty \rho(r')r'\,\mathrm{d}r' - \frac{1}{6}\rho_0 r^2 - \frac{1}{12}\,\rho_1 r^3 - \frac{1}{40}\rho_2 r^4 + \mathcal{O}(r^5) \right\}. \tag{2.170}$$

This shows that $V(r)$ behaves like a parabola, $V_0 + V_1 r^2$, close to the origin. As a consequence, the behaviour of the radial solutions $f(r)$ and $g(r)$ near the origin is determined entirely by the centrifugal potential and not by the electrostatic potential. For $\kappa > 0$ the upper component of the spinor (2.151) carries the orbital angular momentum $l = \kappa$, whilst the lower component carries $\bar{l} = \kappa - 1$. We expect, therefore, the solutions regular at the origin to behave according to

$$\kappa > 0 \quad \begin{cases} g_\kappa(r) \sim r^\kappa \\ f_\kappa(r) \sim r^{\kappa-1} \end{cases} . \tag{2.171a}$$

Similarly, for $\kappa < 0$, the upper component has $l = -\kappa - 1$, the lower component has $\bar{l} = -\kappa$, so that we expect

$$\kappa < 0 \begin{cases} g_\kappa(r) \sim r^{-\kappa-1} \\ f_\kappa(r) \sim r^{-\kappa}. \end{cases} \tag{2.171b}$$

It is not difficult to prove these assertions. The second-order differential equations for $f(r)$ and $g(r)$ alone, which one derives from the system (2.152), contain the centrifugal terms $\kappa(\kappa-1)/r^2$ and $\kappa(\kappa+1)/r^2$, respectively. The characteristic exponents α and β in the ansatz $f(r) = r^\alpha \Sigma a_n r^n$ and $g(r) = r^\beta \Sigma b_n r^n$ are found to satisfy the equations

$$\alpha(\alpha+1) = \kappa(\kappa-1), \quad \beta(\beta+1) = \kappa(\kappa+1),$$

whose solutions are $\alpha_1 = \kappa - 1$, $\alpha_2 = -\kappa$ and $\beta_1 = \kappa$, $\beta_2 = -\kappa - 1$. In fact, the two cases of positive and negative κ, can be written in a particularly simple and compact form if we introduce the definitions:

$$\kappa > 0 \begin{cases} g_\kappa(r) = r^\kappa G_\kappa(r), \\ f_\kappa(r) = r^{\kappa-1} F_\kappa(r), \end{cases} \tag{2.171c}$$

$$\kappa < 0 \begin{cases} g_\kappa(r) = r^{-\kappa-1} F_\kappa(r), \\ f_\kappa(r) = -r^{-\kappa} G_\kappa(r). \end{cases} \tag{2.171d}$$

The function F_κ and G_κ obey the system of first-order differential equations

$$\begin{aligned} \frac{\mathrm{d}F_\kappa}{\mathrm{d}r} &= \{V(r) - E + m\ \mathrm{sign}\,\kappa\}\, r G_\kappa, \\ \frac{\mathrm{d}G_\kappa}{\mathrm{d}r} &= -\frac{1}{r}\{(2|\kappa|+1)G_\kappa + [V(r) - E - m\,\mathrm{sign}\,\kappa]F_\kappa\}. \end{aligned} \tag{2.172}$$

with initial conditions

$$\begin{aligned} F_\kappa(0) = a_0 \neq 0, \quad G_\kappa(0) &= -\frac{V(0) - E - m\,\mathrm{sign}\,\kappa}{2|\kappa|+1} a_0, \\ \frac{\mathrm{d}F_\kappa}{\mathrm{d}r}(0) = \frac{\mathrm{d}G_\kappa}{\mathrm{d}r}(0) &= 0. \end{aligned} \tag{2.173}$$

The system (2.172) with the initial conditions (2.173) is well adapted for numerical integration of the wave functions F and G, from which f and g are then obtained by means of (2.171c or d), respectively. These solutions, which by construction are regular at the origin, must also be regular at infinity. This condition fixes the eigenvalues $E_{n\kappa}$ for given κ. In order to achieve this, it is convenient to use the following trick.

The density $\rho(r)$ has a finite extension, i.e. beyond a certain radius R_M the density is zero or negligibly small. (In case of nuclei R_M is typically 2 to 3 times the nuclear radius.) Therefore, for $r \gtrsim R_\mathrm{M}$ the potential $V(r)$ is practically indistinguishable from $V_c = -Z\alpha/r$, the potential of a point-like charge, and the true wave functions (2.171c, d) must be linear combinations of two linearly independent solutions of the system of differential equations treated above, case (a). One such set of independent solutions for V_c could be the solution (2.165), regular at the origin, together with another solution which is singular at the origin.

The trick now consists in constructing that specific linear combination which, for *arbitrary* energy E, decreases exponentially at infinity (i.e. is regular at infinity), irrespective of its behaviour at $r = 0$. Call this solution $(f_\infty,\ g_\infty)$. It then suffices to vary E until the inner solutions $f_0,\ g_0$ which are regular at $r = 0$ and which are obtained by numerical integration, match the outer solution continuously at the point $r = R_\mathrm{M}$, i.e.

$$[f_\infty(r) g_0(r) - g_\infty(r) f_0(r)]_{r=R_\mathrm{M}} = 0. \tag{2.174}$$

For the construction of $(f_\infty,\ g_\infty)$ we return to (2.158). It is not difficult to verify that a solution independent of the regular one $\phi_2^{\mathrm{R}}(x) = {}_1F_1(a;b;x)$, where $a = \gamma - Z\alpha E/\lambda,\ b = 2\gamma + 1$, is this:

$$\begin{aligned}\phi_2^{\mathrm{I}}(x) &= x^{1-b}\,{}_1F_1(1+a-b;2-b;x)\\ &= x^{-2\gamma}\,{}_1F_1(-\gamma - Z\alpha E/\lambda;-2\gamma+1;x).\end{aligned}$$

According to the defining equations (2.153) and (2.155) ϕ^{R} and ϕ^{I} enter into f and g with the factor $x^\gamma \mathrm{e}^{-x/2}$. Thus, rf and rg are linear combinations of the functions

$$M(\varepsilon, s; x) = x^s \mathrm{e}^{-x/2}\,{}_1F_1(s-\varepsilon;2s+1;x), \tag{2.175}$$

where $\varepsilon := Z\alpha E/\lambda$ and $s = \pm\gamma$.

The asymptotic behaviour of $M(\varepsilon, s, x)$ is determined by the asymptotic form (2.160) of the confluent hypergeometric function. For real and positive x the second term in (2.160) dominates and we have

$$M(\varepsilon, \pm\gamma; x) \underset{x\to\infty}{\sim} \frac{\Gamma(\pm 2\gamma + 1)}{\Gamma(\pm\gamma - \varepsilon)} x^{-\varepsilon-1}\mathrm{e}^{x/2}. \tag{2.176}$$

We now combine the two solutions for positive and negative γ in such a way as to cancel out the exponentially increasing term (2.176). Noting that $\Gamma(\pm 2\gamma + 1) = \pm 2\gamma\Gamma(\pm 2\gamma)$, this is achieved by taking the combination

$$W(\varepsilon, s; x) := \frac{\Gamma(-2\gamma)}{\Gamma(-\gamma-\varepsilon)} M(\varepsilon, \gamma; x) + \frac{\Gamma(2\gamma)}{\Gamma(\gamma-\varepsilon)} M(\varepsilon, -\gamma; x). \tag{2.177}$$

This specific linear combination is regular at infinity for all values $\varepsilon = Z\alpha E/\lambda$.[21]

Repeating some of the steps of case (a) it is straightforward to construct the radial solution (f_∞, g_∞) from (2.177). One finds, barring arbitrary normalization,

$$rf_\infty(r) = \sqrt{m-E}\left\{\left(\kappa + \frac{Z\alpha m}{\lambda}\right)\right\} W(\varepsilon - 1, \gamma; x) - W(\varepsilon, \gamma; x)\Big\}, \tag{2.178a}$$

$$rg_\infty(r) = \sqrt{m+E}\left\{\left(\kappa + \frac{Z\alpha m}{\lambda}\right)\right\} W(\varepsilon - 1, \gamma; x) + W(\varepsilon, \gamma; x)\Big\}, \tag{2.178b}$$

Note that these solutions are indeed regular at infinity for any value of the energy E. (This still remains true when E becomes complex. This occurs if the Dirac equation contains a complex optical potential.)

Remarks. (i) The functions (2.178) are, in general, not regular at the origin because of the factor $x^{-\gamma}$ in the second term of (2.177). (ii) If one imposes regularity at the

[21] W is a Whittaker function, except for an extra factor $x^{1/2}$ on the right-hand side of (2.177).

origin, too, this term must vanish and one recovers the eigenvalue condition of the previous case (a), cf. (2.161).

2.7 Muonic Atoms and Quantum Electrodynamics

In a sense, vacuum polarization is the simplest and most fundamental radiative effect in the quantized theory of photons in interaction with matter. What is this effect and why is it fundamental? In the quantized Maxwell theory the electromagnetic forces which act between charged particles are described by the exchange of photons between these particles. The exchanged photon, through its quantum nature, can go over into all possible intermediate states which are allowed by the conservation laws of the theory. These intermediate states can annihilate and go over into the same photon state again, as sketched in Fig. 2.11. In this diagram the hatched loop stands for the sum of all many-particle and photon states which are allowed by the rules of the theory. The net effect of these insertions is a modification of the photon propagator or, in other words, of the classical forces between charged matter particles. The range of these modifications, in coordinate space, is a function of the masses of the particles in the intermediate states. The phenomenon occurs in any local gauge theory (the photon being replaced by the vector gauge bosons of the theory), and it reflects fundamental properties of the theory. This can be understood qualitatively by cutting the diagram in Fig. 2.11 as indicated by the broken line: If we change the external momenta such that the particle lines at the left and at the right of this figure represent incoming and outgoing particle-antiparticle pairs respectively, then the cut diagrams represent the total pair annihilation cross sections.

In quantum electrodynamics (QED), to lowest order in the fine structure constant α, vacuum polarization is represented by the virtual creation and re-annihilation of all possible pairs of one fermion and its antifermion, as shown in Fig. 2.12(a). This diagram (and likewise all diagrams of higher order in α), when added to the single photon exchange, leads to a modification of the photon propagator and has two basic effects: The first effect is an infinite, logarithmically divergent, contribution which, however, is the same in any diagram where the photon couples to a given particle of charge e_0. The presence of this divergent term indicates that the *bare* charge e_0 (i.e. the coupling constant appearing in the original Lagrangian), is renormalized, by an infinite amount, to the *physical* charge e. The infinity can be circumvented formally by a redefinition of the coupling constant which means replacing the bare

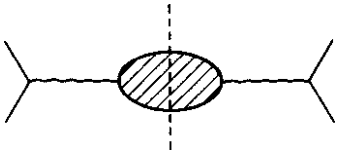

Fig. 2.11 Vacuum polarization in quantum electrodynamics leading to modification of the photon propagator

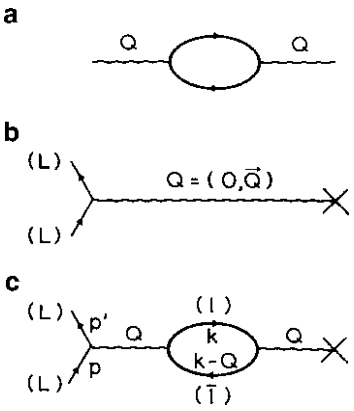

Fig. 2.12 (**a**) Modification of a photon propagator to order α, due to a loop of virtual charged particles (electron, muon, etc). (**b**) Interaction of a charged lepton (L) with an external potential (represented by the cross), (**c**) Modification of the interaction due to diagram (**b**) through lowest-order vacuum polarization

charge e_0 everywhere by the physical charge e. This prescription is formal because e_0 is necessarily infinite and yet the replacement is to be made, order by order, to the order α^n of perturbation theory to which we work, as if everything were finite. As the theory does not allow one to predict the magnitude of the charge e, this first and divergent effect of vacuum polarization is not observable.

In contrast to this, the remainder of vacuum polarization is finite and unique, and does have observable consequences. This second effect can be calculated uniquely, in successive orders of α, and can be confronted with precision measurements, as a test of QED. The physical effect of (finite) vacuum polarization is best understood in the case of an external, electrostatic potential. In this case vacuum polarization leads to a distortion of the given potential, over a distance which is characterized by the Compton wavelength $\lambda\!\!\!^{-}(i) = \hbar/m_i c$ of the particle that runs along the loop in the diagram of Fig. 2.12(a). Clearly, the longest range in this distortion effect is due to the lightest particle, the electron. Thus, vacuum polarization due to virtual *electron–positron pairs* is expected to distort the original potential over a distance of the order

$$\lambda\!\!\!^{-}(\mathrm{e}) \simeq 386\mathrm{fm}. \tag{2.179}$$

If we wish to see the quantitative importance of this effect in atoms we must check whether the orbit radii are large as compared to $\lambda\!\!\!^{-}(\mathrm{e})$ or whether they are comparable to or smaller than $\lambda\!\!\!^{-}(\mathrm{e})$. Hydrogen and muonium ($\mu^+$ e$^-$) have Bohr radii $\sim 1/\alpha m \simeq 5 \times 10^4$ fm and belong to the first class. In those "dilute" systems vacuum polarization is a small effect as compared to other radiative corrections due to vertex modification and to the anomalous magnetic moments. In contrast to these, muonic atoms have sizes which are indeed comparable to $\lambda\!\!\!^{-}(\mathrm{e})$, eq. (2.179). As a consequence, vacuum polarization is the predominant radiative correction in muonic atoms. As far as tests of QED are concerned, weakly bound systems, such as ordinary light atoms and muonium, and muonic atoms are complementary. They test different and complementary predictions of QED. We discuss first the finite and observable parts of vacuum polarization, in lowest order. We then say a little more about higher orders and give some characteristic examples. The discussion

of muonium (which is an example for the complementary situation, similar to hydrogen) can be found e.g. in Scheck (1978).

2.7.1 Observable Part of Vacuum Polarization to Order $O(\alpha)$

For the sake of simplicity let us consider the case where the photon line in Fig. 2.12(a) represents the interaction of a fermion (L) with an *external* electrostatic potential of a *point charge c*. This potential is represented by a cross in Fig. 2.12(b) and (c). According to standard Feynman rules the translation of this cross and the photon line into a formula is

$$\tilde{A}_\mu(\boldsymbol{Q}^2) = \frac{c}{(2\pi)^3}\delta_{\mu 0}\frac{1}{\boldsymbol{Q}^2}, \tag{2.180}$$

which is the Fourier transform of the usual $1/r$-potential[22]

$$\tilde{A}_\mu(\boldsymbol{Q}^2) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3 Q\, \mathrm{e}^{-\mathrm{i}Qx}\frac{c}{4\pi|\mathbf{x}|}\delta_{\mu 0}. \tag{2.180'}$$

Consider now the vacuum loop of a charged lepton l illustrated by Fig. 2.12(c). According to the Feynman rules the sum of diagrams b and c is given by

$$\begin{aligned} M &= 2\pi \mathrm{i} e_0 \overline{u_\mathrm{L}(p')}\gamma_\mu u_\mathrm{L}(p) \\ &\times\left[g^{\mu\nu} + \frac{1}{Q^2}\frac{ie_0^2}{(2\pi)^4}\int \mathrm{d}^4 k\ \mathrm{tr}\left\{\gamma^\mu \frac{1}{\not{k} - m_l + \mathrm{i}\varepsilon}\gamma^\nu \frac{1}{\not{k} - \not{Q} - m_l + \mathrm{i}\varepsilon}\right\}\tilde{A}_\nu\right], \end{aligned} \tag{2.181}$$

where $Q = p - p'$. In the specific case of elastic scattering of the fermion L in an external potential we have $p^0 = p'^0$, so that $Q^2 = -\boldsymbol{Q}^2$. In a more general diagram Q is the momentum carried by the photon lines entering and leaving the closed fermion loop (see Fig. 2.12(a)). In any such case the vacuum polarization loop of order e^2 is represented by the tensor

$$\Pi^{\mu\nu}(Q) := \frac{\mathrm{i}e_0^2}{(2\pi)^4}\int \mathrm{d}^4 k\,\mathrm{tr}\left\{\gamma^\mu \frac{1}{\not{k} - m_\ell + \mathrm{i}\varepsilon}\gamma^\nu \frac{1}{\not{k} - \not{Q} - m_\ell + \mathrm{i}\varepsilon}\right\}. \tag{2.182}$$

As it stands this integral is divergent and we must be very careful in performing algebraic manipulations on it. It is well-defined and finite only in a regularized form of QED. Methods of regularization are described in textbooks on quantum field

[22] In the standard formulation of Feynman rules one chooses natural units so that $e^2/4\pi = \alpha$.

theory. They are essential in identifying the precise nature of the singularities of divergent quantities and in isolating these from the finite parts. If the regularization respects Lorentz invariance and gauge invariance of the theory, these latter, finite parts are unique.

In what follows we assume that the tensor $\prod^{\mu\nu}(Q)$ is already regularized. For instance, using the method of Pauli and Villars, one finds

$$\Pi^{\mu\nu}_{\text{reg}}(Q) = (Q^\mu Q^\nu - Q^2 g^{\mu\nu})\ [C + \Pi(Q^2)], \tag{2.183}$$

where the first term depends on a fictitious regulator mass M,

$$C = \frac{\alpha_0}{3\pi}\ \ln\left(\frac{M}{m_\ell}\right)^2 \tag{2.184}$$

(and hence is logarithmically divergent), whilst the second term

$$\Pi(Q^2) = -\frac{2\alpha}{\pi}\int_0^1 \mathrm{d}z\ z\,(1-z)\ln\left(\frac{m_\ell^2 - Q^2 z\,(1-z)}{m_\ell^2 - \mathrm{i}\varepsilon}\right) \tag{2.185}$$

is finite and independent of the method of regularization used. It can be shown that the logarithmic divergence (2.184) represents a formal renormalization of the charge and that it can be absorbed if the bare charge is replaced by the physical charge,[23]

$$e = e_0\ \sqrt{1 - \frac{\alpha}{3\pi}\ \ln\left(\frac{M}{m_l}\right)^2}\,. \tag{2.186}$$

Finally, we note that the specific convariant form of $\Pi^{\mu\nu}$, (2.183), is a consequence of gauge invariance which requires

$$Q_\mu\ \Pi^{\mu\nu}_{\text{reg}}(Q) = 0 = \Pi^{\mu\nu}_{\text{reg}}(Q)\,Q_\nu.$$

In order to understand the physical content of the finite part $\Pi(Q^2)$ let us transform the integral (2.185) somewhat so that it may easily be transformed to coordinate space, by means of Fourier transformation. Let $z = (1-y)/2$ and, therefore, $1-z = (1+y)/2$. Then

$$\Pi(Q^2) = -\frac{\alpha}{2\pi}\int_0^1 \mathrm{d}y(1-y^2)\ \ln\left\{1 - \frac{Q^2}{m^2 - \mathrm{i}\varepsilon}\ \frac{1-y^2}{4}\right\}\ (m \equiv m_\ell).$$

[23] As we work in second order here, it is consistent to insert $\alpha = e^2/4\pi$, not $\alpha_0 = e_0^2/4\pi$ into (2.185) and the square root in (2.186).

By partial integration we can get rid of the logarithm and obtain

$$\Pi(Q^2) = \frac{\alpha}{\pi}\, Q^2 \int_0^1 \mathrm{d}y\, \frac{y^2(1-y^2/3)}{4m^2 - Q^2(1-y)^2 - \mathrm{i}\varepsilon}. \tag{2.187}$$

An equivalent representation is obtained by means of the substitution

$$\kappa^2 := \frac{4m^2}{1-y^2},$$

i.e.

$$y^2 = 1 - \frac{4m^2}{\kappa^2}, \quad \mathrm{d}\kappa^2 = \frac{\kappa^4}{2m^2}\, y\mathrm{d}y,$$

which gives

$$\Pi(Q^2) = \frac{\alpha Q^2}{3\pi} \int_{4m^2}^{\infty} \mathrm{d}\kappa^2\, \frac{(1+2m^2/\kappa^2)\ \sqrt{1-4m^2/\kappa^2}}{\kappa^2(\kappa^2 - Q^2 - \mathrm{i}\varepsilon)}. \tag{2.188}$$

Let us now return to the example of scattering in an external electrostatic potential. Inserting the result (2.188) into the amplitude (2.181), after having renormalized the charge (to the order at which we work), we obtain

$$\begin{aligned} M &= 2\pi \mathrm{i} e\, \overline{u_{\mathrm{L}}(p')}\, \gamma_\mu\, u_{\mathrm{L}}(p) \left[g^{\mu\nu} + \frac{1}{Q^2}(Q^\mu\, Q^\nu - Q^2 g^{\mu\nu})\ \Pi(Q^2) \right] \tilde{A}_\nu \\ &= 2\pi \mathrm{i} e\, \overline{u_{\mathrm{L}}(p')}\, \gamma^\nu\, u_{\mathrm{L}}\,(p)\ [1 - \Pi(Q^2)]\ \tilde{A}_\nu. \end{aligned}$$

In this example $Q^2 = -\boldsymbol{Q}^2, \tilde{A}_\nu$ is given by (2.180), $\Pi(Q^2 = -\boldsymbol{Q}^2)$ by (2.188). The factor $[1 - \Pi]\tilde{A}$ can be read as a modified external potential and may easily be transformed to coordinate space. Let

$$\tilde{V}(\boldsymbol{Q}^2) := -e[1 - \Pi\,(-\boldsymbol{Q}^2)]\tilde{A}_0(\boldsymbol{Q}^2). \tag{2.189}$$

The potential in coordinate space is the Fourier transform of $\tilde{V}$,

$$V(x) = \frac{C}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{Q} \mathrm{e}^{\mathrm{i}Qx} \left\{ \frac{1}{\boldsymbol{Q}^2} + \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} \mathrm{d}\kappa^2 \frac{(1+2m^2/\kappa^2)\ \sqrt{1-4m^2/\kappa^2}}{\kappa^2(\kappa^2 + \boldsymbol{Q}^2 - \mathrm{i}\varepsilon)} \right\} \tag{2.189$'$}$$

where we set $-ce = C$.

The integrals over $\boldsymbol{Q}$ can be performed by means of the formula

$$\frac{1}{(2\pi)^3} \int \mathrm{d}^3\ Q\ \frac{\mathrm{e}^{\mathrm{i}Qx}}{\boldsymbol{Q}^2 + a^2} = \mathrm{e}^{-ar}/4\pi r \quad (r = |\boldsymbol{x}|),$$

taking $a = 0$ in the first term, $a = \kappa$ in the second term of (2.189′). This gives

$$V(r) = \frac{C}{4\pi}\left\{\frac{1}{r} + \frac{\alpha}{3\pi}\int_{4m^2}^{\alpha} \mathrm{d}\kappa^2 \frac{(1+2m^2/\kappa^2)\sqrt{1-4m^2/\kappa^2}}{\kappa^2}\frac{\mathrm{e}^{-\kappa r}}{r}\right\}. \quad (2.190)$$

Thus, the original $1/r$ potential is modified by a superposition of Yukawa terms with ranges greater or equal $1/2m = \frac{1}{2}\lambda\!\!\!^{-}(\mathrm{e})$.

For practical use one may replace the variable κ^2 by a dimensionless integration variable $\kappa^2 = 4m^2x^2$, so that (Uehling 1935)

$$V(r) = \frac{C}{4\pi r}\left\{1 + \frac{2\alpha}{3\pi}\int_1^{\infty} \mathrm{d}x \mathrm{e}^{-2mxr}\left(1+\frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2}\right\}. \quad (2.190')$$

Remember that C is the external charge. For instance, if this is a proton then $C = |e|$; if it is a point-like nucleus then $C = +Z|e|$. (The case of an extended charge distribution is treated below).

There are two limiting situations where it is easy to estimate the integral in (2.190′). If $rm \gg 1$, i.e. if r is very large compared to $\lambda\!\!\!^{-}(\mathrm{e})$, the integrand is large only close to the lower limit of the integral. We approximate

$$\left(1+\frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2} \simeq \frac{3}{2}\sqrt{2(x-1)}$$

near $x = 1$ and substitute $x - 1 = u$, so that

$$\int_1^{\infty} \mathrm{e}^{-2mxr}\left(1+\frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x} \simeq \mathrm{e}^{-2mr}\int_0^{\infty}\mathrm{d}u\,\mathrm{e}^{-2mru}\frac{3}{2}\sqrt{2u}$$
$$= \frac{3\sqrt{\pi}}{8}\frac{\mathrm{e}^{-2mr}}{(mr)^{3/2}},$$

and therefore

$$V(r) = \frac{C}{4\pi r}\left\{1 + \frac{\alpha}{4\sqrt{\pi}}\frac{\mathrm{e}^{-2mr}}{(mr)^{3/2}}\right\}. \quad r \gg \lambda\!\!\!^{-}(\mathrm{e}). \quad (2.191a)$$

This limit is relevant in normal atoms whose orbital radii are indeed large compared to $\lambda\!\!\!^{-}(\mathrm{e})$. Because of the exponential the correction term is very small. This is in contrast to muonic atoms whose orbital radii are comparable with or smaller than $\lambda\!\!\!^{-}(\mathrm{e})$. Here vacuum polarization is a large effect.

Similarly, if $rm \ll 1$ i.e. if $r \ll \lambda\!\!\!^{-}(\mathrm{e})$ the integral (2.190′) can be solved approximately, too, and is found to be (Blomqvist 1972)

$$V(r) \simeq \frac{C}{4\pi r}\left\{1 - \frac{2\alpha}{3\pi}[\ln(mr) + C_{\mathrm{E}} + 5/6]\right\}, \tag{2.191b}$$

where $C_{\mathrm{E}} = 0.577216$ (Euler's constant).

2.7.2 *Illustration and Interpretation of Vacuum Polarization of Order $\alpha Z\alpha$*

Figure 2.13 illustrates the $1/r$ potential and the vacuum polarization potential of (2.190′): In order to get rid of dimensional quantities and of the numerical factors in front of the integral, we have multiplied $V(r)$, (2.190′), by $(-6\pi^2/\alpha Cm)$. Thus the upper curve represents the reduced vacuum polarization potential

$$\upsilon_{\mathrm{vacpol}}(u) = -\frac{1}{u}\int_1^\infty \mathrm{d}x \mathrm{e}^{-2ux}\left(1 + \frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2} \tag{2.192}$$

as a function of $u = rm \mathrel{\hat{=}} r/\lambda\!\!\!^{-}(\mathrm{e})$. The figure shows that $\upsilon_{\mathrm{vacpol}}$ is small compared to the uncorrected $1/r$ potential for distances $r \gtrsim \lambda\!\!\!^{-}(\mathrm{e})$, but becomes strong at small distances $r \ll \lambda\!\!\!^{-}(\mathrm{e})$. Furthermore, the potential due to vacuum polarization has the same sign as the $1/r$ potential everywhere. Thus, in an attractive $1/r$ potential, vacuum polarization leads to even more attraction. This result contradicts naive expectations if we think of vacuum polarization in analogy to electric polarizability of ordinary matter. Indeed, for a positive point charge, we would expect the virtual positrons to be pushed away from the origin and the virtual electrons to be pulled towards the origin. As the total induced charge is zero, this would mean that the original positive point charge effectively is smeared out over a certain region of space. However, this would lead to an effective screening of the charge and hence to a *reduction* of its field, not to the increase seen in the figure.

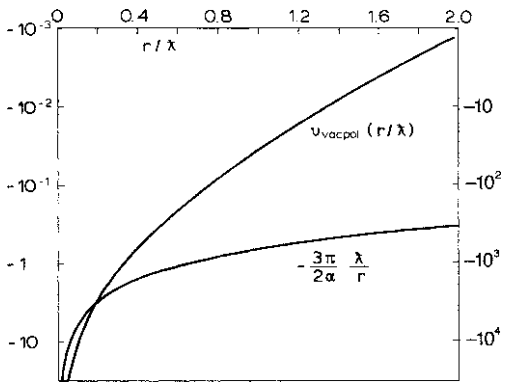

Fig. 2.13 Potential due to vacuum polarization, in dimensionless form, as a function of $r/\lambda\!\!\!^{-}$ where λ is the Compton wavelength of the virtual particle in the loop of Fig. 2.12(a). Also shown is the $1/r$ potential, scaled by the same factor as the former. The right-hand scale holds for the latter

This result becomes even more puzzling if we consider the induced charge density $\rho_{\mathrm{Pol}}(r)$ pertaining to the vacuum polarization potential: $\rho_{\mathrm{Pol}}(r) = -\Delta V_{\mathrm{Pol}}(r)$ (for technical reasons we introduce a convergence factor into the integral),

$$V_{\mathrm{Pol}}(r) = \frac{C}{4\pi r}\frac{2\alpha}{3\pi}\lim_{M\to\infty}\int_1^\infty \mathrm{d}x \mathrm{e}^{-(m/M)x}\mathrm{e}^{-2mrx}\left(1+\frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2}.$$

Using the well-known formula

$$(\Delta-\kappa^2)\frac{\mathrm{e}^{-\kappa r}}{r} = -4\pi\delta(\boldsymbol{r})$$

and taking $\kappa = 2mx$, one finds

$$\rho_{\mathrm{Pol}}(r) = \frac{C}{4\pi}\frac{2\alpha}{3\pi}\lim_{M\to\infty}\left\{4\pi\delta(\boldsymbol{r})\int_1^\infty \mathrm{d}x\mathrm{e}^{-(m/M)x}\left(1+\frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2}\right.$$
$$\left.-\frac{4m^2}{r}\int_1^\infty \mathrm{d}x\mathrm{e}^{-(m/M)x}\mathrm{e}^{-2mrx}\left(1+\frac{1}{2x^2}\right)\sqrt{x^2-1}\right\}. \qquad (2.193)$$

(The convergence factor is needed in the first term of this expression because the integral diverges logarithmically; it is irrelevant in the second term.) This polarization charge density has rather curious properties: For finite argument $r \neq 0$ it has the same sign everywhere. At $r = 0$ it has two singularities, a δ-distribution with a linearly divergent coefficient

$$4\pi\delta(r)\ln(M/m) \qquad (2.194)$$

and a $1/r^3$ pole which comes from the second term [see (2.197a) below] Yet, the integral of ρ_{Pol} over all space vanishes, as it should. This is seen by making use of the formula

$$\int_0^\infty \frac{\mathrm{e}^{-\kappa r}}{r}r^2\mathrm{d}r = 1/\kappa^2$$

with $\kappa = 2mx$:

$$\int \rho_{\mathrm{Pol}}(r)\mathrm{d}^3r = 4\pi\frac{C}{4\pi}\frac{2\alpha}{3\pi}\lim_{M\to\infty}\left\{\int_1^\infty \mathrm{d}x\mathrm{e}^{-(m/M)x}\left(1+\frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2}\right.$$
$$\left.-\int_1^\infty \mathrm{d}x\mathrm{e}^{-(m/M)x}\left(1+\frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2}\right\} = 0. \qquad (2.195)$$

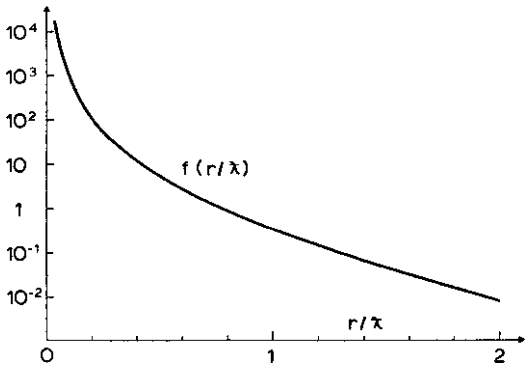

Fig. 2.14 Induced polarization charge density (2.196) in dimensionless form and as a function of $r/\lambda\!\!\!^{-}$. Note that this density is positive (attractive) for all finite r, but is singular at $r = 0$ so that its integral over all space vanishes

Figure 2.14 shows the polarization charge density for $r \neq 0$. Again, in order to get rid of dimensional quantities we have plotted the function

$$f(u) = \left(-\frac{6\pi^2}{\alpha C m^3}\right)\rho_{\mathrm{Pol}}(r)$$
$$= \frac{4}{u}\int_0^\infty \mathrm{d}x \mathrm{e}^{-2ux}\left(1+\frac{1}{2x^2}\right)\sqrt{x^2-1} \qquad (r \neq 0), \qquad (2.196)$$

where $u = mr \mathrel{\hat{=}} r/\lambda\!\!\!^{-}(\mathrm{e})$, as a function of this variable. The connection to the function (2.192) is

$$\frac{1}{u^2}\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\frac{\mathrm{d}\upsilon(u)}{\mathrm{d}u}\right) = -f(u), \quad u \neq 0.$$

From the approximate expressions (2.191a, b) for the potential at large and small r, respectively, we derive the corresponding limiting behaviour of the polarization density (2.196), viz. ($u = mr \mathrel{\hat{=}} r/\lambda\!\!\!^{-}(\mathrm{e})$),

$$r \ll \lambda\!\!\!^{-} \quad f(u) \simeq 1/u^3, \qquad (2.197a)$$

$$r \gg \lambda\!\!\!^{-} \quad f(u) \simeq \frac{3\sqrt{\pi}}{8}u^{-5/2}\mathrm{e}^{-2u}. \qquad (2.197b)$$

Clearly, the polarization charge density is not very intuitive. It has the expected property (2.195) but it is qualitatively different from a polarization density in ordinary matter. The singular term (2.194) at $r = 0$ is reminiscent of the charge renormalization (2.184). The only semi-physical statement one may make is this: When a test charge or a photon probes the field of our point charge at large distances, i.e. at low momentum transfers, then it sees what is called the *physical* charge. The closer it approaches the point charge (i.e. the larger the momentum transfers), the more the test particle sees of the *bare* charge. As the bare charge is larger than the physical charge (see (2.186)) the test particle sees an enhanced field at short

distances. The closer it comes the more enhanced the field. Unfortunately, the bare charge is infinite. The phenomenon of vacuum polarization evades simple analogies to classical polarization phenomena because infinite charge renormalization is not intuitive.

2.7.3 Radiative Corrections in Muonic Atoms

The polarization potential of order $O(\alpha C)$, (2.190′), which we have discussed so extensively, yields the dominant contribution to radiative corrections to the energies of muonic atoms. In this section we give some illustrative examples. We then discuss other radiative corrections such as the Lamb shift and vacuum polarization of higher order.

It is not difficult to generalize the result (2.190′) for a point-like charge to the case of the extended charge density of a nucleus. Here $C = -Ze^2 = -4\pi Z\alpha$ and

$$V(\boldsymbol{r}) = -Z\alpha \int \mathrm{d}^3 r' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \times \left\{ 1 + \frac{2\alpha}{3\pi} \int_1^\infty \mathrm{e}^{-2m|\boldsymbol{r}-\boldsymbol{r}'|x} \left(1 + \frac{1}{2x^2}\right) \frac{\sqrt{x^2-1}}{x^2} \mathrm{d}x \right\}, \tag{2.198}$$

where $\rho(\boldsymbol{r})$ is the charge density normalized to one.

Clearly, the singular properties of the polarization potential and charge density remain unchanged by the integration over the finite charge density.

If the charge density is spherically symmetric the expression (2.198) simplifies somewhat. The first term on the r.h.s. transforms into the uncorrected spherical potential (2.117). The integral over angular variables in the polarization potential can be done by elementary means (exercise!) giving

$$V_{\mathrm{vacpol}}(r) = -Z\alpha \frac{2\alpha}{3m} \int_0^\infty \mathrm{d}r' \frac{r'}{r} \rho(r') \{ I(|r-r'|) - I(r+r') \} \tag{2.199}$$

with

$$I(z) = \int_1^\infty \mathrm{e}^{-2mzx} \left(1 + \frac{1}{2x^2}\right) \frac{\sqrt{x^2-1}}{x^3} \mathrm{d}x.$$

In a deformed nucleus, the multipole expansion (2.133) of the charge density should be inserted into (2.198). (We see from this, in particular, that vacuum polarization will also contribute to electric quadrupole hyperfine structure.)

Let us now illustrate the importance of vacuum polarization of order ($O(\alpha Z\alpha)$ by a few practical examples and let us compare this correction to the remaining radiative corrections.

In the second column of Table 2.4 we give the transition energy $2\mathrm{p}_{1/2} - 1\mathrm{s}_{1/2}$ in four typical atoms. The third column shows the order $\alpha Z\alpha$ vacuum polarization whilst the fourth and fifth columns show vacuum polarization corrections of higher

Table 2.4 Realistic $2p_{1/2} - 1s_{1/2}$ transition energies and radiative corrections in muonic atoms. (The transition energies contain all corrections.) Numbers taken from Engfer et al. (1974)

Nucleus	Trans. energy [keV]	Vac. pol. $\alpha Z\alpha$ [keV]	Vac. pol., higher orders [keV]	Lamb shift [keV]
$^{12}_{6}$C	75.25	0.372	0.002	−0.006
$^{\text{nat}}_{20}$Ca	783.79	6.049	0.044	−0.208
$^{116}_{50}$Sn	3418.99	25.455	0.109	−1.548
$^{208}_{82}$Pb	5778.01	34.804	−0.106	−2.683

Table 2.5 Lamb shifts in muonic hydrogen and helium. All energies in meV (= 10^{-3} eV). Calculated values from Borie and Rinker (1982). Experimental values from Carboni et al. Nucl. Phys. A278(1977)381, Pohl et al. Nature 466 (2010) 213

	$^{1}_{1}$H	$^{4}_{2}$He		
Lamb shift	$2p_{3/2} - 2s_{1/2}$	$2p_{1/2} - 2s_{1/2}$	$2p_{3/2} - 2s_{1/2}$	$3d_{3/2} - 3p_{3/2}$
fine structure	8.4	0	145.7	0
vac. pol. $\alpha Z\alpha$	204.9	1665.8	1666.1	110.560
vac. pol., higher	1.5	12.0	12.0	0.925
vertex correction	−0.6	−11.1	−10.8	−0.069
recoil	~ 0	−0.2	−0.2	0.005
finite size of nucleus	-3.4 ± 0.1	$-103\langle r^2\rangle$[a)]	$-103\langle r^2\rangle$[a)]	~ 0
polarizability	$\sim 2 \times 10^{-2}$	3.1 ± 0.6	3.1 ± 0.6	~ 0
total theoretical	210.8 ± 0.1	1380.9(4.2)	1527.2(4.2)	111.42
experiment	206.295 ± 0.003	1381.3(0.5)	1527.5(0.3)	

[a)] $\langle r^2\rangle^{1/2} = 1.674 \pm 0.012$ fm.

order and the remainder of the Lamb shift, respectively. Clearly, the $O(\alpha Z\alpha)$ term is the dominant correction in all cases.

Table 2.5 shows the radiative corrections for muonic hydrogen and helium 4. The second column contains the radiative and other contributions to the energy splitting of the $2p_{3/2}$ and $2s_{1/2}$ states in hydrogen. The origin of individual contributions is indicated in the first column. The correction labeled "polarizability" is explained in the next section. The third and fourth column give the details of the Lamb shift in the $n = 2$ states of muonic ^{4}He. The last column, finally, shows an example for the Lamb shift in the $n = 3$ levels of helium.

The bottom line shows the experimental results obtained for the $n = 2$ system in helium. The agreement with the theoretical predictions is excellent. However, the uncertainty on the theoretical numbers is about a factor of ten larger than the experimental error bar. As may be seen from the table, this uncertainty stems almost entirely from the finite-size correction, i.e. from the experimental error bar of the r.m.s. radius of ^{4}He. The polarizability shift is also sizeable but is believed to be calculable to at least the accuracy indicated in the table. In this respect, the $n = 3$ states in helium and the Lamb shift in hydrogen are somewhat clearer tests of

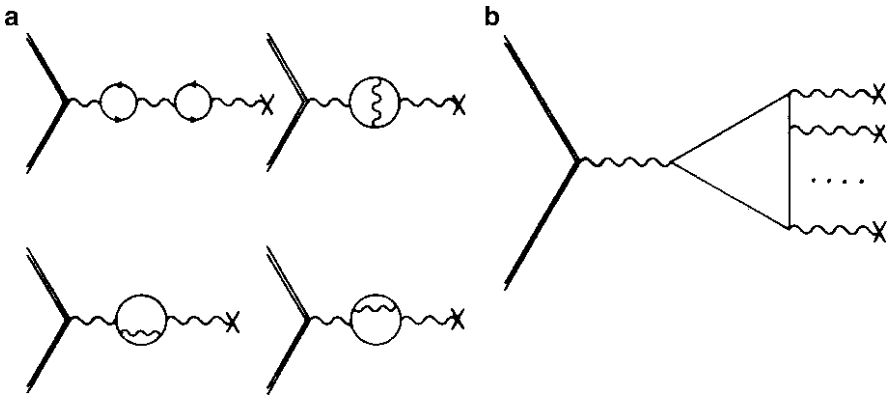

Fig. 2.15 (**a**) Vacuum polarization to order $\alpha^2(Z\alpha)$, Z being the charge of the external potential. The double line represents the external, bound muon. (**b**) Vacuum polarization of order $\alpha(Z\alpha)^{2n+1}$ for $n = 1$ and higher

radiative corrections as both the finite size and polarizability corrections are small. So far, these shifts have not been measured.

In the examples shown in Tables 2.4, 2.5 the corrections due to vacuum polarization of order higher than $\alpha Z\alpha$ are due primarily to

(i) the terms of order $\alpha^2 Z\alpha$ which are depicted in Fig. 2.15(a),
(ii) terms of order $\alpha(Z\alpha)^3$ (more generally $\alpha(Z\alpha)^{2n+1}$) which are illustrated by Fig. 2.15(b). These latter terms, even though they are proportional to α^4, appear enhanced in heavy nuclei because $(Z\alpha)$ is no longer small compared to one. For example, in lead we have $Z\alpha \simeq 0.6$. Here the correction to the $2p - 1s$ transition energy due to diagram (b) is as large as 40 eV. Other corrections such as vacuum polarization due to a virtual *muon* loop are found to be very small.

Finally, the remaining radiative corrections, i.e. the Lamb shift and the anomalous magnetic moment interaction, can be represented by additional potentials in the muon's wave equation. The details and references to the literature are found in Vol. I, Chapter III of [MUP77].

We may summarize this section by saying that vacuum polarization of order $\alpha Z\alpha$ is tested at a level of a few parts per million. The higher order terms of vacuum polarization are tested to about 20%. This provides another piece of evidence for the success of quantum electrodynamics, which is complementary to the very impressive classical tests of QED in electronic atoms and in the g-factor anomaly of electron and muon.

2.8 Deep Inelastic Scattering

2.8.1 Inclusive Electron Scattering

Scattering experiments of electrons on hadronic targets may be designed in yet another, qualitatively different, manner. Suppose the experimental arrangement is such that, while the energy of the incident electron $E \gg m_e$ is held fixed, the energy E' and the angular distribution of the outgoing electron are measured, irrespective of the final state into which the hadron has turned. This means, in other words, that the experiment determines the doubly differential cross section for *inclusive scattering*

$$\frac{d^2\sigma(E - E', \theta)}{dE' d\Omega'}(e + h \to e' + X), \tag{2.200}$$

where h is a nucleon (proton or neutron), or a nucleus, X stands for the sum and integral over all final hadronic states that can be reached by electro-excitation at the energy transfer $E - E'$ and momentum transfer $q^2 = -(k - k')^2$ (corresponding to the scattering angle θ). Both elastic scattering and inelastic scattering to specific final states probe coherent properties such as (ground state or transition) charge and current densities. If both the energy transfer and the momentum transfer are chosen large enough, inclusive scattering becomes insensitive to coherent properties such as wave functions describing the target h but, in turn, probes the "granularity" of the target. As both nuclei and nucleons are made up of constituents whose spatial size is smaller than the target's size, this means that inclusive, deep inelastic scattering probes the constituents rather than the hadron h which they form. This is particularly interesting in the case of nucleonic targets because this is one of the ways to study the nature of quarks, or, more generally, partons in hadrons. Neither quarks nor gluons can exist as free particles. Only composite states with certain well-defined quantum numbers describe physical hadrons. Indeed, historically, deep inelastic scattering was essential in unravelling the constituent structure of proton and neutron.

In the approximation which assumes that electro-excitation is due to the exchange of *one* virtual photon, the cross section (2.200) can be expressed in terms of a few Lorentz scalar, hadronic *structure functions* and some known functions of energy and scattering angle. The structure functions which are extracted from experiment, can then be analyzed in terms of the internal structure of the target h.

For the sake of definiteness take h to be a proton of four-momentum p, and let k and k' denote the four-momenta of the electron before and after the scattering process. In a first step we write down the cross section for the process $e + p \to e + m$ where m is a final state, containing m particles, which carries the right quantum numbers and is in accord with conservation of energy and momentum. A process of this type is sketched in Fig. 2.16. According to the Feynman rules for quantum

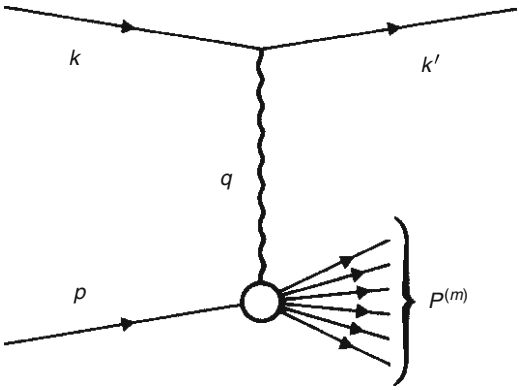

Fig. 2.16 An electron of momentum k converts a proton (momentum p) to an m-particle state, via exchange of a virtual photon

electrodynamics, App. C, the transition amplitude corresponding to the diagram of Fig. 2.16 reads

$$R^{(m)} = -\mathrm{i}\frac{2\pi e^2}{q^2}\overline{u(k')}\gamma_\mu u(k)\langle m|j^\mu(0)|p\rangle\delta(p+k-k'-P^{(m)}).$$

Here, $P^{(m)}$ is the sum of four-momenta in the final state m, i.e. $P^{(m)} = p_1 + p_2 + \ldots + p_m$, $k - k'$ is the momentum transfer delivered by the electron, $j^\mu(x)$ denotes the electromagnetic current operator. Thus, in the approximation of one-photon exchange $q := k - k'$ is the four-momentum of the virtual photon. The T-matrix element needed in the expression (B.3) for the cross section is obtained from $R^{(m)}$ by the defining relation (B.2). In the laboratory system, the struck proton is at rest, $p = (M, \mathbf{0})$, while $k = (E, \boldsymbol{k})$, and $k' = (E', \boldsymbol{k}')$. As the energy of the incident electron is large compared to the rest mass, $E \gg m_\mathrm{e}$, the electron mass can be neglected in the kinematics so that

$$|\boldsymbol{k}| \approx E, \quad |\boldsymbol{k}'| \approx E', \quad \text{and} \quad q^2 = (k-k')^2 \approx -2EE'(1-\cos\theta),$$

where q^2 is the same as t, (2.51b). In the same approximation the flux factor (B.4) is approximately ME. Thus, integrating over the momenta of the hadronic final state and summing over all spin orientations,

$$\begin{aligned}\mathrm{d}^2\sigma^{(m)} = {} & \frac{(2\pi e)^4}{2MEq^4}\left(\frac{1}{2}\sum_{\text{spins}}\ell^*_\mu\ell_\nu\right) \\ & \frac{1}{2}\sum_{\text{spins}}\int\frac{\mathrm{d}^3p_1}{2E_1}\cdots\int\frac{\mathrm{d}^2p_m}{2E_m}\langle p|j^{\mu\dagger}(0)|m\rangle\langle m|j^\nu(0)|p\rangle\delta(p+k-P^{(m)}) \\ & \times\frac{\boldsymbol{k}'^2 d\,|\boldsymbol{k}'|\mathrm{d}\Omega'}{2E'}.\end{aligned}$$

Here ℓ_μ stands for the matrix element of the electromagnetic current taken between electron spinors in momentum space, $\ell_\mu = \overline{u(k')}\gamma_\mu u(k)$, $\boldsymbol{k}'^2 \approx E'^2$, and $\mathrm{d}|\boldsymbol{k}'| \approx \mathrm{d}E'$. Summing over all final states which are allowed by the selection rules and by energy-momentum conservation, one obtains the inclusive cross section (with $e^2/4\pi = \alpha$, in natural units)

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega'} = \alpha^2 \frac{E'}{ME}\frac{1}{q^4} w_{\mu\nu} W^{\mu\nu}. \tag{2.201}$$

In this expression the Lorentz tensors $w_{\mu\nu}$, $W^{\mu\nu}$ are defined as follows. The *leptonic* contribution is contained in the tensor $w_{\mu\nu} := \left(\Sigma_{\text{spins}} \ell^*_\mu \ell_\nu\right)/2$, where the sum over all spin orientations is taken. It is easily evaluated by means of the trace formulae (C.2) and (C.4) of App. C2, viz.

$$w_{\mu\nu} = \frac{1}{2}\mathrm{tr}\{(\not{k} + m_\mathrm{e})\gamma_\mu(\not{k}' + m_\mathrm{e})\gamma_\nu\} \approx 2(k_\mu k'_\nu + k'_\mu k_\nu - (kk')g_{\mu\nu}), \tag{2.202}$$

the second, approximate, equality holding if m_e is neglected. The *hadronic* tensor is defined by

$$\begin{aligned} W^{\mu\nu} := 2\pi^2 \left(\frac{1}{2}\sum_{\text{spins}}\right) {\sum_m}' \langle p|j^\mu(0)|m\rangle\langle m|j^\nu(0)p\rangle (2\pi)^4 \\ \times \delta(P^{(m)} + k' - p - k), \end{aligned} \tag{2.203}$$

where $P^{(m)} = p_1 + \cdots + p_m$, the ${\sum_m}'$ being shorthand for

$${\sum_m}' = \sum_m \int \frac{\mathrm{d}^3 p_1}{2E_1} \cdots \int \frac{\mathrm{d}^3 p_m}{2E_m}, \quad \text{with} \quad E_i = p^{(i)0}.$$

In (2.203) we have made use of the fact that the electromagnetic current operator is hermitean. Note that $W^{\mu\nu}$ depends only on matrix elements of j^μ between on-shell states. Owing to the delta distribution in the four-momenta, the sum Σ'_m can be understood to stand for the completeness relation for states with total energy–momentum equal to $P^{(m)} = (p + k - k') = p + q$. As a consequence $\Sigma'_m|m\rangle\langle m|$, taken between $\langle p|j^\mu$ and $j^\nu|p\rangle$, can be replaced by unity. Furthermore, writing the delta distribution as an integral and making use of a translation formula of the type (2.42′), we have

$$\begin{aligned} (2\pi)^4\delta(p + q - P^{(m)})\langle p|j^\mu(0)|m\rangle &= \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}(p+q-P^{(m)})}\langle p|j^\mu(0)|m\rangle \\ &= \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}qx}\langle p|j^\mu(x)|m\rangle. \end{aligned}$$

Hence, an alternative way of writing (2.203) is the following

$$W^{\mu\nu} = 2\pi^2 \left(\frac{1}{2}\sum_{\text{spins}}\right) \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x} \langle p|j^\mu(x) j^\nu(0)|p\rangle, \quad (q = k - k'). \tag{2.204}$$

In this expression one can replace the product $j^\mu(x)j^\nu(0)$ by the commutator of these operators because the extra term that is added vanishes due to energy–momentum conservation. This is seen as follows: Making use again of a translation formula,

$$\begin{aligned}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x} \langle p|j^\nu(x) j^\mu(x)|p\rangle &= \sum_m{}' \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x} \mathrm{e}^{-\mathrm{i}(p-P^{(m)})\cdot x} \langle p|j^\nu(0)|m\rangle\langle m|j^\mu(0)|p\rangle \\ &= \sum_m{}' (2\pi)^4 \delta^{(4)}(q + P^{(m)} - p) \langle p|j^\nu(0)|m\rangle\langle m|j^\mu(0)|p\rangle.\end{aligned}$$

The 0-component of the delta distribution takes care of energy conservation and reads $\delta^{(1)}(E - E' + P^{(m)0} - M)$, in the laboratory system. As $E - E' \geq 0$ and $P^{(m)0} > M$, the argument of this delta distribution can never be zero, hence the whole expression vanishes. Thus, we find the following, general expression for $W^{\mu\nu}$

$$W^{\mu\nu} = 2\pi^2 \left(\frac{1}{2}\sum_{\text{spins}}\right) \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x} \langle p|[j^\mu(x), j^\nu(0)]|p\rangle. \tag{2.204'}$$

This form of the hadronic tensor is often useful in deriving, for instance, general properties of the hadronic contribution (such as crossing symmetry or Ward identities).

The following result, although not needed in the definition of structure functions and the derivation of cross sections which follows, is nonetheless of intuitive interest.

Remark: One shows that the amplitude (2.204) is intimately related to the Compton amplitude for the scattering of a photon with mass $q^2 \neq 0$, hence a virtual photon, from a proton. The exact relation is

$$\text{Im } T(\gamma(q) + p \to \gamma(q) + p, \theta = 0) = \frac{e^2}{(2\pi)^5} \varepsilon_\mu \varepsilon_\nu W^{\mu\nu},$$

where ε_μ denotes the polarization of the photon. Thus $W^{\mu\nu}$ is proportional to the imaginary part of the Compton amplitude in the forward direction, for a photon that is not on its mass shell. By the optical theorem (Sect. 6.1.3) this means that it is proportional to the total cross section for scattering of virtual photons of mass q^2 on protons.

2.8.2 Covariant Decomposition of $W^{\mu\nu}$ and Cross Section

Clearly, $W^{\mu\nu}$ is a Lorentz tensor in momentum space and, as such, must be decomposable into covariants multiplied by Lorentz scalar functions, in close analogy to the decomposition (2.46) of a current matrix element. Conservation of the electromagnetic current, $\partial_\mu j^\mu(x) = 0$, implies the relations

$$q_\mu W^{\mu\nu} = 0, \qquad W^{\mu\nu} q_\nu = 0. \tag{2.205}$$

The first of these follows if we make the replacement $q_\mu e^{iq\cdot x} = (\partial_\mu e^{iq\cdot x})/i$ and shift the partial derivative to $j^\mu(x)$ by partial integration. The second follows by observing that the argument x in (2.204) can be shifted to the second operator by means of the translation $\langle p|j^\mu(x)j^\nu(0)|p\rangle = \langle p|j^\mu(0)j^\nu(-x)|p\rangle$, and by substituting $x \to -x$. An analysis of physical dimensions in (2.201) shows that $W^{\mu\nu}$ is dimensionless. Indeed, the left-hand side of (2.201) has dimension E^{-3}. The kinematic factor on its right-hand side has dimension E^{-5}, while the electron tensor $w_{\mu\nu}$ has dimension E^2.

The only Lorentz tensors constructed from the available momenta and the invariant tensors $g_{\mu\nu}$ and $\varepsilon_{\mu\nu\sigma\tau}$ which obey the conservation conditions above are

$$\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right), \left(p^\mu - q^\mu \frac{(pq)}{q^2}\right)\left(p^\nu - q^\nu \frac{(pq)}{q^2}\right), \qquad \varepsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau.$$

While the first two of these are genuine Lorentz tensors (i.e. transform with $\Lambda \otimes \Lambda$, where Λ is a Lorentz transformation), the third is a pseudotensor (i.e. transforms with $(\det\ \Lambda)\ (\Lambda \otimes \Lambda)$. As the two current operators in (2.204) both have the same, definite parity, their product is even and $W^{\mu\nu}$ can depend only on the first two tensors, not on the third. Therefore, the most general decomposition of the hadronic tensor must have the form

$$W^{\mu\nu} = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) W_1 + \frac{1}{M^2}\left(p^\mu - q^\mu \frac{(pq)}{q^2}\right)\left(p^\nu - q^\nu \frac{(pq)}{q^2}\right) W_2. \tag{2.206}$$

The functions W_1 and W_2 which are called *structure functions* are Lorentz *scalars* and, as such, can depend only on scalar variables. In inclusive scattering the only Lorentz scalar variables are

$$\begin{aligned} q^2 &= (k - k')^2 \overset{\text{lab}}{=} 2(m_e^2 - EE' + \boldsymbol{k}\cdot\boldsymbol{k}') \approx -4EE'\sin^2(\theta/2), \\ \nu &:= \frac{(pq)}{M} \overset{\text{lab}}{=} E - E'. \end{aligned} \tag{2.207}$$

(The third kinematic invariant p^2 is constant and equals M^2.) The variable ν is defined such as to be equal to the energy loss of the electron in the laboratory

system. The factor M^2 in the denominator of the second term in the defining equation (2.206) is introduced in order to make both structure functions dimensionless. In summary, the Lorentz scalar functions

$$W_1(\nu, q^2),\ W_2(\nu, q^2) \tag{2.208}$$

contain all the information on the structure of the target that can be extracted by means of electron scattering at any transfer of energy and momentum.

The cross section for inclusive scattering is computed as follows. Since q^2 and ν, as defined in (2.207), are the relevant variables, it is convenient to derive the doubly differential cross section with respect to these variables instead of E' and θ. From $E' = E - \nu,\ \cos\theta = 1 + q^2/(2E(E-\nu))$ the Jacobian $\partial(E',\ \cos\theta)/\partial(\nu,\ q^2)$ is found to be $(2EE')^{-1}$. Thus

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu \mathrm{d}q^2} = \frac{\pi}{EE'}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}.$$

Inserting the result (2.202) and the decomposition (2.206) into the expression (2.201) one finds

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu \mathrm{d}q^2} = \frac{\pi}{EE'}\frac{2\alpha^2 E'}{q^4 EM}\left\{W_1\left[(kk') + \frac{2}{q^2}(kq)(k'q)\right] - \frac{W_2}{M^2}\left[(kk')\left(M^2 - \frac{(pq)^2}{q^2}\right)\right.\right.$$
$$\left.\left. -2\left((kp) - \frac{1}{q^2}(qp)(qk)\right)\left((k'p) - \frac{1}{q^2}(qp)(qk')\right)\right]\right\}.$$

Indeed, to take an example, the term that multiplies W_1 is

$$(k_\mu k'_\nu + k'_\mu k_\nu - (kk')g_{\mu\nu})\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) = 2\frac{(qk)(qk')}{q^2} - 3(kk') + 4(kk'),$$

the last term following because $g_{\mu\nu}g^{\mu\nu} = 4$.

Remembering that the electron mass is neglected we have

$$(kq) \approx -(kk') \approx -(k'q), q^2 \approx -2(kk'), (kk') \approx EE'(1-\cos\theta),$$

so that W_1 obtains a factor $2EE'(1-\cos\theta)$ while W_2 appears multiplied with

$$-EE'(1-\cos\theta) - \frac{1}{2}(E-E')^2 + \frac{1}{2}(E+E')^2 = EE'(1+\cos\theta).$$

This gives the final result

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu \mathrm{d}q^2} = \frac{4\pi\alpha^2 E'}{q^4 EM}\left(2W_1(\nu, q^2)\sin^2\frac{\theta}{2} + W_2(\nu, q^2)\cos^2\frac{\theta}{2}\right) \tag{2.209}$$

It should be clear from the discussion above that this general formula contains all scattering processes that occur if an electron with fixed energy E hits a proton. For instance, in elastic scattering the kinematic variables (2.207) are related by $q^2 + 2Mv = 0$. This is easily verified: Energy–momentum conservation requires $p + k = p' + k'$. The process is elastic if $p'^2 = M^2 = p^2$. Thus, with $q = k - k'$ we have $p'^2 = (q + p)^2$ and, hence, $q^2 + 2(qp) = q^2 + 2Mv = 0$. As a consequence we expect the structure functions W_1 and W_2 to contain additive contributions proportional to $\delta(v + q^2/2M)$ which reproduce the expression (2.57′) for elastic scattering that we calculated in Sect. 2.4.2,3. Similarly, if the scattering process leads to an isolated resonance, viz. $e + p \to e + \Delta$, $p'^2 = M_\Delta^2$ where M_Δ is the mass of that resonance. The structure functions should then contain terms proportional to $\delta\left(v + q^2/2M - \left(M_\Delta^2 - M^2\right)/2M\right)$.

The general kinematic situation will be as sketched in Fig. 2.17: In the plane spanned by the variables v and $(-q^2)$ the kinematics of the elastic process is the straight line $q^2 + 2Mv = 0$ and the kinematics of production of a single resonance is the straight line $q^2 + 2Mv - \left(M_\Delta^2 - M^2\right) = 0$. The threshold for the continuum where more than one particle is produced at the hadronic vertex lies somewhere in between. Figure 2.17 also shows typical kinematic lines which are obtained if the energy loss is varied, while the scattering angle is held fixed, $\theta = \text{const}$.

The kinematic domain where both $(-q^2)$ and v are large i.e. the domain far to the right in Fig. 2.17, is called the deep inelastic region. It is this domain which is the most revealing with regard to the constituent structure of the proton.

2.8.3 An Example: Elastic Scattering from the Proton

In the case of elastic scattering the final state is again the proton, the momentum transfer is $q = k - k' = p' - p$. The sum over intermediate states Σ'_m reduces to $m = 1$ and to an integral over the momentum p' of the proton in the final state. The hadronic matrix element has the familiar decomposition (2.46) with $F_3(Q^2) \equiv 0$, or, when transformed by means of the Gordon identity, the covariant form (2.46′). We consider first the integration over the delta distribution in (2.203). With $\{\ldots\}$ denoting the remainder of the integrand, we have

$$\int \frac{\mathrm{d}^3 p'}{2E'_p} \delta^{(4)}(k + p - k' - p')\{\cdots\} = \delta^{(1)}(E + E_p - E' - \omega') \frac{1}{2\omega'} \{\cdots\}_{p'=p+q}$$

$$\overset{\text{lab}}{=} \frac{1}{2\omega'} \delta^{(1)}(v + M - \omega')\{\cdots\},$$

where we have set $\omega' = \sqrt{M^2 + (\boldsymbol{k} - \boldsymbol{k}')^2} = \sqrt{M^2 + v^2 - q^2}$. Since (2.209) depends on v, not on ω', we must rewrite the argument of the delta distribution in terms of the variable v. This is done by means of the formula

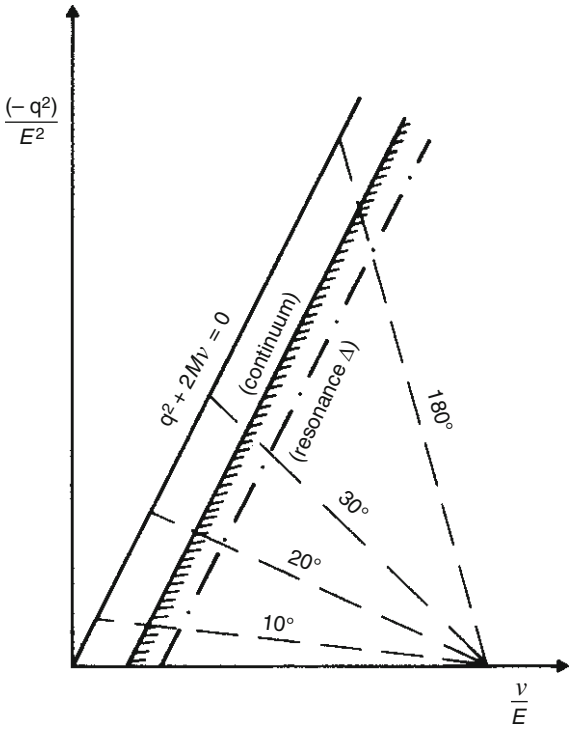

Fig. 2.17 Kinematics of electron–proton scattering to any final state. For a given final hadronic state the variables ν and q^2 are linearly dependent, as shown. Also, lines of constant scattering angle are drawn

$$\delta(f(x)) = \frac{\delta(x_0)}{|f'(x_0)|}, \text{ with } x_0 \text{ a single zero of } f(x).$$

Thus, with $\mathrm{d}(\nu + M - \omega'(\nu))/\mathrm{d}\nu = 1 - \nu/\omega'$,

$$\frac{1}{2\omega'}\delta(\nu + M - \omega') \to \frac{1}{2\omega'}\frac{\omega'}{\omega' - \nu}\delta\left(\nu + \frac{q^2}{2M}\right) \to \frac{1}{2M}\delta\left(\nu + \frac{q^2}{2M}\right).$$

The hadronic tensor for elastic scattering becomes

$$W^{\mu\nu}_{\text{elastic}} = \frac{1}{4}\sum_{r,s=1}^{2} \delta\left(\nu + \frac{q^2}{2M}\right)\frac{1}{2M}\overline{u^{(r)}(p)}\left\{(F_1+F_2)\gamma^\mu - (p+p')^\mu\frac{F_2}{2M}\right\}u^{(s)}(p').$$

$$\overline{u^{(s)}(p')}\left\{(F_1 + F_2)\gamma^\nu - (p + p')\nu\frac{F_2}{2M}\right\}u^{(r)}(p).$$

The traces are the same as in Sect. 2.4.2. and we do not repeat their calculation here. However, instead of expressing the kinematic invariants in terms of s and t (as we did in Sect. 2.4.2.) we write the result such as to conform to the general form (2.206). Using $(pq) = -q^2/2$, and $(pp') = M^2 - q^2/2$ one finds

$$W^{\mu\nu}_{\text{elastic}} = \delta\left(\nu + \frac{q^2}{2M}\right)\frac{1}{2M}\left\{-\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(-\frac{q^2}{2}(F_1 + F_2)^2\right)\right.$$
$$+\frac{1}{M^2}\left(p^\mu - q^\mu\frac{(pq)}{q^2}\right)\left(p^\nu - q^\nu\frac{(pq)}{q^2}\right)2M^2$$
$$\left.\times\left(F_1^2 - \frac{q^2}{4M^2}F_2^2\right)\right\}.$$

It is convenient to replace F_1 and F_2 by the electric and magnetic (Sachs) form factors (2.58) and (2.61), with $t = q^2$ as defined in (2.51b). Then

$$F_1^2 - \frac{q^2}{4M^2}F_2^2 = \frac{1}{1 - q^2/(4M)}\left(G_{\text{E}}^2 - \frac{q^2}{4M^2}G_{\text{M}}^2\right),$$

so that the structure functions describing elastic scattering are found to be

$$W_1^{\text{elastic}} = -\delta\left(\nu + \frac{q^2}{2M}\right)\frac{q^2}{4M}G_{\text{M}}^2 \tag{2.210}$$

$$W_2^{\text{elastic}} = \delta\left(\nu + \frac{q^2}{2M}\right)\frac{M}{1 - q^2/(4M^2)}\left[G_{\text{E}}^2 - \frac{q^2}{4M^2}G_{\text{M}}^2\right]. \tag{2.211}$$

It remains to insert this result into (2.209) and to integrate over the variable ν. Finally, the resulting cross section $\mathrm{d}\sigma/\mathrm{d}q^2$ is easily converted to the form $\mathrm{d}\sigma/\mathrm{d}\Omega'$ through

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'}2\pi\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}q^2}.$$

q^2 is the same as the variable t for which we derived the expression (2.36) in the laboratory system (neglecting m_{e}). There we also calculated the derivative

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} \equiv \frac{\mathrm{d}q^2}{\mathrm{d}(\cos\theta)} = \frac{2E^2}{[1 + (E/M)(1 - \cos\theta)]^2}.$$

Using the relation (in the laboratory system) $E' = E/[1 + E/M(1 - \cos\theta)]$ and collecting results, one indeed rediscovers the formula (2.57′) for the elastic differential cross section due to Rosenbluth (1950).

2.8.4 Counting Quark Generations

The annihilation of positrons on electrons in a *colliding beam* experiment, in principle, is a simple and direct way of discovering new quark generations. Indeed, at the level of the hadronic constituents, e^+e^--annihilation into quark-antiquark pairs may be compared to the well-known QED process $e^+e^- \to \mu^+\mu^-$. For s large as compared to m_μ^2 the production cross section for the latter is calculated to be

$$\sigma^{\text{tot}}(e^+e^- \to \mu^+\mu^-) \simeq \frac{4\pi\sigma^2}{3s}.$$

(For a derivation of this formula see e.g. [QP07], Chap. 10.) All quarks carry nonvanishing electric charge. Therefore, a pair of a quark f of mass M_i and charge e_i, and its antiquark $\bar{f}$ is created in e^+e^--annihilation as soon as the squared center-of-mass energy is larger than the corresponding threshold $s_i = (2M_i)^2$, $s \geq s_i$. The ratio of the cross section for creation of all such pairs up to this threshold to the cross section for creation of a $\mu^+\mu^-$-pair is found to be (see, e.g., [QP07])

$$\frac{\sigma_{\text{tot}}(e^+e^- \to \sum (f_i^+ f_i^-))}{\sigma_{\text{tot}}(e^+e^- \to \mu^+\mu^-)} \simeq 1 + \sum e_i^2 \sqrt{1 - 12M_i^2/s^2 - 16M_i^4/s^3}, \quad (2.212)$$

e_i being the electric charge of the quark f_i. Thus, at every threshold for the production of a new pair $\left(f_i^+ f_i^-\right)$ the ratio (2.212) is expected to increase by an amount roughly proportional to the square e_i^2 of the charge. As an example, it may be instructive for the reader to draw this ratio for the known quark generations, in the order of increasing masses,

$(\bar{s}s)$ (mass 95 MeV, charge $-1/3$),
$(\bar{c}c)$ (mass 1.27 GeV, charge $+2/3$),
$(\bar{b}b)$ (mass 4.68 GeV, charge $-1/3$), and
$(\bar{t}t)$ (mass 171.2 GeV, charge $+2/3$).

Indeed, this stepwise increase of the ratio (2.212) as a function of the variable s is born out by experiment. Up to the inclusion of colour (see Sect. 3.5.1 below) and of corrections that are calculable from Quantum ChromoDynamics (QCD), there is good agreement between data and predictions.

2.8.5 Deep Inelastic Scattering and Parton Structure of Nucleons

The technique of deep inelastic scattering on protons and neutrons is very powerful in unravelling the internal structure of these particles in terms of their constituents. Historically the first empirical evidence for the fact that quarks were indeed genuine constituents of nucleons, and not merely mathematical tools for classifying hadrons in SU(3) multiplets, came from hadronic total cross sections such as $\sigma(pp)$ and

$\sigma(\pi p)$ whose ratio seemed to be determined by the number of quarks or antiquarks contained in protons and pions, respectively[24]. The amazing observation was that, e.g., $\sigma(pp)/\sigma(\pi p)$ at sufficiently high energy was equal to 3/2, reflecting the fact that a proton is made predominantly out of three quarks while a pion is predominantly a quark-antiquark state. In analogy to atomic physics the three quarks *uud* in the proton, and *udd* in the neutron are called *valence quarks*. This picture of nucleons in an *additive quark model*, does not exclude the presence of other stuff in a proton, depending on the momentum scale at which it is being probed. Once the quark structure of nucleons was established, some obvious questions that came up were: How to confirm the third-integral electric charges of quarks? How does the spin of the nucleon arise, from the spins of the constituent quarks, from relative angular momenta, or from other components of its ground state?

Remarkably, deep inelastic scattering of electrons, and also of neutrinos, after the necessary generalization to the charged and neutral weak interactions (see below), allows to probe the nucleon's structure in much more detail. The information that can be obtained by deep inelastic scattering of electrons is contained in the structure functions W_1, W_2, cf. the definition (2.206), and a third such function if spin-sensitive measurements are made. In (2.207) the invariants $q^2 = (k - k')^2$ and $\nu = (pq)/M$ were defined. It is customary to use instead of these the variables

$$Q^2 := -q^2, \tag{2.213a}$$

defined such as to be positive everywhere in the physical region, and

$$x := \frac{Q^2}{2M\nu}, \tag{2.213b}$$

$$y := \frac{(qp)}{(kp)}, \tag{2.213c}$$

both of which have simple physical interpretations. For example, with respect to the nucleon's rest frame,

$$y := \frac{(qp)}{(kp)} = \frac{\nu}{E} = \frac{E - E'}{E}$$

is the fraction of the energy of the electron lost in the collision. The variable x, in turn, is the fraction of the nucleon's momentum carried by the quark which is struck in the process. This can be seen as follows.

The proton or neutron is assumed to be an ensemble of a small number of loosely bound constituents. These constituents which are called *partons*, in the simplest case, are just the quarks of the additive quark model. In this picture the quarks move almost freely along the momentum of the proton so that quark number i carries a

[24] H.J. Lipkin, F. Scheck; Phys. Rev. Lett **16** (1965) 71; E.M. Levin, L.L. Frankfurt; JETP Letters **2** (1965) 65.

fraction $p_i = \xi p$ of the proton's momentum with $0 < \xi < 1$. Neglecting all masses the Mandelstam squared energy variable of the electron-quark system is

$$s^{(i)} = (p_i + k)^2 \simeq 2(p_i k) = 2\xi(pk).$$

If s denotes the corresponding invariant for the electron-*proton* system, $s = (p + k)^2 \simeq 2(pk)$, then $s^{(i)} = \xi s$. It seems natural to assume that the scattering of the electron on quark i is elastic, so that $p_i' = p_i + q$. The mass of the quark in the final state being negligible, too, one has

$$0 \simeq (p_i + q)^2 = 2(p_i q) + q^2 = 2\xi(pq) - Q^2 = 2\xi M\nu - Q^2 = 2M\nu(\xi - x).$$

Therefore, in the limit of vanishing masses $x = \xi$, as was to be shown.

The invariant cross section is obtained from (2.56) above with $F_1 \equiv 1$ and $F_2 \equiv 0$, and by neglecting the masses M and m,

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t^{(i)}} &\simeq \frac{4\pi\alpha^2 e_i^2}{t^{(i)2}} \frac{s^{(i)2} + t^{(i)}s^{(i)}}{s^{(i)2}} \left\{1 + \frac{t^{(i)2}}{2(s^{(i)2} + t^{(i)}s^{(i)})}\right\} \\ &= \frac{2\pi\alpha^2 e_i^2}{t^{(i)2}} \frac{s^{(i)2} + (s^{(i)} + t^{(i)})^2}{s^{(i)2}}. \end{aligned} \tag{2.214}$$

The distribution of the proton's momentum p among the partons/quarks is measured by a *parton distribution function* $f_i(\xi)$ for each species i. These functions describe the internal structure of the proton. Since there is no known method for calculating them, they are usually taken from empirical fits to a set of scattering data.

With $t^{(i)} = q^2 = -Q^2$, $s^{(i)} = \xi s$, and $\xi = x$ the cross section for the proton is obtained from (2.214) by integration over the distribution function and reads

$$\frac{\mathrm{d}\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_i^2}{Q^4} \int_0^1 \mathrm{d}x \; f_i(x) \left\{1 + \left(1 - \frac{Q^2}{xs}\right)^2\right\},$$

from which one obtains the double-differential cross section

$$\frac{\mathrm{d}\sigma}{dQ^2 \mathrm{d}x} = \frac{2\pi\alpha^2 e_i^2}{Q^4} \; f_i(x) \left\{1 + \left(1 - \frac{Q^2}{xs}\right)^2\right\}, \tag{2.215a}$$

This formula exhibits a remarkable feature: If one divides the cross section by the factor

$$\left\{1 + \left(1 - \frac{Q^2}{xs}\right)^2\right\},$$

which, obviously is of a purely kinematic origin, then there remains an expression which depends on the variable x only,

$$\left\{1+\left(1-\frac{Q^2}{xs}\right)^2\right\}^{-1}\frac{\mathrm{d}\sigma}{dQ^2\mathrm{d}x}=\frac{2\pi\alpha^2e_i^2}{Q^4}\ f_i(x). \tag{2.215b}$$

This result from a simple constituent model for the proton, is called *Bjorken scaling*[25]. It rests on the assumption that during the time scale which is typical for the scattering process the quarks in the proton behave as if they were free particles. Interactions between them are neglected at this level.

The analysis of deep inelastic scattering on nucleons with electrons and neutrinos as projectiles, is an important branch of modern elementary particle physics, both in experimental high energy physics and in theory[26]. Here we restrict our summary to electrons only, without terms depending on polarization.

There are many ways of defining the structure functions. A conventional choice adopted in the Review of Particle Physics quoted above is the following. Instead of the structure functions W_i, cf. (2.206), and without polarization dependence, one uses

$$F_1(x,Q^2):=W_1(\nu,Q^2),\qquad F_2(x,Q^2):=\frac{(pq)}{M^2}W_2(\nu,Q^2), \tag{2.216}$$

in terms of which the cross section becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}=\frac{4\pi\alpha^2}{xyQ^2}\left\{xy^2F_1(x,Q^2)+\left(1-y-\frac{x^2y^2M^2}{Q^2}\right)F_2(x,Q^2)\right\}. \tag{2.217}$$

It is instructive to verify that this formula is the same as (2.209). The Jacobian of the transformation from the variables $(x,\ y)$ to the variables $(\nu,\ Q^2)$ is

$$\frac{\partial(x,y)}{\partial(\nu,Q^2)}=\left|\det\begin{pmatrix}-Q^2/(2M\nu^2) & 1/(2M\nu)\\ y/\nu & 0\end{pmatrix}\right|=\frac{y}{2M\nu^2},$$

so that

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}=\frac{2M\nu^2}{y}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2}.$$

For example, in the limit of large energy, neglecting the masses of the electron and the quarks, one has in the laboratory system

$$\nu\simeq E-E',\quad x\simeq\frac{EE'}{M(E-E')}(1-\cos\theta),\quad y\simeq\frac{E-E'}{E}.$$

[25] J.D. Bjorken, E.A. Paschos; Phys. Rev. **185**, 1975 (1969).

[26] A good summary is the review on structure functions by B. Foster, A.D. Martin, M.G. Vincter, in K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010), *Reviews, Tables, and Plots*, p.188 where many recent references are given. The Review of Particle Physics can also be found at http://pdg.lbl.gov/

The cross section (2.217) is then given by

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y} \simeq \frac{4\pi\alpha^2 E'}{Q^4}\{(E-E')2(1-\cos\theta)F_1(x,Q^2) + M(1+\cos\theta)F^2(x,Q^2)\},$$

so that, upon insertion of the definitions (2.216) with $(pq)/M^2 = (E-E')/M$

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\mathrm{d}Q^2} &= \frac{y}{2Mv^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y} \simeq \frac{4\pi\alpha^2E'}{Q^4}\frac{1}{2ME} \\ &\times\left\{2(1-\cos\theta)F_1(x,Q^2) + \frac{M}{E-E'}(1+\cos\theta)F_2(x,Q^2)\right\} \\ &= \frac{4\pi\alpha^2E'}{Q^4EM}\{2\sin^2(\theta/2)W_1(\nu,Q^2) + \cos^2(\theta/2)W_2(\nu,Q^2)\}.\end{aligned}$$

This is what we obtained in (2.209).

In the case of longitudinally polarized electrons there is a third structure function $F_3(x,\ Q^2)$ whose contribution depends on the helicity $\lambda = \pm 1$ of the electron. If weak interactions are taken into account, i.e. instead of photon exchange there is Z^0- and/or $W^\pm$–exchange, and if, alternatively, neutrino scattering is measured, there are additional coupling terms and more structure functions (see reference quoted above).

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Exercises

2.1. Show that (2.17b) can be transformed into (2.17a) by means of a rotation and a reflection. If the interaction is invariant with respect to these operations, the scattering amplitudes must be the same.

2.2. Derive the cross section (2.26,2.35) for electron scattering off a spin-zero target, on the basis of Feynman rules.

2.3. Prove (2.41) starting from (2.39). *Hint*: Consider first the case of an infinitesimal translation. Then make use of

$$\mathrm{e}^x = \lim_{n\to\infty}(1 + x/n)^n.$$

2.4. Prove relation (2.93) and use this relation to calculate the interaction term (2.92).

2.5. Derive the elastic form factor for a homogenous charge density (2.102). Calculate the cross section and discuss the result.

2.6. Prove the representation (2.129). *Hints*: First derive an equation for $(\mathrm{d}/\mathrm{d}r)W(r)$, with $W(r)$ as defined in (2.128), by combining the differential equations (2.105′) for the two potentials. Calculate $W(r = R_\infty)$ at some large radius, making use of the asymptotic forms of the radial functions, and let R_∞ go to infinity.

2.7. An unstable particle of lifetime τ is produced with a given energy E on a fixed target in the laboratory. Over which length can one reasonably hope to transport a beam of such particles? Consider the example of muons and pions at kinetic energies between 10 MeV and 200 MeV.

2.8. For $\kappa < 0$, $l = -\kappa - 1$, show that $rg_{n\kappa}(r) \simeq y_{nl}(r)$. This means that the states $\left(n,\ \kappa,\ j = -\kappa - \frac{1}{2} = l + \frac{1}{2}\right)$ and $\left(n, \kappa' = -\kappa - 1, j = -\kappa - \frac{3}{2} = l - \frac{1}{2}\right)$ have the same radial wave function in the nonrelativistic limit.

2.9. For circular orbits compare the fine structure splitting as predicted by (2.162) with the one obtained in first-order perturbation theory from

$$V_{ls} = \frac{1}{2m^2}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\boldsymbol{l}\cdot\boldsymbol{s}.$$

and nonrelativistic wave functions.

2.10. In a muonic atom consider two potentials V_1 and V_2 which differ only over the nuclear domain. Expand the muonic density according to $|\Psi|^2 \simeq a + br + cr^2$ and show that the eigenvalues of the energy for V_1 and V_2 differ approximately by

$$\Delta E = Ze^2 \frac{2\pi}{3} a \left\{ \Delta \langle r^2 \rangle + \frac{b}{2a} \Delta \langle r^3 \rangle + \frac{3c}{10a} \Delta \langle r^4 \rangle \right\},$$

where $\Delta\langle r^n \rangle$ is the difference in the corresponding nuclear charge moments. Derive an approximate formula for $E_{2p1/2} - E_{2s1/2}$ as compared to the value of this difference in the case of the point charge.

2.11. Prove the following identity, starting from (1.84–1.84'):

$$(p + p')_\alpha \overline{u(\boldsymbol{p}')}\, u(\boldsymbol{p}) = \overline{u(\boldsymbol{p}')} \{ 2M\gamma_\alpha + \mathrm{i}\sigma_{\alpha\beta}(p - p')^\beta \} u(\boldsymbol{p}).$$

Chapter 3
Weak Interactions and the Standard Model of Strong and Electroweak Interactions

This chapter gives an introduction to the phenomenology and the theory of weak interaction processes involving leptons and hadrons. In Sects. 3.1, 3.2 we collect the most prominent and characteristic properties of weak interactions as they follow from the analysis of a set of key experiments, old and recent. The following sections 3.3–3.5 deal with the elements of non-Abelian local gauge theories in general, and with the unified theory of electroweak and strong interactions of Glashow, Salam and Weinberg (GSW), in particular.

The GSW theory is a *minimal* theory in the sense that it incorporates all known phenomenology and successfully describes all known electromagnetic and weak interaction processes, including the masses and properties of the W-and Z^0-bosons. However, there are good reasons to expect that this minimal model is only an approximation to some, more global, theory of leptons and hadrons. The underlying theory could be a larger local gauge theory of all interactions, including the strong interactions and gravitation. In any event, the minimal GSW theory needs to be tested in precision experiments at all available energies. Sections 3.6, 3.7 are devoted to a discussion of such tests and to specific possibilities of searching for possible deviations from its predictions.

Weak interaction processes involving hadrons have another important aspect: Assuming the structure of the weak couplings to be known and to be given by the typical vector and axial vector current couplings, the matrix elements of such currents, taken between hadronic states, can be used to extract information on strong interaction quantities. This is analogous to the case of purely electromagnetic interactions such as electron scattering which allow to determine hadronic form factors, as described in Chap. 2. This aspect of weak interactions at medium energies is dealt with in our discussion of various semileptonic decays of hadrons in Chap. 4.

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3.1 Phenomenological Aspects of Weak Interactions

The theory of electroweak interactions of *leptons* can be formulated in terms of elementary fermion fields and in terms of interaction Lagrangians and equations of motion derived from them, whose solutions can be constructed in straightforward perturbation theory. Weak processes involving *hadrons* are complicated by the fact that any hadronic matrix element of weak interaction vertices is renormalized by strong interactions and is not readily calculable in a perturbation series. In the first place, however, it is not so much the absolute magnitude of weak hadronic matrix elements that matters, but rather their Lorentz-structure, their quantum numbers and selection rules.

The Lorentz structure and form factor decomposition of the hadronic matrix element of a given Lorentz tensor operator $T_{\alpha\beta\ldots}$

$$\langle A|T_{\alpha\beta\ldots}|B\rangle$$

is well defined as soon as the spins and the intrinsic parities of the hadronic states A and B are known. Let us consider some examples: The matrix element of a *vector* current between a one-pion state and the vacuum must vanish,

$$\langle 0|\upsilon_\alpha(x)|\pi(q)\rangle = 0, \tag{3.1}$$

independently of what that vector current is and of what the internal structure of the one-pion state is. This is so because under rotations

$$\begin{aligned} &\upsilon_0 \text{ behaves like an object with spin/parity } 0^+, \\ &\upsilon \text{ behaves like an object with spin/parity } 1^-. \end{aligned} \tag{3.2}$$

The vacuum state carries total angular momentum zero and is assigned positive parity. The pion has no spin and carries negative intrinsic parity. Its orbital state can be decomposed into states of good orbital angular momentum $l = 0, 1, 2, 3, \ldots$ whose parity is $(-)^l$. Combining the intrinsic and orbital properties one sees that the states of good total angular momentum and parity of the pion are

$$(J^P)_\pi = 0^-, 1^+, 2^-, \ldots \tag{3.3}$$

Obviously, none of these possibilities, when combined with the properties (3.2) of the operator υ^α, can yield the $(J^P)_{\text{vac}} = 0^+$ of the vacuum state. Hence the matrix element (3.1) vanishes.

On the other hand, if instead of a vector operator we consider the same matrix element of an *axial vector* operator,

$$\langle 0|a_\alpha(x)|\pi(q)\rangle, \tag{3.4}$$

then this matrix element need not be zero because under rotations

$$a_\alpha \text{ behaves like } \{0^-, 1^+\}, \tag{3.5}$$

which can indeed be coupled with $(J^P)_\pi$ to form the vacuum state 0^+. If the pion state is a plane wave with four momentum q we can translate the operator $a_\alpha(x)$ to $x = 0$, or some other fixed point in space–time. From (2.42) we have

$$\langle 0|a_\alpha(x)|\pi(q)\rangle = e^{-ixq}\langle 0|a_\alpha(0)|\pi(q)\rangle. \tag{3.4'}$$

Whether or not this quantity is different from zero now depends on the internal quantum numbers of a_α. The current $a_\alpha(x)$ must contain a piece proportional to an interpolating pion field, i.e. a field that can indeed create or annihilate a one-pion state. This field, therefore, must carry the quantum numbers of isospin, strangeness, baryon number, etc. of one-pion states. If all this is fulfilled, then the matrix element (3.4) must itself be a *vector*. Since q is the only vector available, the covariant decomposition of (3.4) must be

$$\langle 0|a_\alpha(0)|\pi(q)\rangle = F_\pi(q^2)q_\alpha \frac{1}{(2\pi)^{3/2}}. \tag{3.6}$$

However, $q^2 = m_\pi^2$, so the invariant form factor F_π, in fact, is a constant. Finally, from the behaviour of the pion state and of $a_\alpha(x)$ under time reversal T and charge conjugation one can show that F_π is pure imaginary, $F_\pi = \mathrm{i}f_\pi$ with f_π real. This will be shown below, in Sect. 4.2.

The selection rules which follow from charge conjugation C, G-parity and the internal quantum numbers of the hadronic states and the relevant tensor operators can be kept track of in a transparent manner by writing the weak interaction covariants in terms of elementary quark fields and by making use of the constituent quark model for mesons and baryons.[1]

Therefore we start our discussion of the phenomenology by a summary of the properties of the leptons and of the quark families, for the sake of reference in the following sections.

3.1.1 Basic Properties of Leptons and Quarks

(a) The lepton families

Three families of leptons are known to us: the electron e and its neutrino partner ν_e, the muon μ and its neutrino ν_μ, and the τ-lepton accompanied by yet another neutrino ν_τ:

$$\begin{pmatrix} \nu_e \\ \mathrm{e}^- \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}. \tag{3.7}$$

[1] For an introduction see [KOK 69].

The masses of the charged particles of these families are

$$m_e = 0.510998910(13)\ \mathrm{MeV}/c^2, \tag{3.8a}$$

$$m_\mu = 105.658367(4)\ \mathrm{MeV}/c^2, \tag{3.8b}$$

$$m_\tau = 1776.82(16)\ \mathrm{MeV}/c^2. \tag{3.8c}$$

Actually, the mass *ratio* of the muon to the electron mass is known to an accuracy comparable to that of m_e from the measurement of the muon's magnetic moment and of its g-factor anomaly, viz. (Scheck 1978)

$$m_\mu/m_e = 206.768259(62). \tag{3.9}$$

While the mass $m(\nu_\mathrm{e})$ of the electron's partner is known to be small, i.e. less than about $2\,\mathrm{eV}/c^2$, the *direct* experimental upper limits for $m(\nu_\mu)$ and $m(\nu_\tau)$ are not very stringent and there is indeed the possibility that they have nonzero and measurable masses. For example, for the muon neutrino we know only that $m(\nu_\mu) < 0.27\,\mathrm{MeV}/c^2$. The current upper limit for $m(\nu_\tau)$ is $31\,\mathrm{MeV}/c^2$. For most of what we discuss in the following sections the neutrinos can be assumed to be rigorously massless simply because $m(\nu_\mathrm{e})$ and $m(\nu_\mu)$ are definitely much smaller than the energy released in the decay processes that we consider.

To the best of our knowledge, neutrinos seem to occur in states of definite helicity (cf. Sect. 1.8.3), with ν_f having negative helicity and $\overline{\nu}_\mathrm{f}$ having positive helicity,

$$h(\nu_\mathrm{f}) = -1, \qquad h(\overline{\nu}_\mathrm{f}) = +1. \tag{3.10}$$

The density matrix describing the choice (3.10) is given in (1.159) above. Their *charged* partners, in contrast, can occur in any polarization state of a spin-1/2 particle. Even in kinematical situations where their mass can be neglected they can couple to other particles in states of positive and negative helicity: For instance, a vertex (eeγ) of two external electrons to a real or virtual photon must involve states of positive helicity and states of negative helicity with equal weight, because electromagnetic couplings conserve parity.

Each of the three lepton families (3.7) seems to carry its own additive quantum number L_f which is the only distinctive characteristic for the family (f, ν_f) and which is strictly conserved in all electromagnetic and weak reactions involving leptons. The eigenvalues of these *lepton family numbers* $L_\mathrm{f} = L_\mathrm{e}, L_\mu$, or L_τ are assigned as follows:

$$L_{\mathrm{f}'}(\mathrm{f}^-) = \delta_{\mathrm{ff}'}, \qquad L_{\mathrm{f}'}(\nu_\mathrm{f}) = \delta_{\mathrm{ff}'}. \tag{3.11a}$$

so that for the antiparticles

$$L_{\mathrm{f}'}(\mathrm{f}^+) = L_{\mathrm{f}'}(\overline{\nu}_\mathrm{f}) = -\delta_{\mathrm{ff}'}. \tag{3.11b}$$

The total lepton number L of a given particle ℓ, which is then also strictly conserved, is given by the sum

$$L = \sum_{\mathrm{f}=\mathrm{e},\mu,\tau} L_{\mathrm{f}}(\ell). \tag{3.12}$$

The dynamical origin of these conservation laws is not known; they are fulfilled to a very high degree of accuracy. For instance, processes like

$$\begin{aligned}
\mu^+ &\to \mathrm{e}^+ + \gamma, \\
\mu^+ &\to \mathrm{e}^+\mathrm{e}^-\mathrm{e}^+, \\
\mathrm{K}^+ &\to \pi^+\mathrm{e}^+\mu^-, \\
\mathrm{K}^+ &\to \pi^+\mathrm{e}^-\mu^+,
\end{aligned}$$

in which L is conserved but L_{e} and L_μ are not, have never been observed. They are known to be suppressed at the level of nine to twelve orders of magnitude in branching ratio, as compared to the main decay modes of the parent particle.

(b) The quark families

Except for some exotic states, all hadrons are believed to be composites of quarks and antiquarks. The low-lying mesons, for instance, should be predominantly states of one quark and one antiquark with definite relative orbital angular momentum L, with spin S and total angular momentum $J = L + S, L + S - 1, \ldots, |L - S|$,

$$(\mathrm{q}\bar{\mathrm{q}})_{L,S\to J}.$$

The spin of the meson is J, its intrinsic parity is $P = (-)^{L+1}$. If it is an eigenstate of charge conjugation C, then $C = (-)^{L+S}$. If $P = (-)^J$ one says that the meson has *natural* parity. In this case, and if the meson is an eigenstate of C, then one shows easily that it must have $P = C$. This is a typical prediction of the quark model.

The baryons are described by states of three quarks, with or without resulting internal angular momentum L and coupled to a resulting spin S, so that the spin of the particle J is the vector sum of $\boldsymbol{L}$ and $\boldsymbol{S}$.

The quarks carry *flavour* quantum numbers which add up to the flavour properties of the physical hadrons, such as baryon number B, isospin I, strangeness S, charm C etc. All quarks are assigned baryon number $B = \frac{1}{3}$ so that any state with three quarks (qqq) indeed has $B = +1$. Since baryon number is an additive, charge-like quantum number, and since quarks are fermions, a single antiquark $\bar{\mathrm{q}}$ has $B = -\frac{1}{3}$, a state of three antiquarks ($\bar{\mathrm{q}}\bar{\mathrm{q}}\bar{\mathrm{q}}$) has $B = -1$, and the meson states ($\mathrm{q}\bar{\mathrm{q}}$) have $B = 0$. [Baryon number conservation is an empirical conservation law which is derived from the observation that in all known reactions or decay processes the number of fermionic hadrons – with the correct counting of particles and antiparticles – is conserved. For instance, the stability of the proton and the stability of the hydrogen atom are very good indications of this conservation law. Note, however, that grand

unified theories of *all* interactions as well as the cosmology of the big bang would prefer not to have baryon number as an exact conservation law. In these theories the proton is expected to be very long-lived but not absolutely stable. The present lower limit on the proton's mean life is of the order of 10^{29} years.]

The quarks are denoted u ("up"), d ("down"), s ("strange"), c ("charmed"), b ("bottom" or "beauty"), t ("top" or "truth"). In the older literature on the constituent quark model, u, d, s are sometimes also denoted p, n λ, because, except for baryon number and charge, they have the same properties as the physical states: proton, neutron and lambda Λ ($1116\,\mathrm{MeV}/c^2$). The quark s is the carrier of strangeness S, i.e. it is assigned strangeness -1 while all other quarks have $S = 0$,

$$S(s) = -1, \quad S(\bar{s}) = +1, \quad S(\mathrm{q}) = 0 \quad \forall \mathrm{q} \neq \mathrm{s}.$$

Similarly, c is the carrier of charm, b the carrier of beauty, etc., all of which, like the strangeness, are additive quantum numbers.

In constrast to these, isospin is an internal spectrum symmetry which has the structure of SU(2). Thus, as for angular momentum, the physical hadrons are eigenstates of total isospin $(\boldsymbol{I}^{\mathrm{s}})^2$, with the same eigenvalue I^{s} in a given mass-degenerate multiplet. The members of such a multiplet are classified according to I_3^{s}, the projection of $\boldsymbol{I}^{\mathrm{s}}$ onto the 3-axis in isospin space, with $I_3^{\mathrm{s}} = -I^{\mathrm{s}}, -I^{\mathrm{s}}+1, \ldots, I^{\mathrm{s}}$, as usual. Let us call this symmetry the *strong interaction isospin*, in contrast to the weak interaction isospin to be defined below. The two SU(2)-symmetries should not be confused. This symmetry is indeed an almost exact spectrum symmetry at the level of hadron physics, as may be seen by observing, for instance, that the difference of the proton and the neutron masses

$$m_{\mathrm{n}} - m_{\mathrm{p}} = +\ 1.293318\ (9)\ \mathrm{MeV}/c^2 \tag{3.13}$$

is very small compared to their absolute values and, similarly that

$$m_{\pi^\pm} - m_{\pi^0} = 4.5936\ (5)\ \mathrm{MeV}/c^2 \tag{3.14}$$

amounts to only about 3.3% of m_π. The strong isospin quantum numbers are carried by the u and d quarks: Both have $I^{\mathrm{s}} = \frac{1}{2}$, with $I_3^{\mathrm{s}}(\mathrm{u}) = +\frac{1}{2}$, $I_3^{\mathrm{s}}(\mathrm{d}) = -\frac{1}{2}$. The electric charges finally are

$$Q = +\frac{2}{3} \quad \text{for} \quad \mathrm{u, c, t}, \tag{3.15a}$$

$$Q = -\frac{1}{3} \quad \text{for} \quad \mathrm{d, s, b}. \tag{3.15b}$$

Quarks also carry *colour* quantum numbers with respect to the local gauge theory $\mathrm{SU(3)_c}$ which describes the strong interactions. In contrast to the flavour symmetries mentioned above and in contrast to the local gauge theory of electroweak interactions, this symmetry remains unbroken at all levels of the theory. Electroweak

interactions couple only to the flavour degrees of freedom and are indifferent to colour. In other words, even though each quark flavour state carries an additional SU(3)-colour index,

$$\mathrm{u}^i, \mathrm{d}^k, \mathrm{s}^l, ..., \quad i,k,l,... = 1,2,3 \in \mathrm{SU}(3)_c, \tag{3.16}$$

the electromagnetic and weak coupling constants of the three colour states of the up-quark u^1, u^2, and u^3 are the same (and likewise for d^1, d^2, d^3, etc.). Therefore, in most of our discussion of electromagnetic and weak interactions we shall suppress the SU(3) index describing the colour degrees of freedom. (See however Sect. 3.4.7b below.)

It is not difficult to construct the wave functions of the mesons in flavour space. Denoting the spin projections $m_\mathrm{s} \pm \frac{1}{2}$ by arrows pointing up or down, respectively, we have for instance

$$\mathrm{K}^+ = \frac{1}{\sqrt{2}}(\mathrm{u}\uparrow \bar{\mathrm{s}}\downarrow + \mathrm{u}\downarrow \bar{\mathrm{s}}\uparrow), \tag{3.17a}$$

$$\pi^+ = \frac{1}{\sqrt{2}}(\mathrm{u}\uparrow \bar{\mathrm{d}}\downarrow + \mathrm{u}\downarrow \bar{\mathrm{d}}\uparrow), \tag{3.17b}$$

$$\pi^0 = \frac{1}{\sqrt{4}}(\mathrm{u}\uparrow \bar{\mathrm{u}}\downarrow + \mathrm{u}\downarrow \bar{\mathrm{u}}\uparrow - \mathrm{d}\uparrow \bar{\mathrm{d}}\downarrow - \mathrm{d}\downarrow \bar{\mathrm{d}}\uparrow), \tag{3.17c}$$

where $\mathrm{q}\bar{\mathrm{q}}$, as written in (3.17), means a quark and an antiquark in a relative s-state, coupled to spin 0 and to appropriate isospin, $\frac{1}{2}$ for the kaon, 1 for the pion. Note that we have coupled the spins and isospins according to the scheme

$$(-)^{j-m'}(jm, j-m'|JM)(\mathrm{q})_{jm}(\bar{\mathrm{q}})_{jm'}$$

characteristic for the coupling of particles and antiparticles (or particles and "holes").

For the baryons, the construction of the flavour wave functions is well-defined if we require that (i) the three quarks be in a relative orbital s-state, and, (ii) the wave function in spin and flavour degrees of freedom be totally *symmetric*. For example, a Δ in the state with charge $Q = +2$ and maximal spin projection $m_\mathrm{s} = +\frac{3}{2}$ is described by

$$\Delta^{++}_{m_s=3/2} = \mathrm{u}\uparrow \mathrm{u}\uparrow \mathrm{u}\uparrow . \tag{3.18}$$

Likewise, the Λ (1116), having isospin zero, must contain a pair (ud – du). As the state is required to be symmetric in all quarks, this pair must also be in a state of spin zero. Finally, the strangeness -1 of Λ must be carried by an s-quark. Putting these facts together we have

$$\Lambda_{m_s} = \frac{1}{\sqrt{12}}\sum_\pi (\mathrm{u}\uparrow \mathrm{d}\downarrow - \mathrm{u}\downarrow \mathrm{d}\uparrow)\ \mathrm{s}_{m_s}, \tag{3.19a}$$

where the sum must be taken over all six permutations of the symbols u, d, s. The proton and neutron states are obtained from (3.19a) by replacing simply s by u or d, respectively. The normalization constant changes, however, because, with s replaced by u or d, six permutations out of the twelve terms in (3.19a) are pairwise equal. One obtains

$$p_{m_s} = \frac{1}{\sqrt{18}} \sum_{\pi} (\mathrm{u}\uparrow \mathrm{d}\downarrow - \mathrm{u}\downarrow \mathrm{d}\uparrow)\, \mathrm{u}_{m_s}, \tag{3.19b}$$

$$n_{m_s} = \frac{1}{\sqrt{18}} \sum_{\pi} (\mathrm{u}\uparrow \mathrm{d}\downarrow - \mathrm{u}\downarrow \mathrm{d}\uparrow)\, \mathrm{d}_{m_s}. \tag{3.19c}$$

It is easy to verify that the states (3.17–3.19) reproduce the correct spin, parity and interal quantum numbers of the corresponding mesons and baryons. However, for the baryons there remains a problem: if the decomposition of baryon states in terms of quark states is to make sense at all, we would expect the *radial* wave functions of these states to be *symmetric*, too. [Unless the potential has a weird radial dependence, ground states in quantum mechanics have symmetric spatial wave functions.] This obvious clash with Fermi–Dirac statistics is repaired by introducing the colour degrees of freedom: Indeed, it is assumed that all physical hadron states are singlets with respect to $\mathrm{SU}(3)_\mathrm{c}$. For the baryons this means that each product $(\mathrm{q}^1\mathrm{q}^2\mathrm{q}^3)$ in (3.18, 3.19) is to be replaced by

$$\frac{1}{\sqrt{6}} \sum_{ikl=1}^{3} \varepsilon_{ikl}\, \mathrm{q}^{1i}\mathrm{q}^{2k}\mathrm{q}^{3l}. \tag{3.20}$$

This is a singlet with respect to $\mathrm{SU}(3)_\mathrm{c}$. Obviously, it is antisymmetric in the three quarks so that the complete baryon wave functions with symmetric orbital functions are indeed antisymmetric.

3.1.2 Empirical Information on Weak Interactions

In this section we summarize some characteristic properties of the weak interactions. In quoting experimental information on these properties we do not follow the historical development. Instead, we mention those experiments which illustrate the point under discussion in the simplest and most self-evident manner.

(a) Range of weak interactions
All purely leptonic and semileptonic weak scattering processes or decays that have been observed in the laboratory up to now involve four external fermionic particles, leptons and/or quarks. For purely leptonic processes such as

$$\mu^- - e^-\overline{\nu}_e\nu_\mu,$$
$$\nu_\mu + e^- \to e^- + \nu_\mu,$$

or semileptonic processes involving baryons, such as

$$\begin{aligned} \mathrm{n} &\to \mathrm{p}\mathrm{e}^{-}\nu_{\mathrm{e}}, \\ \overline{\nu}_{\mu} + \mathrm{p} &\to \mathrm{n} + \mu^{+}, \end{aligned}$$

this is evident. For semileptonic decays of mesons, such as

$$\begin{aligned} \pi^{+} &\to \mu^{+}\nu_{\mu}, \\ \pi^{+} &\to \pi^{0}\mathrm{e}^{+}\nu_{\mathrm{e}}, \\ \mathrm{K}^{+} &\to \pi^{0}\mu^{+}\nu_{\mu}, \end{aligned}$$

this is seen most easily by considering these processes in the framework of the quark model, as discussed above. For instance, in the decay $\pi^{+} \to \mu^{+}\nu_{\mu}$, the fundamental four-fermion process is

$$\mathrm{u} + \overline{d} \to \mu^{+} + \nu_{\mu},$$

The quark and the antiquark of the pion annihilate at one vertex, the lepton pair is created at another vertex. Thus a typical weak amplitude contains *four* fermionic field operators.

There is clear experimental evidence that the weak interactions are of very short range. All known weak amplitudes at low and intermediate energies contain essentially only s- and p-waves in their partial wave decomposition. This indicates that the interaction is very close to being a *contact* interaction, effective only when the four particles are all at the same point of space and time. Indeed, if the external particles had spin zero, then a contact interaction would yield only s-wave amplitudes. As they carry spin one-half, such an interaction can also yield p-wave amplitudes because of spin-orbit or spin-momentum coupling.

As an example, let us consider the scattering process

$$\nu_{\mu} + e \to e + \nu_{\mu}, \tag{3.21}$$

Suppose, this process is due to the exchange of a heavy photon-like boson whose mass M_B is large as compared to the invariant momentum transfer, $M_B^2 \gg |q^2|$. In this case, its propagator can be approximated as

$$\frac{-g^{\alpha\beta} + q^{\alpha}q^{\beta}/M_B^2}{q^2 - M_B^2} \simeq \frac{g^{\alpha\beta}}{M_B^2} \tag{3.22}$$

We can make use of (2.56) provided we replace the photon propagator $1/q^2$ by $1/M_B^2$ and replace $4\pi\alpha$ by $g_{\mathrm{e}} \cdot g_{\mu}$, the product of the coupling constants of the boson to the lepton pairs (e,e) and (ν_{μ}, ν_{μ}) [we assume these to be equal up to factors of order unity]. Let E^* be the neutrino energy in the c.m. system, z^* the cosine of the c.m. scattering angle, and assume $E^* \gg m_{\mathrm{e}}$. From (2.56) with $M = m_{\mathrm{e}}$,

$m \equiv m_\nu = 0$, we have

$$s = 2E^{*2} + m^2 + 2E^* \sqrt{m^2 + E^{*2}} \simeq 4E^{*2},$$
$$t = -2E^{*2}(1 - z^*).$$

Setting $F_1(t) \equiv 1$, $F_2(t) \equiv 0$, equation (2.56) gives

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{c.m.}} \simeq \left(\frac{g^2}{M_B^2}\right)^2 \frac{E^{*2}}{4\pi^2} \left\{1 + \frac{t}{s} + \frac{1}{2}\left(\frac{t}{s}\right)^2\right\}$$
$$= \left(\frac{g^2}{M_B^2}\right)^2 \frac{E^{*2}}{4\pi^2} \left\{1 - \frac{1}{2}(1 - z^*) + \frac{1}{8}(1 - z^*)^2\right\}.$$

If this expression is integrated over all angles one obtains an expression for the total elastic cross section of the form

$$\sigma = \int \mathrm{d}\Omega \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{c.m.}} = \text{const.}\, E^{*2} \simeq \text{const.}\frac{s}{4}, \tag{3.23a}$$

i.e. proportional to the square of the c.m. energy of the neutrino. When expressed in the laboratory system where $s = m^2 + 2mE_\nu \simeq 2mE_\nu$ this means that the cross section is linear in the laboratory energy of the neutrino,

$$\sigma = \text{const.}\frac{1}{2} m(E_\nu)_{\text{lab}}. \tag{3.23b}$$

This linear increase of neutrino cross sections is indeed verified experimentally up to laboratory energies of the order several 100 GeV. It is found to hold also for neutrino reactions such as $\nu_\mu + N \to N' + \mu$ etc. Up to energies of this order of magnitude the weak interactions effectively behave like contact interactions of the form

$$a_\Gamma \frac{g^2}{M_B^2} \overline{\Psi(1)}\, \Gamma_\alpha \Psi(2) \overline{\Psi(3)}\, \Gamma^\alpha \Psi(4) \tag{3.24}$$

where Γ_α denotes certain Dirac matrices such as γ^α, $\gamma^\alpha\gamma_5$ etc. and a_Γ is a dimensionless constant of order unity. If this is so then the widths Γ of weakly decaying states are proportional to $(a_\Gamma g^2/M_B^2)^2$ which has the dimension E^{-4}. As the decay width has the dimension of an energy, for dimensional reasons, we expect it to be proportional to the fifth power of the released energy. Except for some special cases (such as the decay $\pi \to \mathrm{e}\nu_\mathrm{e}$) in which there are additional hindrance factors due to angular momentum conservation, this is indeed what one finds empirically. The most striking proof is the great variety of lifetimes in nuclear β-decay which cover many orders of magnitude, from fractions of seconds to many thousands of years.

As an example, let us compare the lifetimes of the muon and of the neutron in the processes

$$\mu \to \mathrm{e}\nu\bar{\nu},$$
$$\mathrm{n} \to \mathrm{p}\mathrm{e}\bar{\nu}.$$

In the first case the energy release is $(\Delta E)_\mu \simeq m_\mu/2$, in the second case it is $(\Delta E)_\mathrm{n} \simeq m_\mathrm{n} - m_\mathrm{p}$. If the widths are proportional to $(\Delta E)^5$, the lifetimes are proportional to $(\Delta E)^{-5}$. The measured lifetimes are

$$\tau_\mu = 2.2 \times 10^{-6}\,\mathrm{s},$$
$$\tau_\mathrm{n} = 885.7\,\mathrm{s},$$

their ratio being $\tau_\mu/\tau_\mathrm{n} = 2.5 \times 10^{-9}$. This is to be compared to

$$\left(\frac{m_n - m_p}{m_\mu/2}\right)^2 = 8.7 \times 10^{-9}.$$

Additional remarks. More detailed expressions for neutrino cross sections as well as their absolute magnitude will be given below. We note, in particular, that the behaviour (3.23) cannot hold indefinitely since, from a certain energy on, it violates unitarity. At which energy this will happen is worked out below. Finally, the quantitative analysis will show that the actual range of weak interactions is of the order of 10^{-16} cm.

(b) Charged current and neutral current vertices
We distinguish two classes of weak interactions:
(i) The *charged current* (CC) interactions which are described by the exchange of heavy, *charged* bosons $\mathrm{W}^\pm$. They are characterized by vertices of the kind shown in Fig. 3.1 (a–c) at which either a lepton state f is converted into a neutrino state ν_f of the same lepton number L_f, or a quark of charge Q is converted into a quark of charge $Q + 1$. Examples are

$$\nu_\mu + \mathrm{e}^- \to \mu^- + \nu_e,$$

where ν_μ is converted to μ^-, through emission of a virtual W^-, and where e^- is converted to ν_e, through absorption of the same W^-. Similarly, in the reaction

$$\bar{\nu}_\mathrm{e} + \mathrm{p} \to \mathrm{n} + \mathrm{e}^+$$

the $\bar{\nu}_\mathrm{e}$ is converted into a e^+, a u-quark into a d-quark. Clearly, the incoming (outgoing) fermion can also be replaced by the corresponding outgoing (incoming) antifermion. In these cases pairs of the type $(\mathrm{f}^-, \bar{\nu}_\mathrm{f})$, $(\mathrm{f}^+, \nu_\mathrm{f})(\mathrm{u}, \bar{\mathrm{d}})$, $(\mathrm{u}, \bar{\mathrm{s}})$ etc. are

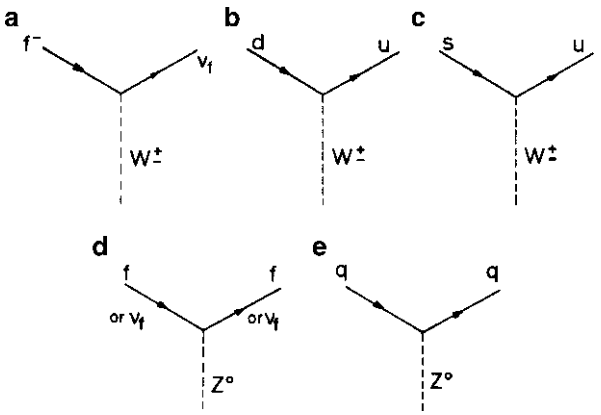

Fig. 3.1 Charged current vertices (**a–c**) and neutral current vertices (**d,e**) describing the coupling of a fermion pair (quarks or leptons) to vectors bosons $W^{\pm}$ and Z^0, respectively

created or annihilated with simultaneous absorption or emission of a W^+ or W^-. Examples are

$$\mu^- e^- \overline{\nu}_e \nu_\mu, \quad \text{where } \mu^- \to \nu_\mu W^-, \quad W^- \to e^- \overline{\nu}_e,$$
$$\Lambda \to p \mu^- \overline{\nu}_\mu, \quad \text{where } s \to u W^-, \quad W^- \to \mu^- \overline{\nu}_\mu,$$
$$\pi^+ \to \mu^+ \nu_\mu \quad \text{where } u\overline{d} \to W^+, \quad W^+ \to \mu^+ \nu_\mu,$$
$$K^- \to \mu^- \overline{\nu}_\mu \quad \text{where } \overline{u}s \to W^-, \quad W^- \to \mu^- \overline{\nu}_\mu.$$

(ii) The *neutural current* (NC) *interactions* which we describe by the exchange of the neutral boson Z^0. Typical vertices are shown in Fig. 3.1(d,e). These interactions are somewhat analogous to electromagnetic interactions with the exchange of a photon, except for the following differences: Unlike the photon which is massless, the Z^0-boson is massive and very heavy; the couplings at vertices of the kind shown in Fig. 3.1 simultaneously involve vector and axial-vector currents; neutrinos which have zero electric charge, have nonvanishing couplings to Z^0-bosons. Because the Z^0 is electrically neutral, amplitudes due to exchange of virtual Z^0 can interfere with amplitudes due to photon exchange. This is discussed in more detail below.

Examples of pure NC reactions are

$$\nu_\mu + e^- \to e^- + \nu_\mu,$$
$$\overline{\nu}_\mu + p \to p + \overline{\nu}_\mu,$$

while the interference of NC couplings with electromagnetic vertices can be detected in parity-violating spin-momentum correlations.

(c) Parity violation, vector and axial-vector currents

It is well known that the strong and electromagnetic interactions are both invariant under the operation of space inversion (parity). In fact, the parity transformation is defined in such a way that it leaves invariant the Lagrangian of strong and

of electromagnetic interactions. The weak interactions, however, are not invariant under this parity operation, defined with respect to strong and electromagnetic interactions. Parity violation in weak interactions manifests itself, for instance, in nonvanishing, observable correlations between a spin $\boldsymbol{S}$ and a momentum $\boldsymbol{q}$, $\boldsymbol{S} \cdot \boldsymbol{q}$, in reactions and decay processes where there is no such correlation in the initial state. We give some examples:
(i) In charged pion decay, $\pi^- \to \mu^- \overline{\nu}_\mu$, experiment shows that the muon is fully polarized along its momentum. A measurement based on polarization studies in muonic atoms (formed with muons from pion decay), as well as measurements of polarization quantities in muon capture, show that the longitudinal polarization $P_\ell \equiv (\boldsymbol{\sigma} \cdot \boldsymbol{q})/q$ is 1 within about 10% (Abela et al. 1982, Roesch et al. 1982).
(ii) The electron (positron) from $\mu^-(\mu^+)$ decay $\mu \to e\nu\overline{\nu}$ is found to be fully polarized along its momentum, even if the initial muon is unpolarized. The latest result from μ^+-decay (Burkhard et al. 1985) is

$$P_\ell = 1.00 \pm 0.04. \tag{3.25}$$

(iii) In the decay of polarized muons the electron is emitted anisotropically with respect to the muon spin direction. If x denotes the electron energy E in units of its maximum W, $x = E/W$, and if θ is the angle between the muon spin and the electron momentum, the double differential decay probability is proportional to [cf. (4.58) below summed over the electron spin].

$$\frac{\mathrm{d}^2\Gamma(\mu^- \to \mathrm{e}^-\overline{\nu}_\mathrm{e}\nu_\mu)}{\mathrm{d}(\cos\theta)\mathrm{d}x} \propto x^2\{(3-2x) - \mathrm{P}_\mu\xi(2x-1)\cos\theta\}, \tag{3.26}$$

where P_μ is the longitudinal polarization of the muon that stems from pion decay.

Thus, the parameter ξ measures the correlation between the muon spin and the electron momentum. Experiment gives the value (Beltrami et al. 1987, Jamieson et al. 2006)

$$\xi = 1.0007 \pm 0.0035 \tag{3.27}$$

Thus, for x close to 1, the electron is emitted preferentially in the direction opposite to the muon polarization ζ_μ, but is never emitted along the direction ζ_μ. This is in agreement with the observation (3.25) and with the fact that neutrinos are left-handed and antineutrinos right-handed: For $x = 1$, i.e. maximal energy, the neutrinos must emerge with parallel momenta but opposite to the electron momentum. This is illustrated by Fig. 3.2 which holds for $\theta = 180°$ and which shows that conservation of angular momentum favours this emission angle but forbids $\theta = 0$.
(iv) Analogous studies were performed with polarized τ leptons for which the formula (3.26) is applicable up to (small) mass corrections when there is a muon in the final state. The analysis of the leptonic decays $\tau^- \to \mu^-\overline{\nu}_\mu\nu_\tau$ and the analogous reaction with μ replaced by e, $\overline{\nu}_\mu$ replaced by $\overline{\nu}_\mathrm{e}$ also gave ξ equal to 1, with an error bar of about 6%.

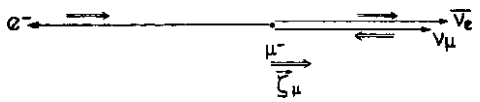

Fig. 3.2 Decay of negative muon at maximal energy of the decay electron at $\theta = 180$. The neutrino spins compensate, the electron takes over the spin projection ζ_μ of the decay muon

(v) Precise polarization data also comes from nuclear β-decay. For instance, the longitudinal polarization of β-particles from Gamow-Teller transitions is found to be (Koks et al. 1976)

$$P_\ell = (1.001 \pm 0.008)\, v/c. \tag{3.28}$$

All of these examples show not only that parity is violated, but that it is violated in a *maximal* way: the longitudinal polarizations and correlation parameters assume their maximal theoretical value. This is true for leptonic CC interactions. For hadrons, maximal parity violation is sometimes hidden and attenuated by strong interaction effects. The intermediate bosons $W^\pm$ and Z^0, very much like the photon, carry spin 1. In contrast to the photon, however, they couple to Lorentz vector as well as axial vector currents. For a long time it was not possible to disprove the alternative possibility that the CC weak interactions were due exclusively to Lorentz scalar, pseudoscalar and tensor, pseudotensor couplings. We now have clear evidence that such a pure situation is not realized in the weak interactions. There may be several bosons with spin 1, of the kind of the W. There may even be, in addition to these, bosons with spin 0 and, possibly, bosons with spin 2. However, the interactions due to the exchange of W-bosons must be predominant over the spin 0 or spin 2 exchanges. The evidence for this comes from an experiment which measures the transfer of leptonic helicity in the following inclusive reaction of $\bar{\nu}_\mu$ on iron (Jonker et al. 1979, 1983)

$$\bar{\nu}_\mu + \mathrm{Fe} \to \mathrm{X} + \mu^+, \tag{3.29}$$

in which X stands for any final state that can be reached in this reaction. This needs some explanation: A pair of external fermions f_1, f_2 couples to an intermediate boson B with spin J, via a covariant of the form

$$\overline{\Psi_{f_1}(x)}\, \Gamma^{(J)} \Psi_{f_2}(x). \tag{3.30}$$

If the boson has spin zero $J = 0$, the matrix $\Gamma^{(0)}$ is a linear combination of the unit matrix and of γ_5,

$$\Gamma^{(J=0)} = x_0 \mathbb{1} + y_0 \gamma_5. \tag{3.31}$$

This can also be written as follows:

$$\begin{aligned} \Gamma^{(J=0)} &= (x_0 + y_0)\frac{1}{2}(\mathbb{1} + \gamma_5) + (x_0 - y_0)\frac{1}{2}(1 - \gamma_5) \\ &= P_+(x_0 + y_0)P_+ + P_-(x_0 - y_0)P_-, \end{aligned} \tag{3.31'}$$

where we have denoted the projection operators (1.76) by P_+, P_- and have made use of their property of being idempotent. In the experiment (3.29) the muon is produced at an average, squared momentum around 4 GeV^2. As this is large as compared to m_μ^2 the muon mass may be neglected in the analysis. However, the matrices P_+, P_-, when applied to states of *massless* external fermions, project onto eigenstates of helicity. Using, for example, the standard representation we have

$$P_\pm := \frac{1}{2}(\mathbb{1}_4 \pm \gamma_5) = \frac{1}{2}\begin{pmatrix} \mathbb{1} & \pm\mathbb{1} \\ \pm\mathbb{1} & \mathbb{1} \end{pmatrix} \tag{3.32}$$

and, from the explicit spinor solutions (1.90–1.91),

$$P_\pm u(\boldsymbol{p}) = \frac{1}{2}\sqrt{p^0}\begin{pmatrix} (\mathbb{1} \pm h)\chi^{(\mathrm{r})} \\ (\pm\mathbb{1} + h)\chi^{(\mathrm{r})} \end{pmatrix}, \tag{3.33a}$$

$$P_\pm\, v(\boldsymbol{p}) = \frac{1}{2}\sqrt{p^0}\begin{pmatrix} \pm(\mathbb{1} \mp h)\chi^{(s)} \\ (\mathbb{1} \mp h)\chi^{(\mathrm{s})} \end{pmatrix}, \tag{3.33b}$$

where $h = (\boldsymbol{\sigma}\cdot\boldsymbol{p})/|\boldsymbol{p}|$ and $p^0 = |\boldsymbol{p}|$. [In deriving (3.33b) one must remember that in the spinor $\upsilon(\boldsymbol{p})$ the Pauli spinor $\chi^{(\mathrm{s})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents "spin up".] Thus, P_+ projects onto incoming *particle* states with *positive* helicity, and onto incoming *antiparticle* states with *negative* helicity. Likewise, P_- projects onto particle states with negative helicity and onto antiparticle states with positive helicity.

For outgoing states the same statements hold with the sign of the helicity reversed. We prove this by way of an example

$$\overline{u(\boldsymbol{p})}P_+ = \frac{1}{2}u^\dagger(\boldsymbol{p})\gamma^0(\mathbb{1} + \gamma_5) = \frac{1}{2}u^\dagger(\boldsymbol{p})(\mathbb{1} - \gamma_5)\gamma^0 = (P_-u(\boldsymbol{p}))^\dagger\gamma^0.$$

The action of the operator $\Gamma^{(0)}$, (3.31'), on the helicities at a vertex with, say, an incoming neutrino ν_f and an outgoing charged lepton f^- is illustrated by Fig. 3.3a, b. Specifically, we note that if the incoming neutrino is fully left-handed, the outgoing charged lepton must be fully right-handed (in the limit of neglecting its mass m_f).

It is not difficult to see that the same situation applies to the case of an intermediate boson with spin 2. In this case

$$\begin{aligned} \Gamma^{(J=2)} &= x_2\sigma^{\alpha\beta} + y_2\sigma^{\alpha\beta}\gamma_5 \\ &= (x_2 + y_2)\sigma^{\alpha\beta}P_+ + (x_2 - y_2)\sigma^{\alpha\beta}P_-, \end{aligned} \tag{3.34}$$

where $\sigma^{\alpha\beta} = (\mathrm{i}/2)(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$. As both P_+ and P_- commute with a product of two γ-matrices, and hence with $\sigma^{\alpha\beta}$, this is equal to

$$\Gamma^{(2)} = P_+(x_2 + y_2)\sigma^{\alpha\beta}P_+ + P_-(x_2 - y_2)\sigma^{\alpha\beta}P_-. \tag{3.34'}$$

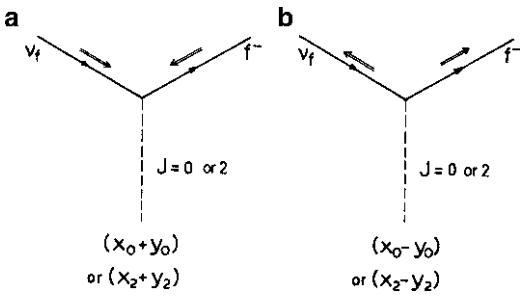

Fig. 3.3 Pattern of helicity transfer at a vertex of two massless fermions which couple to a boson carrying spin 0 or 2. (**a**): (hypothetical) neutrino is right-handed, (**b**): (physical) neutrino is left-handed

$\Gamma^{(2)}$ has the same structure as $\Gamma^{(0)}$, (3.31'), and therefore, the transfer of helicity at a vertex with a boson of spin 2 is the same as for the case of spin 0; see Fig. 3.3.

The situation is different for the case of an intermediate boson with spin 1. The covariant (3.30) is now a linear combination of vector and axial vector terms, so that the matrix $\Gamma^{(1)}$ has the form

$$\begin{aligned}\Gamma^{(1)} &= x_1\gamma^\alpha + y_1\gamma^\alpha\gamma_5 \\ &= (x_1+y_1)\gamma^\alpha P_+ + (x_1-y_1)\gamma^\alpha P_-. \end{aligned} \tag{3.35}$$

Again, we replace P_+ and P_- by their squares $(P_\pm)^2$ and then move one factor to the left in both terms of (3.35). However, as γ_5 anticommutes with every γ^α, cf. (1.80) one has now

$$\gamma^\alpha P_+ = P_-\gamma^\alpha, \quad \gamma^\alpha P_- = P_+\gamma^\alpha,$$

so that

$$\Gamma^{(1)} = P_-(x_1+y_1)\gamma^\alpha P_+ + P_+(x_1-y_1)\gamma^\alpha P_-. \tag{3.35'}$$

Upon comparison with (3.31') and (3.34') one sees that the transfer of helicity is different from the two previous cases, as indicated in Fig. 3.4. In particular, if the incoming neutrino is fully left-handed, the outgoing (approximately massless) lepton is also fully left-handed. For *massive* fermions the same reasoning applies with *helicity* replaced by *chirality*, or *handedness:* P_+ projects onto righ-handed (R) states, P_- projects onto left-handed (L) states.

Equipped with this knowledge let us now analyze the results of reaction (3.29). The incoming $\bar{\nu}_\mu$ beam originates primarily from π^- and K^- decays and, therefore, is polarized *along* its momentum: $h(\bar{\nu}_\mu)$ is known to be $+1$ to within, say, 10%, from the experiments mentioned above. In the experiment the average polarization of the outgoing μ^+ was measured. Integrating (3.26) over the variable x from 0 to 1, and reversing the sign of the correlation term in $\cos\theta$ [(3.26) holds for μ^--decay], one obtains

$$\int_0^1 \mathrm{d}x \frac{\mathrm{d}^2\Gamma(\mu^+ \to \mathrm{e}^+ \nu_e \bar{\nu}_\mu)}{\mathrm{d}(\cos\theta)\mathrm{d}x} \propto \frac{1}{2}\left\{1 + \frac{1}{3}\xi\cos\theta\right\}. \tag{3.36}$$

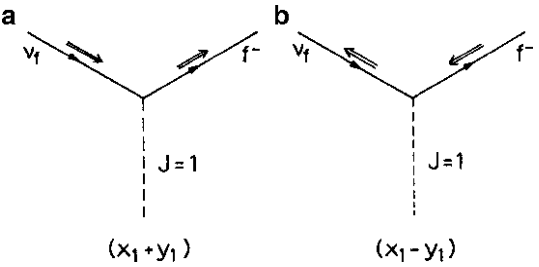

Fig. 3.4 Pattern of helicity transfer at a vertex of two massless fermions which couple to a boson with spin 1

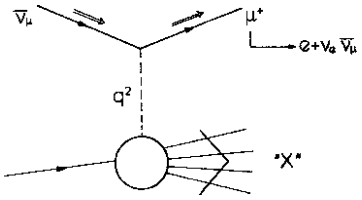

Fig. 3.5 Helicity transfer from the incoming muonic antineutrino to the positive muon in the inclusive charged current interaction (3.29). The longitudinal polarization of the outgoing μ^+ is determined from the decay asymmetry in its decay

From the measured asymmetry (3.36) it was deduced that the μ^+ has a longitudinal polarization of $+1$, to within an error of about 20%. Thus, the helicity transfer was found to be predominantly of the type shown in Fig. 3.5. This important result tells us that the reaction must be due predominantly to exchanges of spin 1.

(d) Helicity transfer at weak and electromagnetic vertices: Additional remarks

In the context of the discussion of helicity transfer in weak CC and NC reactions we wish to add some further remarks of general interest.

(i) Our analysis holds strictly only for massless external fermions. If the mass of the charged leptons is taken into account, then the helicities ± 1 are to be replaced by longitudinal polarizations $\pm v/c$, respectively. In many situations, very much like in the case discussed above, the kinematical situation is such that the mass may be neglected to a good approximation.

(ii) The pattern of helicity transfer depends only on the spin of the exchanged boson, and not on the parameters x_i and y_i in (3.31, 3.34, 3.35). Therefore, the situations depicted in Figs. 3.3, 3.4 are independent of whether or not there is parity violation in the interaction. As an example, consider (3.35) with $x_1 = 1$, $y_1 = 0$. The covariant (3.30) is then a Lorentz vector $\overline{\Psi_{f_1}(x)}\gamma^\alpha\Psi_{f_2}(x)$ and may represent, for $f_1 = f_2$, the electromagnetic current coupling to a photon line. The transfer of helicity is still as described above. For example, in electron scattering at high energies, the helicity of the incoming electron is transferred to the outgoing electron according to the same pattern as shown in Fig. 3.4. The two situations have the same coupling constants and, in case the incoming electron is unpolarized, they occur with equal weights. There is no helicity flip. This is a result that we had already found in Sect. 2.2 in the context of a somewhat different approach.

(iii) Consider the decay of a particle with spin zero and mass M into two leptons with masses m_1 and m_2. Examples are

$$\begin{aligned} &\eta^0(547) \to \mu^+\mu^-, \quad \eta^0(547) \to \mathrm{e}^+\mathrm{e}^-, \\ &\pi^0 \to \mathrm{e}^+\mathrm{e}^-, \\ &\pi^+ \to \mu^+\nu_\mu, \quad \pi^+ \to \mathrm{e}^+\nu_\mathrm{e}, \\ &\mathrm{K}^+ \to \mu^+\nu_\mu, \quad \mathrm{K}^+ \to \mathrm{e}^+\nu_\mathrm{e}. \end{aligned} \tag{3.37}$$

In the rest system of the decaying meson the two leptons are emitted with equal and opposite 3-momentum. Let the direction of this c.m. momentum be the 3-axis. The total angular momentum J as well as its projection J_3 onto that axis are conserved. As the outgoing particles are described by plane waves propagating parallel to the 3-axis, the projection l_3 of the relative *orbital* angular momentum vanishes. Conservation of J_3 then means that the projection S_3 of the total spin is also conserved. As $J_3 = 0$ before the decay, the two leptons must emerge with the *same* longitudinal polarization or helicity.

Suppose now that the interacion responsible for the decay is due to the exchange of bosons with spin 1 ($\mathrm{W}^\pm$ or photons, respectively) and that the leptons in the final state are massless. From the analysis given above we know that they must be emitted with *opposite* helicity, cf. Fig. 3.4. As this is in conflict with the conservation of angular momentum we conclude:

A state of spin zero cannot decay into two massless leptons if the interaction is due to bosons with spin 1.

In reality, the charged leptons f^- are not massless and the decays (3.37) are not strictly forbidden. For a massive particle the projection operators $P_\pm$, when acting on the particle's spinor wave function, yield nonvanishing components of either longitudinal polarization. However, the component required by conservation of angular momentum in the decays (3.37) is found to be proportional to m_f, the mass of the charged lepton. For example, if the $\mathrm{W}^\pm$ boson couples with equal strength to $(\mathrm{e}, \nu_\mathrm{e})$ and to (μ, ν_μ), the ratio of the decay width $\pi \to \mathrm{e}\nu_\mathrm{e}$ and $\pi \to \mu\nu_\mu$ is found to be (see (4.93) below)

$$\begin{aligned} R = \frac{\Gamma(\pi^+ \to \mathrm{e}^+\nu_\mathrm{e})}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} &= \frac{m_\mathrm{e}^2}{m_\mu^2}\left(\frac{1-(m_\mathrm{e}/m_\pi)^2}{1-(m_\mu/m_\pi)^2}\right)^2 \\ &= 2.34 \times 10^{-5} \cdot 5.49 = 1.28 \times 10^{-4}. \end{aligned} \tag{3.38}$$

The second factor in (3.38) is the squared ratio of the c.m. momenta in the two decays. It is larger than 1 reflecting the fact that from kinematics alone, the decay into the lighter fermion is preferred over the heavier one. The first factor $(m_\mathrm{e}/m_\mu)^2$ is the suppression factor discussed above. It is due to the conflict between the helicity transfer as required by the exchange of a boson with spin 1 and conservation of angular momentum. We emphasize, in particular, that this result does not depend on the specific form of $\Gamma^{(1)}$, (3.35), and holds for any choice of the parameters x_1, y_1.

The result (3.38) receives a small additional contribution from radiative corrections, so that the theoretical prediction is

$$R_{\text{theor}} = 1.233 \times 10^{-4}, \tag{3.39}$$

to be compared with the experimental value

$$R_{\text{exp}} = (1.230 \pm 0.004) \times 10^{-4}. \tag{3.40}$$

(e) Universality of weak interactions and some properties of semileptonic processes

There is good empirical evidence for the hypothesis that the weak interactions of leptons and of hadrons are universal in the following sense: All weak interactions are mediated by heavy, charged or neutral bosons carrying spin 1. External fermions couple to them via Lorentz vector and axial vector currents with coupling constants which are the same for leptons and quarks except for effects of state mixing and possible renormalization effects from strong interactions in the quark sector. Up to these modifications in the hadronic sector, a precise definition of which will be given below, semileptonic processes are governed by the same effective interaction. The decay widths for, say, a Fermi transition in a nucleus, for pion β-decay $\pi^+ \to \pi^0 \mathrm{e}^+ \nu_\mathrm{e}$, and for muon decay, are all determined by the same effective coupling constant. The exact relationship between charged interaction and neutral interaction coupling constants for leptons and quarks will be formulated in some detail below, in discussing the GSW unified theory.

Nevertheless, a few empirical properties of semileptonic weak processes can already be formulated at this point without needing the formalism of a unified gauge theory.

A semileptonic process, by definition, involves both hadronic and leptonic states. The empirical selection rules and regularities that we now discuss, always refer to the hadronic part of the process under consideration. Regarding the additive quantum numbers of hadrons, the weak interactions conserve neither strangeness S, nor charm, nor any of the other, new quantum numbers associated with the b-, t-quarks. Furthermore, they are not scalars with respect to isospin: the charged weak currents which do not change the strangeness, $\Delta S = 0$, carry isospin 1, as demonstrated, for instance, by the decay modes

$$\pi^+ \to \mu^+ \nu_\mu, \qquad \pi^+ \to \pi^0 \mathrm{e}^+ \nu_\mathrm{e}. \tag{3.41}$$

Weak neutral currents with $\Delta S = 0$, contain both isoscalar and isovector pieces. If the strangeness is changed, $\Delta S \neq 0$, the weak current carries isospin 1/2, as is clear from the existence of the kaonic decay modes, analogous to (3.41),

$$\mathrm{K}^+ \to \mu^+ \nu_\mu, \quad \mathrm{K}^+ \to \pi^0 \mathrm{e}^+ \nu_\mathrm{e}. \tag{3.42}$$

A most remarkable property of such interactions with $\Delta S \neq 0$ is this: the change in strangeness is always equal to the change of electric charge in the hadronic states,

$$\Delta S = (\Delta Q)_{\text{hadrons}}. \tag{3.43}$$

As an example, the decays

$$\Sigma^+ \to \mathrm{n f}^+ \nu_{\mathrm{f}},$$

where $\Delta S = -1$ but $(\Delta Q)_{\text{hadrons}} = +1$, have never been observed, in contrast to

$$\Sigma^- \to \mathrm{n f}^- \overline{\nu}_{\mathrm{f}},$$

which are allowed by the selection rule (3.43) and which have indeed been seen.

The selection rule (3.43) implies, in particular, that there are no strangeness changing processes with $\Delta Q = 0$, or, in other words, that hadronic *neutral currents do not change the strangeness*. For example, the decay processes

$$\mathrm{K}_{\mathrm{L}}^0 \to \mu^+\mu^-, \quad \mathrm{K}^+ \to \pi^+\mathrm{e}^+\mathrm{e}^- \tag{3.44}$$

are found to be strongly suppressed as compared to the other kaonic decay modes. The experimental branching ratios are

$$R(\mathrm{K}_{\mathrm{L}}^0 \to \mu^+\mu^- / \mathrm{K}_{\mathrm{L}}^0 \to \text{all}) = 6.84 \pm 0.11 \times 10^{-9}, \tag{3.45a}$$

$$R(\mathrm{K}^+ \to \pi^+\mathrm{e}^+\mathrm{e}^- / \mathrm{K}^+ \to \text{all}) = 3.00 \pm 0.09 \times 10^{-7}, \tag{3.45b}$$

[A more detailed analysis shows that the experimental results (3.45) can, in fact, be described by processes of higher than first order.]

3.2 *Vector and Axial-Vector Covariants: Effective Lagrangians with V and A Couplings

In this section we analyze the covariants (3.30) for the case of vector and axial vectors in more detail. Although, historically, this analysis of the empirical information about weak interactions provided the basis for the minimal (GSW) standard model, this subsection is not essential for its actual construction and, hence, may be skipped in a first reading. We construct the general effective four-fermion Lagrangian for the case of vector (V) and axial vector (A) couplings, i.e. the limit of the interaction due to exchange of bosons with spin 1, taken at momentum transfers which are small compared to the boson masses. The behaviour of this Lagrangian under parity P, charge conjugation C, and time reversal T is investigated. The full fermion-boson interaction that leads to the effective Lagrangians will be derived in Sects. 3.3.4 below, whilst the case of more general covariants is deferred to the discussion of possible deviations from V and A couplings in Sect. 4.1.

3.2.1 *Vectors and Axial Vectors*

Let us start by recalling the definition of $\overline{\Psi(x)}$ from (1.77):

$$\overline{\Psi(x)} = (\chi^{*b}(x)\phi_B^*(x)) = (\phi^{*a}(x)\chi_A^*(x))\gamma^0, \tag{3.46}$$

where we had defined

$$\phi^{*a}(x) := \phi_A^*(x)(\hat{\sigma}^0)^{Aa}, \tag{3.46a}$$

$$\chi_A^*(x) := \chi^{*a}(x)(\sigma^0)_{aA}, \tag{3.46b}$$

and γ^0 was given by (1.71'). In the four-component formalism (3.46) is often written, somewhat inaccurately, as

$$\overline{\Psi(x)} = \Psi^\dagger(x)\gamma^0 \tag{3.47}$$

without explicit reference to the class of representations for the matrices γ^μ that is being used. Clearly, in the high-energy representation we may write this relation, provided we keep in mind the correct position of indices in (3.46). If we wish to use any other representation which is obtained from the high-energy representation via a linear, nonsingular transformation S, then the relation (3.47) can only be maintained if S is *unitary*. This is seen very easily by observing that with $\Psi' = S\Psi$ and $\gamma'^\mu = S\gamma^\mu S^{-1}$, (3.47) transforms as follows

$$\overline{\Psi(x)} = \Psi'^\dagger(S^{-1})^\dagger S^{-1}\gamma'^0 S.$$

This is equal to $\overline{\Psi'(x)}S$ only if S is unitary, in which case $\overline{\Psi'(x)}\Psi'(x) = \overline{\Psi(x)}\Psi(x)$.

In Sect. 1.3 we showed that under a proper, orthochronous Lorentz transformation $x' = \Lambda x$, $\Lambda \in \mathrm{L}_+^\uparrow$ the matrices $\sigma^\mu\partial_\mu$ and $\hat{\sigma}^\mu\partial_\mu$ behave as follows:

$$(\sigma^\alpha\partial'_\alpha) = A(\Lambda)(\sigma^\beta\partial_\beta)A^\dagger(\Lambda), \tag{3.48a}$$

$$(\hat{\sigma}^\alpha\partial'_\alpha) = A(\Lambda)(\hat{\sigma}^\beta\partial_\beta)A^{-1}(\Lambda). \tag{3.48b}$$

As the derivative transforms according to

$$\partial'_\alpha = \Lambda_\alpha{}^\beta\partial_\beta,$$

it is clear that both

$$\chi^{*a}(x)(\sigma^\alpha)_{aB}\chi^B(x) \quad \text{and} \quad \phi_A^*(x)(\hat{\sigma}^\alpha)^{Ab}\phi_b(x) \tag{3.49}$$

transform like contravariant vectors with respect to $L_+^\uparrow$, i.e.

$$\chi^{*a}(\Lambda x)(\sigma^\alpha)_{aB}\chi^B(\Lambda x) = \Lambda^\alpha{}_\beta \chi^{*c}(x)(\sigma^\beta)_{cD}\chi^D(x),$$

and similarly for the second covariant. Let us then see how these vectors transform under the parity operation P. From (1.97) we know that under P

$$\begin{aligned}\phi_a(x) &\underset{P}{\to} (\sigma^0)_{aB}\chi^B(Px), \\ \chi^A(x) &\underset{P}{\to} (\hat{\sigma}^0)^{Ab}\phi_b(Px).\end{aligned} \tag{3.50}$$

Inserting this into the expressions (3.49) and observing that

$$\begin{aligned}\hat{\sigma}^0\sigma^\alpha\hat{\sigma}^0 &= \begin{pmatrix}\hat{\sigma}^0 \\ -\hat{\sigma}^i\end{pmatrix}, \\ \sigma^0\hat{\sigma}^\alpha\sigma^0 &= \begin{pmatrix}\sigma^0 \\ -\sigma^i\end{pmatrix},\end{aligned}$$

we find that the sum

$$\chi^*\sigma^\alpha\chi + \phi^*\hat{\sigma}^\alpha\phi =: \upsilon^\alpha \tag{3.51}$$

behaves like a Lorentz *vector*, i.e.

$$\upsilon^\alpha(x) \underset{P}{\to} (-)^{1+\delta_{\alpha 0}}\,\upsilon^\alpha(Px), \tag{3.52}$$

while the difference

$$-\chi^*\sigma^a\chi + \phi^*\hat{\sigma}^\alpha\phi =: a^\alpha \tag{3.53}$$

behaves like an *axial vector*, i.e.

$$a^\alpha(x) \underset{P}{\to} (-)^{\delta_{\alpha 0}}a^\alpha(Px). \tag{3.54}$$

It is easy to verify that the quantities υ^α and a^α, when expressed in terms of Dirac spinors, are given by

$$\upsilon^\alpha(x) = \overline{\Psi(x)}\,\gamma^\alpha\Psi(x), \tag{3.51'}$$

$$a^\alpha(x) = \overline{\Psi(x)}\,\gamma^\alpha\gamma_5\Psi(x). \tag{3.53'}$$

Having derived the behaviour of the currents (3.51) and (3.53) under proper, orthochronous Lorentz transformations and parity, let us now investigate how they transform under *charge conjugation* and *time reversal*.

The transformation properties of spinors under C was derived in Sect. 1.5, viz.

$$\phi_a(x) \underset{C}{\to} \chi_a^*(x),$$
$$\chi^A(x) \underset{C}{\to} -\phi^{*A}(x),$$

and, therefore, $\chi^{*a} \to \phi^a$, $\phi_A^* \to -\chi_A$. Thus

$$\chi^{*a}(\sigma^\alpha)_{aB}\chi^B \to -\phi^a(\sigma^\alpha)_{aB}\phi^{*B} = +\phi_a(\hat{\sigma}^{\alpha *})^{aB}\phi_B^* = -\phi_B^*(\hat{\sigma}^{\alpha *})^{Ba}\phi_a, \quad (3.55a)$$

where we have used relation (1.60) and have made use of the fact that ϕ_a and ϕ_B^* anticommute and that the matrices $\hat{\sigma}^\alpha$ are unitary. Similarly,

$$\phi_A^*(\hat{\sigma}^\alpha)^{Ab}\phi_b \to -\chi_A(\hat{\sigma}^\alpha)^{Ab}\chi_b^* = +\chi^A(\sigma^{\alpha *})_{Ab}\chi^{*b} = -\chi^{*b}(\sigma^\alpha)_{bA}\chi^A. \quad (3.55b)$$

This implies that υ^α is odd, a^α is even under charge conjugation. We note, in particular, that the field operators $\overline{\Psi(x)}$ and $\Psi(x)$ in (3.53',51') are interchanged in taking the charge conjugate of υ^α and a^α.

Time reversal, finally, has the following effect:

$$\phi_a(x) \underset{T}{\to} \phi_a^*(Tx) = (\sigma^0)_{aB}\varepsilon^{BD}\phi_D^*(Tx),$$
$$\chi^A(x) \underset{T}{\to} \chi^{*A}(Tx) = (\hat{\sigma}^0)^{Ab}\varepsilon_{bd}\chi^{*d}(Tx),$$

so that

$$\chi^{*a}(x) \to \chi^B(Tx)\varepsilon_{BD}(\hat{\sigma}^0)^{Da},$$
$$\phi_A^*(x) \to \phi_b(Tx)\varepsilon^{bd}(\sigma^0)_{dA}.$$

In applying T to a product of operators, the prescription is to reverse the order of factors without regard to the fermion character of the fields. Thus we have

$$\begin{aligned}\chi^{*a}(x)(\sigma^\alpha)_{aA}\chi^A(x) &\to \varepsilon_{BD}(\hat{\sigma}^0)^{Da}(\sigma^\alpha)_{aA}(\hat{\sigma}^0)^{Ab}\varepsilon_{bd}\,\chi^{*d}(Tx)\chi^B(Tx) \\ &= \varepsilon_{BD}\begin{pmatrix}\hat{\sigma}^0 \\ -\hat{\sigma}^i\end{pmatrix}^{Db}\varepsilon_{bd}\chi^{*d}(Tx)\chi^B(Tx) \qquad (3.56a) \\ &= \chi^{*d}(Tx)\begin{pmatrix}\sigma^0 \\ -\sigma^i\end{pmatrix}_{dB}\chi^B(Tx).\end{aligned}$$

In the first step we inserted the relation mentioned above, in the second we used $(\sigma^{\alpha *})_{Bd} = (\sigma^\alpha)_{dB}$, i.e. the unitarity of σ^α. In a similar manner

$$\phi_A^*(\hat{\sigma}^\alpha)^{Ab}\phi_b \to \phi^*{}_B(Tx)\begin{pmatrix}(\hat{\sigma}^0)^{Ba} \\ (-\hat{\sigma}^i)^{Ba}\end{pmatrix}\phi_a(Tx). \quad (3.56b)$$

Thus, v^α and a^α behave as follows under T:

$$v^\alpha(x) \to (-)^{1+\delta_{\alpha 0}} v^\alpha(Tx), \tag{3.57}$$

$$a^\alpha(x) \to (-)^{1+\delta_{\alpha 0}} a^\alpha(Tx). \tag{3.58}$$

Note, however, that here again the fields $\Psi(x)$ and $\overline{\Psi(x)}$ are interchanged.

[It is instructive to compare the behaviour of v^α under P, C, and T, with the behaviour of a classical vector current, given by the product of a charge and a four-velocity, under the same transformations.]

Finally, from this discussion we can deduce the behaviour under parity, charge conjugation and time reversal, of vector and axial vector currents which are composed of two *different* fields, i.e.

$$v^\alpha(x; i \to k) := \overline{\Psi^{(k)}(x)}\, \gamma^\alpha\, \Psi^{(i)}(x), \tag{3.59}$$

$$a^\alpha(x; i \to k) := \overline{\Psi^{(k)}(x)}\, \gamma^\alpha \gamma_5 \Psi^{(i)}(x). \tag{3.60}$$

The transformation behaviour of the current operators (3.59, 3.60) with respect to proper, orthochronous Lorentz transformations $\Lambda \in \mathrm{L}_+^\uparrow$ is the same as for the operators (3.51', 53') where particle field i and particle field k were identical that is

$$v'^\alpha(\Lambda x; i \to k) = \Lambda^\alpha{}_\beta\, v^\beta(x; i \to k)$$

and similarly for a^α.

Regarding parity we must keep in mind that the fields i and k may have different intrinsic parities, in which case there is an extra minus sign in (3.52, 3.54). Intrinsic parity of a particle field is always defined relative to the vacuum state which by convention is assigned positive parity. The intrinsic parities of some particles in nature are fixed and given by experiment (examples are the photon field, and the neutral pion for which the intrinsic parity can be measured).

Similarly, if the intrinsic parity of a fermion is fixed then we know from the theory of Dirac fields that the corresponding antiparticle has the opposite parity. The comparison of relative intrinsic parities for two different particle fields i and k is not possible whenever an absolute conservation law forbids the transformation of one particle of type i into one particle of type k, via a parity conserving reaction. For example, the relative parity of proton and neutron cannot be determined from the parities of the vacuum, the π^0 and the photon alone because charge conservation forbids vertices such as (pnπ^0) or (pnγ). However, if, by convention, we define the intrinsic parity of the charged pions π^+, π^- to be the same as for the neutral one, experiments such as π^- p $\to \pi^0$n or πd $\to$ nn will show that p and n then have even relative parity (or vice versa, the assignment of the same parity to p and n will fix the relative parity of $\pi^\pm$ and π^0). Similarly, the relative parity of electron and proton, or even of electron and muon, is not fixed and cannot be taken from experiment as long as electron and muon family numbers and baryon number are absolutely conserved.

Thus, fixing the relative intrinsic parities of particle fields needs a certain number of conventions (beyond the experimental information as in the case of π^0 etc.). We do not enter this discussion in any more detail but just mention that it is possible to assign the same (positive) intrinsic parity to the lepton families (3.7) and the quark families (3.15), without running into conflict with experiment. The intrinsic parity of composite states such as (3.17)–(3.19), of course, then is determined by the parity of the orbital quark–antiquark and three quark states, respectively. The physical current operators that we shall consider, connect either leptons of the same family with one another, our quarks within the flavour families (3.15). So, even though in the case of two different fields the relations (3.52) and (3.54) must be multiplied by a factor

$$\eta_P^{(k)*}\eta_P^{(i)},$$

where $\eta_P^{(i)} = \mathrm{e}^{i\pi\alpha(i)}$ with $\alpha(i) = 0$ or 1, we can choose conventions such that this factor is always $+1$.

In a similar way, charge conjugation C and time reversal T, in general, will give additional signs, or phase factors, when applied to the operators (3.59) and (3.60):

$$\overline{\Psi^{(k)}(x)}\,\gamma^\alpha\,\Psi^{(i)}(x) \underset{C}{\to} -\eta_C^{(k)*}\eta_C^{(i)}\,\overline{\Psi^{(i)}(x)}\,\gamma^\alpha\,\Psi^{(k)}(x), \tag{3.61a}$$

$$\overline{\Psi^{(k)}(x)}\,\gamma^\alpha\gamma_5\Psi^{(i)}(x) \underset{C}{\to} \eta_C^{(k)*}\eta_C^{(i)}\,\overline{\Psi^{(i)}(x)}\,\gamma^\alpha\gamma_5\Psi^{(k)}(x), \tag{3.61b}$$

$$\overline{\Psi^{(k)}(x)}\,\gamma^\alpha\,\Psi^{(i)}(x) \underset{T}{\to} \eta_T^{(k)*}\eta_T^{(i)}(-)^{1+\delta_{\alpha 0}}\overline{\Psi^{(i)}(x)}\,\gamma^\alpha\Psi^{(k)}(Tx), \tag{3.62}$$

the T-transformation of $a^\alpha(i \to k)$ being the same as that of $v^\alpha(i \to k)$. Again, for the physical currents of interest in electroweak interactions the additional phase factors

$$\eta_C^{(k)*}\eta_C^{(i)} \quad \text{and} \quad \eta_T^{(k)*}\eta_T^{(i)} \tag{3.63}$$

can be chosen to be $+1$ by a suitable choice of conventions. If a composite state is an eigenstate of C (or of G-parity $G = C\exp\{i\pi I_2\}$), its charge conjugation phase is completely determined by the properties of the bound state wave functions (see exercises 3.1 and 3.2).

3.2.2 *Effective Vector (V) and Axial-Vector (A) Interactions*

On the basis of the empirical information discussed in Sect. 3.1.2 and equipped with the technical tools of the preceding Sect. 3.2.1 we can now write down the most general, effective Lagrangian for four external fermions interacting via vector and axial vector currents. It reads

$$\begin{aligned}-\mathscr{L}_{\mathrm{VA}} = \frac{G}{\sqrt{2}}\{&(\overline{\Psi^{(k)}(x)}\,\gamma^\alpha\,\Psi^{(i)}(x))\\ &\times [C_{\mathrm{V}}(\overline{\Psi^{(m)}(x)}\,\gamma_\alpha\Psi^{(n)}(x) + C'_{\mathrm{V}}(\overline{\Psi^{(m)}(x)}\,\gamma_\alpha\gamma_5\Psi^{(n)}(x))]\end{aligned}$$

$$+ (\overline{\Psi^{(k)}(x)}\,\gamma^{\alpha}\gamma_5\Psi^{(i)}(x))[C_A(\overline{\Psi^{(m)}(x)}\,\gamma_{\alpha}\gamma_5\Psi^{(n)}(x))$$
$$+ C'_A(\overline{\Psi^{(m)}(x)}\,\gamma_{\alpha}\Psi^{(n)}(x))] + \text{hermitean conjugate}\}, \tag{3.64}$$

where $C_{\mathrm{V}}, \ldots, C'_{\mathrm{A}}$ are four complex constants. For the sake of convenience a real constant $G/\sqrt{2}$ has been factorized in (3.64), so, that, in fact, C_{V} (if it does not vanish) can be chosen to be a simple phase, or if that phase is not relevant in the interference of the interaction (3.64) with other interactions, to be $+1$. Even though there is this redundance, we keep C_{V} explicitly in order to preserve the symmetry of (3.64) in the four types of couplings. Let us rewrite (3.64), in a more compact form and omitting the space-time argument x which is the same in all current operators,

$$-\mathscr{L}_{\mathrm{VA}} = \frac{G}{\sqrt{2}}\{\upsilon^{\alpha}(i \to k)[C_{\mathrm{V}}\upsilon_{\alpha}(n \to m) + C'_{\mathrm{V}}a_{\alpha}(n \to m)]$$
$$+ [C^*_{\mathrm{V}}\upsilon^{\alpha}(m \to n) + C'^*_{\mathrm{V}}a^{\alpha}(m \to n)]\upsilon_{\alpha}(k \to i)$$
$$+ a^{\alpha}(i \to k)[C_{\mathrm{A}}a_{\alpha}(n \to m) + C'_{\mathrm{A}}\upsilon_{\alpha}(n \to m)]$$
$$+ [C^*_{\mathrm{A}}a^{\alpha}(m \to n) + C'^*_{\mathrm{A}}\upsilon^{\alpha}(m \to n)]a_{\alpha}(k \to i)\}. \tag{3.64'}$$

Here we have used $\gamma^0(\gamma^{\alpha})^{\dagger}\gamma^0 = \gamma^{\alpha}$ and $\gamma^0(\gamma^{\alpha}\gamma_5)^{\dagger}\gamma^0 = \gamma^{\alpha}\gamma_5$. By construction, $\mathscr{L}_{\mathrm{VA}}$ is invariant under all $\Lambda \in \mathrm{L}^{\uparrow}_{+}$. In order to make it conserve electric charge and all other, additively conserved, quantum numbers such as L_{f} and B, the fields i, k, m, n must be chosen such that the net balance of these "charges" is zero.

For example, in the case of a charged current (CC) leptonic weak interaction we have

$$k \equiv f,\ i \equiv \nu_{\mathrm{f}}, \tag{3.65}$$
$$m \equiv \nu_{\mathrm{f}'}, n \equiv \mathrm{f}',$$

Indeed, the combination

$$(\overline{\Psi^{(\mathrm{f})}(x)}\,\Gamma^{\alpha}\Psi^{(\nu_{\mathrm{f}})}(x))\,\overline{\Psi^{(\nu_{\mathrm{f}'})}(x)}\,\Gamma'_{\alpha}\Psi^{(\mathrm{f}')}(x)) + \text{h.c.} \tag{3.65'}$$

of charged leptons (f,f') and uncharged leptons (ν_{f}, $\nu_{\mathrm{f}'}$) preserves electric charge, as well as the lepton family numbers L_{f} and L'_{f}. In the case of neutral current (NC) interactions, i and k must clearly be identical, as must m and n. For example, leptonic NC interactions will have

$$i \equiv k = \mathrm{f} \qquad \text{or} \qquad i \equiv k = \nu_{\mathrm{f}},$$

and similarly, $m \equiv n = \mathrm{f}'$ or $\nu_{\mathrm{f}'}$.

Let us then study the behaviour of (3.64'), under P, C and T.

(a) *Parity P*: The parity operation leaves invariant the product of two like operators ($v \cdot v$ or $a \cdot a$) but changes the sign of the product of two unlike operators ($v \cdot a$ or $a \cdot v$), up to a common phase factor $\eta_P = \eta_P^{(k)*} \eta_P^{(i)} \eta_P^{(m)*} \eta_P^{(n)}$. As pointed out before this factor can be chosen to be $+1$ in all cases of interest here. Thus the terms with unprimed coefficients C_i are even, the terms multiplied by primed constants C_i' are odd, or, written symbolically,

$$\{C_i, C_i'\} \underset{P}{\rightarrow} \{C_i, -C_i'\}. \tag{3.66}$$

(b) *Charge conjugation C*: From (3.61a) one sees that under C the terms $v \cdot v$ and $a \cdot a$ are transformed into their hermitean conjugates without extra signs, whilst the products $v \cdot a$ and $a \cdot v$ are transformed into minus their hermitean conjugates. In all cases this holds up to the phase factor $\eta_C = \eta_C^{(k)*} \eta_C^{(i)} \eta_C^{(m)*} \eta_C^{(n)}$ which, however, can be chosen to be $+1$. Thus, the action of C on $\mathscr{L}_{\mathrm{VA}}$ can be summarized by

$$\{C_i, C_i'\} \underset{C}{\rightarrow} \{C_i^*, -C_i'^*\}. \tag{3.67}$$

(c) *Time reversal T*: As is evident from (3.62), all terms in $\mathscr{L}_{\mathrm{VA}}$ are transformed into their hermitean conjugate, irrespective of whether they are products of like or unlike current densities. This holds again up to a common phase factor $\eta_T = \eta_T^{(k)*} \eta_T^{(i)} \eta_T^{(m)*} \eta_T^{(n)}$. This phase factor can be taken to be $+1$ in all cases of interest. Thus $\mathscr{L}_{\mathrm{VA}}$ transforms according to

$$\{C_i, C_i'\} \underset{T}{\rightarrow} \{C_i^*, C_i'^*\}. \tag{3.68}$$

We conclude this section with some remarks on these results.

(i) Let us combine the three operators P, C, and T to their product

$$\Theta := PCT. \tag{3.69}$$

From the results a)–c), equations (3.66–3.68), we see that the interaction (3.64) is invariant under the combined operation (3.69),

$$\Theta \mathscr{L}_{\mathrm{VA}} \Theta^{-1} = \eta_P \eta_C \eta_T \mathscr{L}_{\mathrm{VA}} \text{ or } \{C_i, C_i'\} \underset{\Theta}{\rightarrow} \eta_P \eta_C \eta_T \{C_i, C_i'\}. \tag{3.70}$$

This is a special case of the *PCT*-theorem of Lüders and Jost (Lüders 1957, Jost 1957, 1963). Thus, if $\mathscr{L}$ is not invariant under one of the three symmetries P, C and T, it must break at least one more of them: For example, P and C could be violated but T conserved.

Parity invariance is broken if simultaneously some C_i as well as some C_i' are different from zero. On the other hand if $\mathscr{L}_{\mathrm{VA}}$ contains only unprimed couplings, or primed couplings, it is invariant under parity. [In this case parity could still be broken through interference with some other interaction

Lagrangian whose behaviour under P is different from the behaviour of $\mathscr{L}_{\mathrm{VA}}$ (3.64).]

For time reversal invariance to be broken at least some of the coupling constants must be relatively complex, cf. (3.68). For violation of C-invariance either of the two previous conditions, or both, must be fulfilled.

(ii) An important special case, relevant for leptonic CC interactions, is one where all C_i and C_i' are relatively real and where $C_i' = -C_i$. In this case $\mathscr{L}_{\mathrm{VA}}$ is invariant under time reversal while invariance under parity and charge conjugation is violated in a *maximal* way: the P-odd terms in $\mathscr{L}_{\mathrm{VA}}$ have the same magnitude as the P-even terms, the C-odd terms have the same magnitude as the C-even terms.

(iii) In a contact interaction of the product form (3.65') the order of the fields is not fixed uniquely. For instance the fields $\Psi^{(\mathrm{vf})}$ and $\Psi^{(\mathrm{f})}$ in (3.65') can be interchanged by means of Fierz reordering (Fierz, 1936), at the price of expressing the product of two specific covariants in one ordering by a sum over S, P, V, A, and T covariants in the other ordering. For example (cf. Sect. 4.1.1d)

$$
\begin{aligned}
&(\overline{\Psi^{(1)}}\gamma^{\alpha}\Psi^{(2)})(\overline{\Psi^{(3)}}\,\gamma_{\alpha}\Psi^{(4)}) \\
&\quad = -(\overline{\Psi^{(1)}}\mathbb{1}\Psi^{(4)})(\overline{\Psi^{(3)}}\mathbb{1}\Psi^{(2)}) - (\overline{\Psi^{(1)}}\gamma_5\Psi^{(4)})(\overline{\Psi^{(3)}}\gamma_5\Psi^{(2)}) \\
&\qquad + \frac{1}{2}(\overline{\Psi^{(1)}}\gamma^{\alpha}\Psi^{(4)})(\overline{\Psi^{(3)}}\gamma_{\alpha}\Psi^{(2)}) + \frac{1}{2}(\overline{\Psi^{(1)}}\gamma^{\alpha}\gamma_5\Psi^{(4)})(\overline{\Psi^{(3)}}\gamma_{\alpha}\gamma_5\Psi^{(2)}). \qquad (3.71)
\end{aligned}
$$

A specific ordering is singled out only when the effective Lagrangian $\mathscr{L}_{\mathrm{VA}}$ is the limiting form of an interaction due to the exchange of a heavy vector boson which couples to specific and well-defined currents (3.59) or (3.60).

(iv) Finally, we note that the C, P and T transformations can also be applied to interactions containing products of other covariants (3.30), such as scalars, pseudoscalars or tensors. We retun to this in Sect. 4.1.

3.2.3 *Charged Current and Neutral Current V and A Interactions Due to Exchange of Heavy Vector Bosons*

Suppose the CC leptonic weak interactions are due to the exchange of charged vector mesons $\mathrm{W}^{\pm}$ of mass m_{w}. Write the complex fields describing these particles as linear combinations of two mass degenerate real fields $A_{\alpha}^{(1)}$ and $A_{\alpha}^{(2)}$, in the spherical basis,

$$
W_{\alpha}^{\pm} = \mp\frac{1}{\sqrt{2}}(A_{\alpha}^{(1)} \pm \mathrm{i}A_{\alpha}^{(2)}). \qquad (3.72)
$$

If these fields couple exclusively to the left-handed current $\gamma^{\alpha}\frac{1}{2}(\mathbb{1}-\gamma_5) = \gamma^{\alpha}P-$, and if the coupling constant is denoted by g, the exchange of a $\mathrm{W}^{\pm}$ between two pairs of leptons gives rise to amplitudes of the form

$$(\nu_{\mathrm{f}}\mathrm{f}\,|\,T\,|\,\mathrm{f}'\nu_{\mathrm{f}'}) \propto \frac{1}{\sqrt{2}} g(\overline{u}_{\mathrm{f}}\gamma^{\alpha} P_{-} u_{\nu_{\mathrm{f}}}) \frac{-g_{\alpha\beta} + q_{\alpha}q_{\beta}/m_{\mathrm{w}}^2}{q^2 - m_{\mathrm{w}}^2} \frac{1}{\sqrt{2}} g(\overline{u_{\nu_{\mathrm{f}'}}}\gamma^{\beta} P_{-} u_{\mathrm{f}'}), \tag{3.73}$$

where q is the four-momentum transferred in the reaction. In the limit of small momentum transfer, $|\,q^2\,| \ll m_{\mathrm{w}}^2$, the W-propagator may be replaced by $g_{\alpha\beta}\,/\,m_{\mathrm{w}}^2$. In this approximation the amplitude (3.73) (as well as the analogous amplitudes with the external particles replaced by antiparticles), may be viewed as being due to the effective contact interaction

$$-\mathscr{L}_{\mathrm{CC}}^{eff} = \frac{g^2}{8m_{\mathrm{w}}^2} (\overline{\Psi^{(\mathrm{f})}(x)}\,\gamma^{\alpha}(\mathbb{1} - \gamma_5)\Psi^{(\nu_{\mathrm{f}})}(x))\,(\overline{\Psi^{(\nu_{\mathrm{f}'})}(x)}\gamma_{\alpha}(\mathbb{1} - \gamma_5)\Psi^{(\mathrm{f}')}(x)) + \mathrm{h.c.} \tag{3.74}$$

This has the form of (3.64, 64') with the identification (3.65) and the following special values of the coupling constants:

$$G/\sqrt{2} = g^2/8m_{\mathrm{w}}^2, \tag{3.75}$$

$$C_{\mathrm{V}} = C_{\mathrm{A}} = 1, C_{\mathrm{V}}' = C_{\mathrm{A}}' = -1. \tag{3.76}$$

It follows from the general discussion in Sect. 3.2.2 that the interaction (3.74) is invariant under time reversal but that it violates both parity and charge conjugation invariance. The violation of invariance under parity, in fact, is maximal: the interaction (3.74) produces neutrinos in purely left-handed states, antineutrinos in purely right-handed states. The interaction (3.74) is generally referred to as the *effective V–A four-fermion interaction*.

Suppose further that the NC weak interactions of leptons (and quarks) are due to amplitudes in which a heavy neutral vector boson Z^0 is exchanged between two pairs of external fermions of like charge. In the simplest case the field Z_{α} and the photon field A_{α} are orthogonal linear combinations of a field $A_{\alpha}^{(3)}$, the neutral partner of $A_{\alpha}^{(1)}$ and $A_{\alpha}^{(2)}$, and still another neutral vector field $A_{\alpha}^{(0)}$, viz.

$$A_{\alpha}(x) = \frac{1}{\sqrt{g^2 + g'^2}}\{g' A_{\alpha}^{(3)}(x) + g\, A_{\alpha}^{(0)}(x)\}, \tag{3.77a}$$

$$Z_{\alpha}(x) = \frac{1}{\sqrt{g^2 + g'^2}}\{g A_{\alpha}^{(3)}(x) - g' A_{\alpha}^{(0)}(x)\}. \tag{3.77b}$$

There is, of course, some arbitrariness in this ansatz. We have written these equations in anticipation of the results of a local gauge theory of electroweak interactions which is based on the gauge group $\mathrm{U}(2) \simeq \mathrm{SU}(2) \times \mathrm{U}(1)$. The fields $A_{\alpha}^{(0)}$ and $\{A_{\alpha}^{(1)}, A_{\alpha}^{(2)}, A_{\alpha}^{(3)}\}$ are then gauge fields and, therefore, fall into the adjoint representations of U(1) and SU(2), respectively.

These gauge fields, which are defined with respect to the underlying group, are not observable as such. The physical vector mesons with electric charge ± 1 are defined by (3.72), whilst the electrically neutral photon and Z^0 boson fields are given by (3.77). In terms of the latter the gauge fields $A_{\alpha}^{(3)}$ and $A_{\alpha}^{(0)}$ are given by

$$A_\alpha^{(3)}(x) = \frac{1}{\sqrt{g^2 + g'^2}}\{g Z_\alpha(x) + g' A_\alpha(x)\}, \tag{3.78a}$$

$$A_\alpha^{(0)}(x) = \frac{1}{\sqrt{g^2 + g'^2}}\{-g' Z_\alpha(x) + g A_\alpha(x)\}. \tag{3.78b}$$

At this stage and without having developed that theory, it is not possible to guess the detailed form of physical couplings between fermions and the neutral vector bosons except for the following input conditions:

(i) The coupling of the charged leptons and quarks to the photon must be of the form $\overline{\Psi^{(i)}}\gamma_\alpha \Psi^{(i)} A^\alpha$, i.e. must conserve parity; the coupling constant must be the electric charge of the external fermion.
(ii) The neutrinos, being electrically neutral, must not couple to the photon field.
(iii) If only lefthanded neutrinos couple in weak interactions, they must couple to the Z^0 via the V–A current $\overline{\Psi^{(\nu_f)}}\gamma_\alpha(1 - \gamma_5)\Psi^{(\nu_f)}$.

Furthermore, if ν_f and f form an isopinlike doublet, (3.7), with respect to the gauge group, we may guess that the field $A_\alpha^{(3)}$ couples to this doublet via pure V–A operators $\gamma^\alpha P_-$ and via an operator of the type σ_3. The parameter g being the coupling constant to $A_\alpha^{(1)}$ and $A_\alpha^{(2)}$, this means that ν_f and f couple to $A_\alpha^{(3)}$ with equal and opposite coupling constants, $\pm\frac{1}{2}g$. Let us first consider the neutrino couplings:

$$\mathscr{L}_{NC}^{(\nu)} = \{\frac{1}{2}g A_\alpha^{(3)}(x) + \kappa A_\alpha^{(0)}(x)\}\overline{\Psi^{(\nu_f)}(x)}\,\gamma^\alpha P_- \Psi^{(\nu_f)}(x). \tag{3.79}$$

A glance at (3.78) shows that the curly brackets exclude the photon field if and only if $\kappa = -\frac{1}{2}g'$. In this case (3.79) becomes

$$\mathscr{L}_{NC}^{(\nu)} = \frac{1}{2}\sqrt{g^2 + g'^2}\, Z_\alpha^0(x)\,\overline{\Psi^{(\nu_f)}}\,\gamma^\alpha P_- \Psi^{(\nu_f)}(x). \tag{3.79'}$$

The couplings of its charged partner f^- to the neutral gauge fields must have the form

$$\mathscr{L}_{\text{neutral}}^{(f)} = \overline{\Psi^{(f)}(x)}\{-\frac{1}{2}g A_\alpha^{(3)}(x)\gamma^\alpha P_- - \frac{1}{2}g' A_\alpha^{(0)}(x)(\lambda\gamma^\alpha P_+ + \gamma^\alpha P_-)\}\,\Psi^{(f)}(x). \tag{3.80}$$

We have fixed the second term of the coupling to $A_\alpha^{(0)}$ on the basis of the following consideration: $A_\alpha^{(0)}$ being a singlet field, we expect this field to couple to the left-handed current $\overline{\Psi^{(f)}}\gamma^\alpha P_- \Psi^{(f)}$ with the same strength $\kappa = -\frac{1}{2}g'$ as to the neutrino field, cf. (3.79). The constant λ, finally, must be chosen such as to meet the requirement (i) above. This is guaranteed if and only if $\lambda = 2$, as in this case the terms with a $\gamma^\alpha\gamma_5$ coupling to the photon field A_α cancel out. Setting $\lambda = 2$, (3.80) becomes

$$\mathscr{L}^{(\mathrm{f})}_{\mathrm{neutral}} = \frac{1}{\sqrt{g+g'^2}}\overline{\Psi^{(\mathrm{f})}(x)}\,\gamma^\alpha\,\{-gg'\,(P_- + P_+)\,A_\alpha\,(x) + \frac{1}{2}[g'^2(2P_+ + P_-) - g^2\,P_-]Z^0_\alpha(x)\}\Psi^{(\mathrm{f})}(x). \tag{3.80'}$$

Of course, $P_+ + P_-$ is unity. Therefore, the elementary charge is to be identified as follows:

$$Q_{\mathrm{f}} = -|\,e\,| = gg'/\sqrt{g^2+g'^2}. \tag{3.81}$$

As $e^2 \leq g^2$, one can set

$$e^2/g^2 =: \sin^2\theta_{\mathrm{w}}, \tag{3.82}$$

$$g' = -g\,tg\,\theta_{\mathrm{w}}. \tag{3.83}$$

Equation (3.82) defines the Weinberg angle θ_{w}.

With this parametrization the neutral interactions can be written as follows:

$$\mathscr{L}^{(\mathrm{f})}_{\mathrm{neutral}} = -Q_{\mathrm{f}}\,A_\alpha\,(x)\overline{\Psi^{(\mathrm{f})}(x)}\,\gamma^\alpha\,\Psi^{(\mathrm{f})}\,(x) + \frac{g}{\cos\theta_{\mathrm{w}}}\,Z_\alpha\,(x)\{\overline{\Psi^{(\mathrm{f})}\,(x)}[-\frac{1}{2}\,\gamma^\alpha\,P_- + \sin^2\,\theta_{\mathrm{w}}\,\gamma^\alpha]\,\Psi^{(\mathrm{f})}(x) + \overline{\Psi^{(\nu_{\mathrm{f}})}\,(x)}\frac{1}{2}\,\gamma^\alpha\,P_-\,\Psi^{(\nu_{\mathrm{f}})}\,(x)\}, \tag{3.84}$$

where $e = |e|$ and $Q_{\mathrm{f}} = -e$.

As in the case of the charged bosons $\mathrm{W}^\pm$, the exchange of a Z^0 with squared momentum transfer small as compared to its mass, $|\,q^2\,| \ll m^2_{\mathrm{Z}}$, gives rise to an effective current–current interaction of the form

$$-\mathscr{L}^{\mathrm{eff}}_{\mathrm{NC}} = \frac{1}{2}\frac{g^2}{16\cos^2\,\theta_{\mathrm{w}}}\frac{1}{m^2_{\mathrm{Z}}}\,K^\dagger_\alpha\,(x)\,K^\alpha\,(x), \tag{3.85}$$

where

$$K^{(\mathrm{f})}_\alpha\,(x) = \overline{\Psi^{(\mathrm{f})}(x)}[-\gamma_\alpha\,(1\,-\,\gamma_5) + 4\,\sin^2\,\theta_{\mathrm{w}}\,\gamma_\alpha]\,\Psi^{(\mathrm{f})}\,(x) + \overline{\Psi^{(\nu_{\mathrm{f}})}(x)}\gamma_\alpha\,(1-\gamma_5)\Psi^{(\nu_{\mathrm{f}})}\,(x), \tag{3.86}$$

(the factor $\frac{1}{2}$ in (3.85) accounting for double counting).

The effective coupling strength in (3.85) can also be written in terms of G, (3.75)

$$\frac{g^2}{16\cos^2\,\theta_{\mathrm{w}}}\frac{1}{m^2_{\mathrm{Z}}} = \frac{1}{2}\,\rho\,\frac{G}{\sqrt{2}}, \tag{3.87}$$

with the definition

$$\rho := \frac{m_{\mathrm{W}}^2}{m_{\mathrm{Z}}^2 \cos^2 \theta_{\mathrm{w}}}. \tag{3.88}$$

The effective interaction (3.85) is of V and A character with real coupling constants. Therefore, it is invariant under time reversal but breaks both parity and charge conjugation symmetry.

These semiempirical results constitute the leptonic sector of the Glashow–Salam–Weinberg unified theory of electroweak interactions (Glashow 1963, Salam 1967, Weinberg 1967). They will be derived below in the mathematical framework of local gauge theories.

In anticipation of later results we note at this point that the phenomenological analysis of the data yields $G = 1.16639(2) \times 10^{-5} \mathrm{GeV}^{-2}$, $\sin^2 \theta_{\mathrm{w}} = 0.2319(5)$, $\rho \simeq 1$. This suffices to estimate the masses of $\mathrm{W}^{\pm}$ and Z^0. From (3.75) we have

$$m_{\mathrm{W}} = \sqrt{\frac{\pi\alpha}{G\sqrt{2}}} \frac{1}{\sin\theta_{\mathrm{w}}} = 37.28/\sin\theta_{\mathrm{w}}\,\mathrm{GeV}, \tag{3.89a}$$

and from (3.88)

$$m_{\mathrm{Z}} = \frac{m_{\mathrm{W}}}{\cos\theta_{\mathrm{W}}} = 37.28/\sin\theta_{\mathrm{W}}\,\cos\theta_{\mathrm{W}}\mathrm{GeV}. \tag{3.89b}$$

[These expressions hold up to radiative corrections.] Thus m_{w} is found to be of the order of 80 GeV, m_{z} of the order of 90 GeV, or expressed in Compton wavelengths, $\lambda\!\!\!^{-}(\mathrm{W}) \simeq 2.5 \times 10^{-16}$cm, and $\lambda\!\!\!^{-}(\mathrm{Z}^0) \simeq 2.2 \times 10^{-16}$cm. This gives a measure for the range of weak CC and NC interactions. Indeed, the experimental masses are found to be

$$m_{\mathrm{W}} = (80.399 \pm 0.023)\,\mathrm{GeV}, \quad m_{\mathrm{Z}} = (91.1876 \pm 0.0021)\,\mathrm{GeV}. \tag{3.90a}$$

Their total widths are

$$\Gamma_{\mathrm{W}} = (2.085 \pm 0.042)\,\mathrm{GeV}, \quad \Gamma_{\mathrm{Z}} = (2.4952 \pm 0.0023)\,\mathrm{GeV} \tag{3.90b}$$

3.2.4 Difficulties of the Effective Current-Current Theory

The effective current–current interactions (3.74) and (3.85), as well as their hadronic counterparts, are very useful in practical applications to weak processes at low and intermediate energies, provided they are treated in lowest order perturbation theory. However, if one applies them to weak scattering processes at high energies (say $\gg 100$ GeV) or if one tries to compute higher order effects, one runs into two kinds of difficulties. A contact interaction of the current–current form (3.74) or (3.85) leads to neutrino cross sections which increase linearly with increasing neutrino (laboratory) energy, cf. (3.23b). This linear increase cannot hold indefinitely and

is in contradiction to unitarity. Furthermore, such a theory is not renormalizable, that is higher order diagrams cannot be made finite by means of a *finite* number of renormalization constants. Clearly, the two problems are intimately related. Since we do not know how to handle a nonrenormalizable field theory, we cannot decide whether or not a theory with contact interactions makes mathematical sense, in an exact way. That lowest order diagrams (generalized Born terms) lead to conflict with unitarity at high energies is not new, of course. In the frame of perturbation theory unitarity of the S-matrix is restored by diagrams of higher orders. These, however, cannot be computed in a theory which is not renormalizable.

In order to obtain a more quantitative feeling of where this conflict with unitarity is to be expected let us work out the following very simple example: Consider the elastic scattering process

$$\nu_e + e^- \to e^- + \nu_e \tag{3.91}$$

on the basis of the CC contact interaction (3.74) [and neglecting the NC interaction (3.85) that contributes to this process, too]. From the formulae in App. B we have

$$d\sigma = \frac{(2\pi)^{10}}{4(pq)} \frac{1}{2} \sum |T|^2 \, \delta(p + q - p' - q') \frac{d^3q'}{2q'_0} \frac{d^3p'}{2p'_0},$$

where p, p' denote the initial and final neutrino momenta, respectively, q, q' those of the electron before and after the scattering.

Let $s = (p + q)^2$ and $d^3q' = x^2 dx d\Omega^*$, where $x \equiv |\boldsymbol{q}'|$. Integrating over d^3p' one finds

$$\frac{d\sigma}{d\Omega^*} = \frac{(2\pi)^{10}}{16(s - m_e^2)} \int_0^\infty x dx \, \frac{\delta\left(\sqrt{x^2 + m_e^2} + x - \sqrt{s}\right)}{\sqrt{x^2 + m_e^2}} \sum |T|^2.$$

The integral over x gives the integrand at $x_0 = (s - m_e^2)/2\sqrt{2}$ times a factor $\sqrt{x_0^2 + m_e^2}/\sqrt{s}$ which stems from the derivative of the argument of the δ-distribution,

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{32s} (2\pi)^{10} \sum |T|^2.$$

The squared T-matrix element for the interaction (3.74), when summed over the spins is

$$\begin{aligned}(2\pi)^{12} \sum |T|^2 &= \frac{1}{2} G^2 \mathrm{tr}(\gamma^\alpha (1 - \gamma_5) \not{p} \gamma^\beta (1 - \gamma_5)(\not{q}' + m_e)\} \\ &\quad \times \mathrm{tr}\{\gamma_\alpha (1 - \gamma_5)(\not{q} + m_e) \gamma_\beta (1 - \gamma_5) \not{p}'\} \\ &= 128\, G^2\, (pq)(p'q') = 32 G^2\, (s - m_e^2)^2,\end{aligned}$$

so that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{G^2}{4\pi^2}\frac{(s-m_\mathrm{e}^2)^2}{s} = 1.34\times 10^{-39}\,\mathrm{cm}^2\,\mathrm{GeV}^{-2}\frac{(s-m_\mathrm{e}^2)^2}{s} \tag{3.92}$$

and the integrated cross section is

$$\sigma = \frac{G^2}{\pi}\frac{(s-m_\mathrm{e}^2)^2}{s}. \tag{3.93}$$

Let us now analyze these results in some detail. The differential cross section in the c.m. system (3.92) as well as the integrated cross section (3.93), for $s \gg m_\mathrm{e}^2$, increase like s, i.e. like the *square* of the neutrino energy in the c.m. frame. This same cross section (3.93) can also be evaluated in the laboratory system, where

$$s = (p+q)^2 = m_\mathrm{e}^2\,(1 + 2E_\nu^{\mathrm{lab}}/m_\mathrm{e}) \equiv m_\mathrm{e}^2(1+2\omega)$$

With $\omega := E_\nu^{\mathrm{lab}}/m_\mathrm{e}$. This gives

$$\sigma = \frac{2G^2m_\mathrm{e}^2}{\pi}\frac{2\omega^2}{1+2\omega} = 8.8\times 10^{-45}\,\mathrm{cm}^2\frac{2\omega^2}{1+2\omega}. \tag{3.93'}$$

For $\omega \gg 1$ this cross section increases *linearly* with E_ν^{lab}. For $E_\nu^{\mathrm{lab}} \simeq 400\,\mathrm{GeV}$ it is of the order of $7\times 10^{-39}\,\mathrm{cm}^2$. Returning to the c.m. system, we can write (3.92) in terms of the standard scattering amplitude f as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{2}\sum |f|^2,$$

where the relation between the scattering amplitude and the T-matrix element is $f = (8\pi^5/\sqrt{s})T$ so that

$$\sum |f|^2 = \left(\frac{G}{\pi\sqrt{2}}\frac{s-m_\mathrm{e}^2}{\sqrt{s}}\right)^2.$$

This spin-average of the squared amplitude is isotropic and, therefore, behaves like a scalar (i.e. spinless) s partial wave. On the basis of unitarity it must have the general form of a scattering amplitude for $l = 0$

$$\left(\sum |f|^2\right)^{1/2} \sim f_{l=0} = \frac{1}{2\mathrm{i}k}(\eta \mathrm{e}^{2\mathrm{i}\varepsilon} - 1),$$

where ε is a real phase and η is the inelasticity bounded by $0 \le \eta \le 1$; k is the c.m. momentum and is given by $k = (s-m_\mathrm{e}^2)/2\sqrt{s}$. Unitarity implies an upper bound on the cross section,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} \leq \frac{1}{8k^2}(\eta+1)^2 \leq \frac{1}{2k^2} = \frac{2s}{(s-m_\mathrm{e}^2)^2}. \tag{3.94}$$

The calculated differential cross section (3.92) reaches the unitarity bound (3.94) at a critical value $s = s_\mathrm{c}$ which is so large that m_e^2 can be neglected as compared to s_c, viz.

$$s_\mathrm{c} \simeq \frac{2\pi\sqrt{2}}{G} = 7.62 \times 10^5\,\mathrm{GeV}^2. \tag{3.95}$$

The value (3.95) corresponds to a c.m. energy of the neutrino of the order of $E_\nu \simeq \sqrt{s}/2 \simeq 440\,\mathrm{GeV}$. This critical energy is very large indeed. Therefore, at low and intermediate energies the effective current–current interaction (3.74) and (3.85) is a very good approximation to the interaction due to exchange of $\mathrm{W}^\pm$ and Z^0. The result (4.64) for muon decay below may serve as an example to illustrate the magnitude of the correction due to a W-propagator. So in many practical applications one can neglect the typical effects of W- and Z-propagators. The full unified gauge theories are renormalizable, as was first shown by 't Hooft ('t Hooft 1971), and lead to unitary and calculable S-matrices. The current–current contact interactions then appear as the effective interactions, valid at low and intermediate energies.

3.3 Elements of Local Gauge Theories Based on Non-Abelian Groups

The section deals with the principles of constructing field theories which, in addition to being Lorentz covariant, are invariant under a group of *local* symmetry transformations. Depending on whether the underlying symmetry group is Abelian or non-Abelian, we talk about *Abelian* or *non-Abelian gauge theories*. Quantum electrodynamics (QED) is an example of a Lorentz covariant theory which, in addition, is invariant under local gauge transformations of the photon field, cf. (1.204), and of the matter fields, cf. (1.202).

In this case, as is evident from (1.202), the group of transformations is a one-parameter continuous group, i.e. it is an Abelian group and has the structure of U(1). We develop the more general non-Abelian case in close analogy to QED. We define generalized vector potentials $A_\mu^{(k)}(x)$, generalized field tensors $F_{\mu\nu}^{(k)}(x)$, and generalized covariant derivatives, (1.205), of matter fields. These notions and definitions provide the tools for the construction of a rather general class of local gauge theories. We develop these theories in a constructive but still elementary way. Although we try to render the main properties and results as transpartent as possible by invoking the geometrical interpretation of the basic elements of the theory, we do not enter the mathematical properties of gauge theories in their full rigour. [These form an important and rich topic by themselves; however, they go beyond the scope of this book.]

3.3.1 Groups of Local Gauge Transformations

Let G be a compact Lie group. We shall always assume that G is simple, or is the direct product of a finite number of simple Lie groups. Examples of interest in particle physics are the unitary unimodular groups SU(n) (i.e. the groups defined by the unitary matrices with determinant 1 in n complex dimensions), such as

$$\mathrm{U}(1),\ \mathrm{SU}(2),\ \mathrm{SU}(3),$$

or direct products thereof

$$\mathrm{SU}(2)\times\mathrm{U}(1),\ \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1),\ \text{etc.}$$

The generators of infinitesimal transformations in G are written as T_i in abstract notation, i.e. when no reference to a specific representation is made. They obey the commutators

$$[T_i,\ T_j] = \mathrm{i}\sum_{k=1}^{N} C_{ijk}\,T_k, \quad i,j = 1,2,\ldots,N, \tag{3.96}$$

where C_{ijk} are the structure constants. Definitions here are such that the generators are hermitean. As is well known the structure constants C_{ijk} can be chosen totally antisymmetric and fulfill the identity [HAM 62, RAC 64]

$$\sum_l \{C_{ikl}\,C_{lmn} + C_{kml}\,C_{lin} + C_{mil}\,C_{lkn}\} = 0, \tag{3.97}$$

which follows from the Jacobi identity for T_i, T_k, and T_m. N finally, is the dimension of the Lie algebra (3.96) of the group G, $N = \dim(G)$[2]. Let us consider some examples:

U(1): Here $N = 1$ and T_1 is the unit element.

SU(2): Here we have $N = 3$ and $C_{ikl} = \varepsilon_{ikl}$, the totally antisymmetric Levi-Civita tensor in three real dimensions.

A concrete realization of U(1) and SU(2) is obtained by considering the group U(2) of unitary matrices in two complex dimensions. Such matrices $u \in \mathrm{U}(2)$ have the form

$$u = \mathrm{e}^{i\alpha}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with } |a|^2 + |b|^2 = 1,$$

so that $uu^\dagger = \mathbb{1}$. Any such matrix depends on four real parameters and can be written as an exponential series

$$u = \exp\left\{\mathrm{i}\sum_{\mu=0}^{3}\Lambda_\mu h_\mu\right\}, \tag{3.98}$$

[2]For SU(n), as is well-known, $N = n^2 - 1$.

in the hermitean matrices

$$h_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; h_i = \frac{1}{2}\sigma^{(i)}, \; i = 1, 2, 3, \tag{3.99}$$

Λ_μ being arbitrary real parameters. The commutators of these matrices are

$$[h_0, \, h_\mu] = 0, \quad \mu = 0, \, 1, \, 2, \, 3, \tag{3.100a}$$

$$[h_i, \, h_j] = \mathrm{i} \sum_{k=1}^{3} \varepsilon_{ijk} \, h_k, \quad j = 1, 2, 3. \tag{3.100b}$$

As h_0 commutes with h_i, we can write (3.98) equivalently as follows

$$u = \exp\{\mathrm{i} \, \Lambda_0 \, h_0\} \, \exp\left\{\mathrm{i} \sum_{k=1}^{3} \Lambda_k \, h_k\right\}. \tag{3.98'}$$

The first factor of (3.98') defines an Abelian subgroup of U(2) and, therefore, forms a U(1) group. As to the second factor we note that $H := \Sigma_{k=1}^{3} \, \Lambda_k \, h_k$ is not only hermitean but also traceless, $\mathrm{tr}\{H\} = 0$. Therefore, the matrices $\exp\{\mathrm{i}H\}$ are unitary and have determinant 1 (see exercise 3.3). Thus, the second factor in (3.98') defines SU(2), the group of all unitary, unimodular matrices in two complex dimensions. In abstract notation we write the generators of U(2) as T_μ with $\mu = 0, \, 1, \, 2, \, 3$ and obtain the commutators

$$[T_0, \, T_\mu] = 0, \; \forall \mu, \tag{3.101a}$$

$$[T_i, \, T_j] = \mathrm{i} \sum_{k} \varepsilon_{ijk} \, T_k. \tag{3.101b}$$

SU (3): Here we have $N = 8$ and the structure constants are $C_{ijk} = f_{ijk}$ with f_{ijk} as indicated in the following scheme.

ikl	123	147	156	246	257	345	367	458	678
f_{ikl}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$

Those structure constants f_{stu} for which *stu* is not an even or odd permutation of the indices *ikl* as listed in this table vanish.

A concrete realization of SU(3) is obtained by considering all unitary, unimodular matrices in 3 complex dimensions. As for SU(2) we write

$$u = \exp\left\{\mathrm{i} \sum_{k=1}^{8} \Lambda_k h_k\right\}, u \in \mathrm{SU}(3), \tag{3.102}$$

with

$$uu^\dagger = 1, \quad \det(u) = 1,$$
$$h_k = h_k^\dagger, \quad \mathrm{tr}(h_k) = 0,$$
$$h_k = \frac{1}{2}\lambda_k,$$

where λ_k are eight linearly independent matrices whose explicit form can be found in the literature [GAS66], or in Appendix D. The constants Λ_k are arbitrary real parameters, as before. In abstract notation the generators of infinitesimal SU(3) transformations are written as $T_k, k = 1, 2, \ldots, 8$. They obey the Lie algebra

$$[T_i,\, T_j] = \mathrm{i} \sum_{k=1}^{8} f_{ijk}\, T_k, \tag{3.103}$$

with f_{ijk} as defined above.

In all of these examples we are dealing with *compact* groups. The group parameters Λ_k are then generalized angles of rotation. If one wishes, one can choose these angles such that their domain of variation is either the interval $[0, \pi]$ or the interval $[0, 2\pi]$. For example, SU(2) can be parametrized by means of three Euler angles, cf. (1.41, 1.141), with

$$0 \le \psi \le 2\pi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi,$$

Transformations of the kind of (3.98) or (3.102), are called *global* transformations. These must be distinguished from *local* transformations where the group parameters are allowed to be functions of space and time to which we now turn. A group element of G is said to be a *local* transformation or a *gauge transformation* if it has the form

$$g(x) = \exp\left\{ \mathrm{i} \sum_{k=1}^{N} \Lambda_k(x) T_k \right\}, \tag{3.104}$$

where $x \equiv \{x^\mu\}$ is a point in Minkowski space. In contrast to the case of global transformations the group parameters are now taken to be (in general, infinitely differentiable) functions $\Lambda_k(x)$ of space and time.

This construction looks rather natural because gauge transformations act on particle states which may be localized states in the sense that one is studying a set of particles in a given finite space volume and within a certain finite time interval. There is then no need to transform simultaneously other particles far away from that volume and at times outside that time interval. Having the group parameters depend on a space-time argument allows to restrict the gauge transformation to the localized states at stake.

The group of *global* transformations (i.e. not depending on space–time) is called the *structure group*, while the *local* transformations belong to the *gauge group*.

It is easily verified that the local transformations (3.104), for fixed x, form a group with respect to the group multiplication

$$g_1(x) \cdot g_2(x).$$

Thus the definition (3.104) provides a copy $G(x)$ of the original Lie group G, for every x in Minkowski space.

Consider now a set of M *matter* fields

$$\phi(x) := \{\phi_n(x)\}, \quad n = 1, 2, \ldots, M, \tag{3.105}$$

which form an M-dimensional representation of the group G. This representation must be unitary but need not be irreducible. For simplicity we take the fields ϕ_n to be spin-zero boson fields, at least for the moment, but note that in the discussion of their transformation under G the spin character is irrelevant. The transformation properties under G apply equally well to fermion fields or fields with spin 1. Let L_k denote the matrix representatives of the generators T_k in the space of the fields ϕ_n,

$$(L_k)_{ij} = U_{ij}(T_k). \tag{3.106}$$

The action of a local transformation $g(x)$ on the fields is then given by

$$\phi(x) \underset{g(x)}{\to} \phi'(x) = \exp\left\{ \mathrm{i} \sum_{k=1}^{N} \Lambda_k(x) L_k \right\} \phi(x). \tag{3.107}$$

Of course, we can write this transformation in representation-free notation by defining the abstract group element $g(x)$ as an exponential series in terms of the generators T_k and the parameter functions $\Lambda_k(x)$,

$$g(x) = \exp\{\mathrm{i}\, \Lambda(x)\} \quad \text{with} \quad \Lambda(x) = \sum_{k=1}^{N} \Lambda_k(x) T_k. \tag{3.108}$$

In the space of the functions $\phi(x)$ the transformation (3.108) is represented by the matrix

$$\begin{aligned} U(\Lambda(x)) \equiv U(g(x)) &= \exp\{\mathrm{i} \Sigma \Lambda_k(x) U(T_k)\} \\ &= \exp\{\mathrm{i} \Sigma \Lambda_k(x) L_k\}. \end{aligned}$$

Suppose we wish to construct a field theory which describes the equations of motion of the fields $\phi(x)$ and which is invariant under the symmetry transformations of the structure group G. The group G defines an internal symmetry of this field theory and

the indices n on the fields $\phi_n(x)$, with $n = 1, 2, \ldots, M$, refer to a "charge" space which is given by unitary representations of G. As an example consider the three pion fields $\phi_m(x), m = +1, 0, -1$, which are distinguished by the projection I_3^s of the strong interaction isospin $I^s = 1$. In this example G is the SU(2) of isospin, the representation formed by the pion fields is three dimensional and irreducible. Under a transformation g of G which is specified by three parameters (Euler angles), the pion fields transform according to

$$\phi'_m(x) = \sum_{\mu=1}^{3} D^{(1)}_{m\mu}(\Lambda_1, \Lambda_2, \Lambda_3)\phi_\mu(x).$$

If G is to be a *global* symmetry then the Lagrangian of the pion fields $\phi_m(x)$ can depend only on products of the fields and of their derivatives which are scalars under G. The kinetic energy term, in particular, must have the Lorentz invariant and G-invariant form

$$(\partial_\alpha \phi, \partial^\alpha \phi), \tag{3.109}$$

where the parentheses (,) imply, symbolically, coupling of the bilinear $\phi_m \phi_{m'}$ to an invariant. [In the present example $(\partial_\alpha \phi, \partial^\alpha \phi) \equiv \Sigma_m (-)^{1-m} \partial_\alpha \phi_m \partial^\alpha \phi_{-m}$.]

If G is to be a *local* symmetry, i.e. if the parameters Λ_i depend on space and time, then a new aspect emerges: As we said before the prescription (3.104) defines an infinity of copies $G(x)$ of the original abstract group G, one for each point of space–time. Regarding the matter fields ϕ this prescription implies that each point x in Minkowski space is endowed with a local charge space ${}^x H$. As the derivative $\partial_\alpha \phi$ connects the fields in neighbouring points, x and $x + \mathrm{d}x$, it relates at the same time the charge space ${}^x H$ to the charge space ${}^{x+\mathrm{d}x} H$. A component $\phi_n(x)$ of $\phi(x)$ with respect to a given basis in the charge space ${}^x H$ in x is not simply the same component n with respect to ${}^{x+\mathrm{d}x} H$. Therefore, the kinetic energy term (3.109) cannot be invariant under local gauge transformations and must be replaced by a more general, $G(x)$-invariant form.

This problem can be solved in two steps: First, one derives the transformation that carries $\phi_n(x)$ in the space ${}^x H$ into the same component ϕ_n in ${}^{x+\mathrm{d}x} H$ (parallel transport). Second, one constructs the covariant derivative D_α which replaces ∂_α in (3.109) and which makes this form locally gauge invariant.

3.3.2 *Vector Potentials and Their Transformation Properties*

Let us consider a given component $\phi_n(x)$ of the representation formed by the fields $\phi(x)$. The field is taken at a fixed point x of space–time. The given index n, in fact, refers to a basis in the internal symmetry space ${}^x H$, in which case we write ${}^x\phi_n(x)$, or the analogous basis in the symmetry space ${}^{x+\mathrm{d}x} H$ attached to a neighbouring point $(x + \mathrm{d}x)$, in which case we write ${}^{x+\mathrm{d}x}\phi_n(x)$, for the sake of clarity. The two fields are not the same. However, it must be possible to relate them by an

infinitesimal local gauge transformation, for which we make the following ansatz

$$ {}^{x+\mathrm{d}x}\phi_n(x) - {}^{x}\phi_n(x) = -\sum_m U_{nm}(A_\alpha)\cdot \mathrm{d}x^\alpha\, {}^{x}\phi_m(x), \tag{3.110} $$

with

$$ U_{nm}(A_\alpha) = \mathrm{i}e\sum_{k=1}^{N} A_\alpha^{(k)}(x)(L_k)_{nm}, \tag{3.111} $$

or, in abstract notation,

$$ A_\alpha(x) := \mathrm{i}e\sum_{i=1}^{N} A_\alpha^{(k)}(x)T_k. \tag{3.112} $$

Note that the fields $A_\alpha^{(k)}$ carry both an internal symmetry index k and a Lorentz vector index α. The Lorentz vector behaviour is needed in order to obtain a Lorentz invariant form $A_\alpha \mathrm{d}x^\alpha$ in (3.110). Thus, A_α as defined by (3.112) has a dual nature: on the one hand it transforms like an ordinary Lorentz vector field, on the other hand, through its dependence on the generators T_k, it is an operator in the internal symmetry space. In other words, $A_\alpha(x)$ is an object which transforms like a Lorentz vector field and takes its values in the Lie algebra of the structure group. Equation (3.112) defines A_α in abstract form, whilst (3.111) defines its matrix representation in the space of the matter fields ϕ_n.

Of course, we can apply an arbitrary local gauge transformation $g(x)$ and $g(x+\mathrm{d}x)$ to ${}^{x}\phi_n$ and ${}^{x+\mathrm{d}x}\phi_n$, respectively. Equation (3.110) describes the parallel transport of the field ϕ_n provided it commutes with the local transformation g,

$$ U(g(x+\mathrm{d}x))(\mathbb{1} - U(A_\alpha(x))\mathrm{d}x^\alpha)^{x}\phi_n = (\mathbb{1} - U(A'_\alpha(x))\mathrm{d}x^\alpha)U(g(x))^{x}\phi_n. $$

We can derive the transformation behaviour of the vector fields $A_\alpha(x)$, (3.111), under $g(x)$ from this requirement: As it must hold in all possible representations we can work it out in an abstract, i.e. representation-free form

$$ g(x+\mathrm{d}x)(\mathbb{1} - A_\alpha(x)\mathrm{d}x^\alpha) \stackrel{!}{=} (\mathbb{1} - A'_\alpha(x)\,\mathrm{d}x^\alpha)g(x). $$

Writting $g(x+\mathrm{d}x) \simeq g(x) + \partial_\alpha g(x)\mathrm{d}x^\alpha$ and collecting all terms linear in $\mathrm{d}x^\alpha$, we have

$$ \partial_\alpha g(x) - g(x)A_\alpha(x) = -A'_\alpha(x)g(x) $$

or

$$ A'_\alpha(x) = g(x)A_\alpha(x)g^{-1}(x) - (\partial_\alpha g(x))g^{-1}(x). $$

Since $\partial_\alpha(g(x)g^{-1}(x)) = (\partial_\alpha g)g^{-1} + g(\partial_\alpha g^{-1}) = 0$, this can also be written as follows:

$$ A'_\alpha(x) = g(x)A_\alpha(x)g^{-1}(x) + g(x)\partial_\alpha g^{-1}(x). \tag{3.113} $$

This fixes the transformation behaviour of the quantity $A_\alpha(x)$, (3.111), under a local gauge transformation $g(x) \in G(x)$.

Let us analyze in more detail the meaning of (3.113) and of the fields $A_\alpha^{(k)}(x)$. The transformation $A_\alpha \to A'_\alpha$, (3.113), contains two elements: the first term is a conjugation, i.e. the familiar transformation behaviour of an operator with respect to global transformations g in G. Indeed, if g does not depend on x, the derivative $\partial_\alpha g^{-1}$ vanishes and $A'_a = gA_\alpha g^{-1}$. The second term is a generalized gauge transformation, as may be seen by considering the example of G being a U(1) group. In this case

$$g(x) = \mathrm{e}^{\mathrm{i}\Lambda(x)}, \quad g^{-1}(x) = \mathrm{e}^{-\mathrm{i}\Lambda(x)},$$
$$g(x)\partial_\alpha g^{-1}(x) = -\mathrm{i}\partial_\alpha \Lambda(x),$$

so that (3.113) reduces to

$$A'_\alpha(x) = A_\alpha(x) - \mathrm{i}\partial_\alpha \Lambda(x).$$

Furthermore, the gauge group being U(1), the sum on the r.h.s. of (3.111) contains only one term, $A_\alpha = \mathrm{i}eA_\alpha^{(1)}\mathbb{1}$, so that

$$A_\alpha^{(1)\prime}(x) = A_\alpha^{(1)}(x) - \frac{1}{e}\partial_\alpha \Lambda(x). \tag{3.114}$$

This is precisely the expression for a gauge transformation in electrodynamics, cf. (1.204). Thus, (3.113) provides the generalization of the familiar gauge transformations of electrodynamics to the case of non-Abelian local gauge groups.

At the same time this comparison suggests that the vector fields $A_\alpha^{(k)}(x)$, of which there are $N = \dim G$ types, are generalizations of the vector potential of electrodynamics. In this context it is instructive to work out the transformation behaviour of these fields under infinitesimal transformations $g(x)$,

$$g(x) \simeq \mathbb{1} + \mathrm{i}\sum \Lambda_k(x) T_k.$$

This gives

$$\begin{aligned} gA_\alpha g^{-1} &\simeq \left(\mathbb{1} + \mathrm{i}\sum_k \Lambda_k T_k\right) \mathrm{i}e \sum_i A_\alpha^{(i)} T_i \left(\mathbb{1} - \mathrm{i}\sum_j \Lambda_j T_j\right) \\ &\simeq \mathrm{i}e \sum_i \left\{ T_i + \mathrm{i}\sum_k \Lambda_k [T_k, T_i] \right\} A_\alpha^{(i)}(x) \\ &\simeq \mathrm{i}e \sum_i \left\{ T_i - \sum_{k,l} \Lambda_k C_{kil} T_l \right\} A_\alpha^{(i)}(x) \end{aligned}$$

and

$$g(x)\partial_\alpha g^{-1}(x) \simeq -\mathrm{i}\sum_j T_j\,\partial_\alpha \Lambda_j(x).$$

Inserting these formulae into (3.113) and comparing the coefficients of T_i on either side, one obtains, with $|\Lambda_k| \ll 1$,

$$A_\alpha^{(i)\prime}(x) \simeq A_\alpha^{(i)}(x) - \sum_{kl} C_{ikl}\Lambda_k(x)A_\alpha^{(l)}(x) - \frac{1}{e}\partial_\alpha \Lambda_i(x). \tag{3.115}$$

For $G = \mathrm{U}(1)$ this reduces to the result (3.114) above. For $G = \mathrm{SU}(2)$ one has, in an obvious vector notation,

$$\boldsymbol{A}'_\alpha(x) \simeq \boldsymbol{A}_\alpha(x) - \boldsymbol{\Lambda}(x)\times\boldsymbol{A}_\alpha(x) - \frac{1}{e}\partial_\alpha\boldsymbol{\Lambda}(x).$$

The constant e which appears in the definition (3.112), in principle, is arbitrary. Like in electrodynamics it plays the role of a coupling constant of the matter fields ϕ to the vector bosons represented by the gauge fields $A_\alpha^{(i)}$. For each irreducible component G_i of $G = G_1 \times G_2 \times \cdots$ there is one such constant e_i which can be chosen arbitrarily (cf. exercise 3.4). We return to this arbitrariness in more detail below.

We summarize the results of this section: The very definition of parallel transport requires the introduction of a set of vector gauge fields $\{A^{(k)}(x); k = 1, \ldots, N\}$ which form the adjoint representation of the group G. Therefore, once G is given, say

$$G = G_1 \times G_2, \tag{3.116a}$$

where G_i are simple, then

$$N = N_1 + N_2 \quad \text{with} \quad N_i = \dim(G_i). \tag{3.116b}$$

With respect to infinitesimal local gauge transformations the fields $A_\alpha^{(k)}$ transform according to (3.115) which behaviour is the generalization of the familiar gauge transformations of electrodynamics to the non-Abelian case. In the expression for the parallel transport, the gauge fields appear in the form of the operator A_α, (3.112) whose transformation behaviour under *finite* $g(x)$ is defined by (3.113) (equivalent to (3.115) for infinitesimal gauge transformations). In those gauges which preserve the manifest covariance of the theory, A_α is a Lorentz vector. At the same time it is an operator with respect to the group G and takes its values in the Lie algebra of G. Equation (3.111) gives its matrix representation in the space of a set of matter fields which form a unitary representation of G.

3.3.3 Covariant Derivatives

Our aim is to construct a generalized derivative of the matter fields $D_\alpha\phi_n(x)$ such that bilinears of two such forms can be coupled to an invariant, generalized, kinetic energy $(D_\alpha\phi, D^\alpha\phi)$. For this purpose let us consider the difference ${}^A\phi_n(x + \mathrm{d}x) - {}^A\phi_n(x)$ of the field ϕ_n for two infinitesimally separated arguments but expressed in the *same* symmetry space ${}^AH$, attached to the point $A = x$ in space–time. This infinitesimal difference can be expressed as the sum of a term containing the ordinary derivative and a parallel transport. Let

$$A \equiv x \quad \text{and} \quad B \equiv x + \mathrm{d}x.$$

Then

$${}^A\phi_n(B) - {}^A\phi_n(A) = [{}^B\phi_n(B) - {}^A\phi_n(A)] + [{}^A\phi_n(B) - {}^B\phi_n(B)],$$

where the first term is given by[3]

$${}^B\phi_n(B) - {}^A\phi_n(A) \simeq \partial_\alpha {}^A\phi_n(A)\mathrm{d}x^\alpha,$$

while the second term is a parallel transport of ${}^B\phi_n(B)$ to the point $A = B - \mathrm{d}x$. This is obtained from (3.110) with $\mathrm{d}x$ replaced by $-\mathrm{d}x$, viz.

$${}^A\phi_n(B) - {}^B\phi_n(B) = \sum_m U_{nm}(A_\alpha)\mathrm{d}x^\alpha\, {}^B\phi_m(B).$$

Thus, to first order in $\mathrm{d}x$,

$$\begin{aligned} {}^A\phi_n(B) - {}^A\phi_n(A) &\simeq \sum_m \{\delta_{nm}\partial_\alpha + U_{nm}(A_\alpha)\}\mathrm{d}x^\alpha\, {}^A\phi_m(A) \\ &\equiv D_\alpha(A)^A\phi(A)\,\mathrm{d}x^\alpha, \end{aligned}$$

which defines the covariant derivative $D_\alpha(A)\phi$ with

$$D_\alpha(A) := \mathbb{1}\partial_\alpha + U(A_\alpha). \tag{3.117}$$

Clearly, these considerations do not depend on the nature of the fields ϕ. In particular, they hold equally well for fermion fields Ψ. For example, specializing again to electrodynamics by taking $G = \mathrm{U}(1)$, we see that the definition (3.117) reduces to the covariant derivative D_α of (1.205).

[3]Note that the x-dependence appears in the argument *and* in the basis.

The generalized derivative $D_\alpha(A)\phi$ is called covariant because under a local gauge transformation $g(x) \in G$ it transforms according to the law

$$D_\alpha(A'(x)) = U(g(x))D_\alpha(A(x))U^{-1}(g(x)), \tag{3.118}$$

where $U(g)$ is the matrix representation of g in the space of the fields ϕ. Equation (3.118) says that $D_\alpha(A)$ transforms like a tensor operator. Therefore, on the basis of these tensors, it will be easy to form invariants under $G(x)$.

The transformation behaviour (3.118) follows from the construction given above. It may also be verified by explicit calculation as follows:

$$\begin{aligned} D_\alpha(A')\phi'(x) &= (\partial_\alpha + U(A'_\alpha))\, U(g(x))\phi \\ &= (\partial_\alpha U(g))\phi + U(g)\partial_\alpha\phi \\ &\quad + U(g)[U(A_\alpha)U^{-1}(g) + (\partial_\alpha U^{-1}(g))]\, U(g)\phi \\ &= U(g)(\partial_\alpha\phi + U(A_\alpha))\phi \\ &\quad + \{(\partial_\alpha U(g))\phi + U(g)[\partial_\alpha(U^{-1}(g)U(g)) \\ &\quad - U^{-1}(g)(\partial_\alpha U(g))]\phi\}. \end{aligned}$$

As $U^{-1}U = \mathbb{1}$ is independent of x, the term in curly brackets is zero, while the first is precisely $U(g)\, D_\alpha(A)\phi$. Thus we obtain

$$D_\alpha(A')\phi'(x) = U(g)D_\alpha(A)\phi,$$

from which (3.118) follows immediately.

3.3.4 *Field Tensor for Vector Potentials*

The field tensor $F^{\mu\nu}$ in non-Abelian gauge theories which generalizes the field strength tensor (1.190) of electrodynamics, is obtained, for instance, by studying two successive, infinitesimal, parallel translations (3.110) from a point x to a point $z = x + \mathrm{d}x + \mathrm{d}y$. This transformation can be effected in either of the two following ways which are not equivalent:

(a) $$x \to x + \mathrm{d}x = y \to z = y + \mathrm{d}y = x + \mathrm{d}x + \mathrm{d}y,$$

(b) $$x \to x + \mathrm{d}y = y' \to z = y' + \mathrm{d}x = x + \mathrm{d}x + \mathrm{d}y.$$

Expanding $A(x + \mathrm{d}x)$, $A(x + \mathrm{d}y)$ around the point x, their difference to second order in $\mathrm{d}x\mathrm{d}y$ is found to be

$$
\begin{aligned}
&(\mathbb{1} - A_\beta(x + \mathrm{d}x)\mathrm{d}y^\beta)(\mathbb{1} - A_\alpha(x)\mathrm{d}x^\alpha) \\
&\quad -(\mathbb{1} - A_\alpha(x + \mathrm{d}y)\mathrm{d}x^\alpha)(\mathbb{1} - A_\beta(x)\mathrm{d}y^\beta) \\
&= -\{\partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x) + A_\alpha(x)A_\beta(x) - A_\beta(x)A_\alpha(x)\}\mathrm{d}x^\alpha \mathrm{d}y^\beta\} \\
&=: -F_{\alpha\beta}(x)\mathrm{d}x^\alpha \mathrm{d}y^\beta,
\end{aligned}
$$

where the tensor $F_{\alpha\beta}(x)$ is defined by

$$
F_{\alpha\beta}(x) := \partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x) + [A_\alpha(x), A_\beta(x)]. \tag{3.119}
$$

$F_{\alpha\beta}(x)$ is a tensor with respect to Lorentz transformations. At the same time it has the properties of a tensor operator with respect to the symmetry transformations g. This is easy to see if we notice that $F_{\alpha\beta}$ can be related to the commutator of the covariant derivatives D_α and D_β. In the space of the matter fields

$$
\begin{aligned}
[D_\alpha(A), D_\beta(A)] &= \partial_\alpha U(A_\beta) - \partial_\beta U(A_\alpha) + [U(A_\alpha), U(A_\beta)] \\
&= U(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]) = U(F_{\alpha\beta}(x)).
\end{aligned} \tag{3.120}
$$

Thus the matrix representative of $F_{\alpha\beta}$ in the space of the fields ϕ, transforms like the commutator of D_α and D_β whose transformation behaviour, in turn, is given by (3.118). This proves the tensor character of $F_{\alpha\beta}$ with respect to the gauge transformations of $G(x)$.

Finally, in analogy to the decomposition (3.112) of A_α, we can write $F_{\alpha\beta}$ as a linear combination of the generators T_k of G, viz.

$$
F_{\alpha\beta}(x) = \mathrm{i}e \sum_{k=1}^{N} T_k F^{(k)}_{\alpha\beta}(x), \tag{3.121}
$$

where $F^{(k)}_{\alpha\beta}(x)$ are ordinary Lorentz tensor fields. Their explicit form is obtained from the definition (3.119) by inserting the decomposition (3.112):

$$
\begin{aligned}
\mathrm{i}e \sum_{k=1} T_k F^{(k)}_{\alpha\beta}(x) &= \mathrm{i}e \sum T_k (\partial_\alpha A^{(k)}_\beta(x) - \partial_\beta A^{(k)}_\alpha(x)) \\
&\quad + (\mathrm{i}e)^2 \sum_{i,j} [T_i, T_j] A^{(i)}_\alpha(x) A^{(j)}_\beta(x).
\end{aligned}
$$

The commutator in this expression is given by (3.96). As the generators T_k are linearly independent, we can compare the coefficients of T_k to find

$$
F^{(k)}_{\alpha\beta}(x) = f^{(k)}_{\alpha\beta}(x) - e \sum_{i,j} C_{kij} A^{(i)}_\alpha(x) A^{(j)}_\beta(x), \tag{3.122}
$$

with

$$f^{(k)}_{\alpha\beta}(x) := \partial_\alpha A^{(k)}_\beta(x) - \partial_\beta A^{(k)}_\alpha(x). \tag{3.123}$$

Equation (3.122) is the direct generalization of the electromagnetic field-strength tensor (1.190) to the case of non-Abelian gauge theories.

3.3.5 *How to Construct Locally Gauge Invariant Theories*

In the previous sections we have established three types of operators which have a simple transformation behaviour under local gauge transformations: the generalized vector potentials $A_\alpha(x)$, the corresponding field tensors $F_{\alpha\beta}(x)$, and the covariant derivative $D_\alpha(A)$ of matter fields ϕ. As to the latter, we noted previously that the spin content of the field ϕ is irrelevant. Therefore, the covariant derivative of *spinor* fields (or any other field) is given by exactly the same definition (3.117), where $U(A_\alpha)$ is now the matrix representation of the operator A_α in the space of the spinor fields. These operators are the tools which we need to construct Lagrangians which are invariant under local gauge transformations.

Suppose the theory is to contain a set of boson fields

$$\phi(x) = \{\phi_n(x); n = 1, \ldots, M\}, \tag{3.124a}$$

as well as a set of spinor fields

$$\Psi(x) = \{\Psi_p(x); p = 1, \ldots, P\}. \tag{3.124b}$$

If we require the Lagrangian to be invariant under *global* transformations $g \in G$ then it must have the form

$$\mathscr{L}_0 = \frac{1}{2}(\partial_\alpha\phi, \partial^\alpha\phi) + \frac{1}{2}\mathrm{i}(\overline{\Psi}, \gamma^\alpha \overleftrightarrow{\partial}_\alpha \Psi) - (\overline{\Psi}, m\Psi) - (\overline{\Psi}, g\,\phi\,\Psi) - V(\phi). \tag{3.125}$$

The parentheses (X, Y) are meant to indicate that X and Y are coupled to a scalar with respect to G. In the third term of (3.125), in particular, m is a matrix in the space of the Ψ_p (mass matrix). Likewise, the fourth term represents a G-invariant Yukawa coupling of the fields Ψ and ϕ, while the last term ("potential" term) denotes a group-invariant self-interaction of the fields ϕ as well as possible mass terms for that field. The explicit form of these invariants depends on the nature of the Lie algebra (3.96) and of the multiplets $\phi(x)$ and $\Psi(x)$. In particular, it may happen that it is not possible to construct a group invariant mass term for the fermion fields. [This case will be encountered in the GSW model].

On the basis of the results obtained in Sects. 3.3.2–4 it is easy to construct a new version of the theory (3.125) which is invariant under *local* gauge transformations $g(x) \in G(x)$ as well: Let $U(A_\alpha)$ and $V(A_\alpha)$ be the matrix representatives of the operator A_α, (3.112), in the space of the boson fields (3.124a) and in the space of

the fermion fields (3.124b), respectively. In order to obtain gauge invariant kinetic terms of these matter fields, the ordinary derivatives $\partial_\alpha \phi$, $\partial_\alpha \Psi$ must be replaced by the covariant derivatives

$$D_\alpha \phi = (\mathbb{1}\partial_\alpha + U(A_\alpha))\phi, \tag{3.126a}$$

$$\overrightarrow{D}_\alpha \Psi = (\mathbb{1}\partial_\alpha + V(A_\alpha))\Psi, \tag{3.126b}$$

$$\overline{\Psi}\overleftarrow{D}_\alpha = \partial_\alpha \overline{\Psi}\mathbb{1} + \overline{\Psi}\, V^\dagger(A_\alpha) = \partial_\alpha \overline{\Psi}\mathbb{1} - \overline{\Psi} V(A_\alpha). \tag{3.126c}$$

In addition, a term of the form $(F_{\alpha\beta}, F^{\alpha\beta})$ must be added to the Lagrangian which generalizes the well-known kinetic energy term (1.189) of the Maxwell fields, viz.

$$\begin{aligned} -\mathscr{L}_A &= \frac{c}{4}(F_{\alpha\beta}, F^{\alpha\beta}) \\ &= e^2 \frac{c}{4} \sum_{i,k} \mathrm{tr}(T_i T_k)\, F^{(i)}_{\alpha\beta} F^{(k)\alpha\beta}. \end{aligned}$$

Now, with

$$\mathrm{tr}(T_i T_k) = \kappa \delta_{ik} \tag{3.127}$$

c must be chosen to be

$$c = 1/\kappa e^2$$

for the derivative terms $f^{(i)}_{\alpha\beta} f^{(i)\alpha\beta}$ to obtain the same factor $-\frac{1}{4}$ as in the case of the Maxwell field, eq. (1.189). The mass terms, coupling terms and generalized potentials which do not contain derivatives, remain the same as in the globally invariant version of the theory.

The full Lagrangian describing the interacting matter fields ϕ and Ψ, in interaction with the gauge fields $A^{(i)}_\alpha$, is then given by

$$\begin{aligned} \mathscr{L} = {} & \frac{1}{4\kappa e^2}(F_{\alpha\beta},\ F^{\alpha\beta}) + \frac{1}{2}(D_\alpha \phi,\ D^\alpha \phi) + \frac{\mathrm{i}}{2}(\overline{\Psi},\ \gamma^\alpha \overleftrightarrow{D}_\alpha \Psi) \\ & -(\overline{\Psi}, (m + g\phi)\Psi) - V(\phi). \end{aligned} \tag{3.128}$$

Theories of this kind which are invariant under a non-Abelian group of local gauge transformations have a number of remarkable general properties.

(i) As in the Abelian case of electrodynamics, the Lagrangian cannot contain a mass term $m^2 A^{(i)}_\alpha A^{(i)\alpha}$ for the gauge fields because such a term is not invariant under gauge transformations. Therefore, if the gauge symmetry remains unbroken or, if no additional unphysical fields are introduced (Stückelberg fields), the gauge fields $A^{(i)}_\alpha$ describe *massless* vector bosons. In turn, if we wish to give some of these bosons finite masses, the symmetry must be broken to some extent. In fact, we shall see that it is possible to partially break the

gauge symmetry while maintaining the global invariance of $\mathscr{L}$ with respect to the structure group.

(ii) In the non-Abelian case the first term in $\mathscr{L}$, (3.128), contains not only the kinetic energy

$$-\frac{1}{4}\sum_i f^{(i)}_{\alpha\beta} f^{(i)\alpha\beta}$$

of the gauge fields, but also coupling terms of the kind

$$\sum_{ijk} C_{ijk} f^{(i)}_{\alpha\beta}(x) A^{(j)\alpha}(x) A^{(k)\beta}(x)$$

and

$$\sum_{i\cdots q} C_{ijk} C_{ipq} A^{(j)}_{\alpha}(x) A^{(k)}_{\beta}(x) A^{(p)\alpha}(x) A^{(q)\beta}(x),$$

i.e. cubic and quartic interactions of the gauge fields among themselves.

(iii) Owing to the dependence of $D_\alpha(A)$ on the fields $A^{(i)}_\alpha$ the generalized "kinetic" terms of the boson and fermion fields also yield the couplings of the matter fields to the gauge vector bosons. The gauge bosons are seen to couple to currents of the type $\phi^\dagger U(T_i) i \overleftrightarrow{\partial}_\alpha \phi$ and $\overline{\Psi} V(T_i) \gamma^\alpha \Psi$, respectively. In particular, if T_i is diagonal in the boson multiplet or fermion multiplet, this implies that the physical coupling constants of individual members of this multiplet, i.e. their "charges", are proportional to each other. They are given by the constant e multiplied by the diagonal matrix elements of $U(T_i)$ or $V(T_i)$, respectively. This is a new element of universality which does not occur in Abelian theories.

(iv) There is a beautiful geometric interpretation for each of the building blocks of local gauge theories that we could not describe in any detail without rendering this chapter exceedingly long. We refer to the more specialized literature on this topic. While the structure of $G(x)$, for fixed x in space–time, is known as soon as the structure group G is given, it needs the vector potential $A_\alpha(x)$ to connect two different copies $G(x)$ and $G(y)$ of G, for $x \neq y$. In particular, A_α is needed for the transport of geometric objects from the internal symmetry space in x to the internal space in y, an operation which is very similar to parallel transport of vectors in ordinary differential geometry. It turns out that $A_\alpha(x)\mathrm{d}x^\alpha$ is an explicit (coordinate) representation of what is called a *connection form* in geometry. Furthermore, as we saw above, the field strength tensor $F_{\alpha\beta}(x)$ appears if one compares the result of parallel transport from x to z along two different paths. This is reminiscent of the notion of curvature in differential geometry. Indeed, it turns out that $F_{\alpha\beta}(x)\mathrm{d}x^\alpha \mathrm{d}x^\beta$ is the coordinate representation of what is called the *curvature form* pertaining to the connection form from which one started.

3.4 Glashow–Salam–Weinberg Model for Leptons and Quarks

The GSW model whose phenomenology was summarized in Sect. 3.2.3 above, is based on the gauge group

$$G = U(2), \tag{3.129}$$

whose algebra reduces to the algebras of SU(2) and U(1). More precisely, the group U(2) is locally isomorphic to $\mathrm{U}(1) \times \mathrm{SU}(2)$ but such a decomposition is not unique and, therefore, not canonical. In physics parlance this means that while U(2), as a local gauge theory, provides two neutral vector fields $A_\alpha^{(0)}(x)$ and $A_\alpha^{(3)}(x)$, associated to the generator of the U(1) one has selected and the neutral partner of $T_\pm$, respectively, their association with the physical photon and Z^0-fields is not fixed automatically. The two physical vector bosons can be arbitrary mixtures of the fields $A_\alpha^{(0)}(x)$ and $A_\alpha^{(3)}(x)$. Let T_0 denote the generator of the Abelian factor U(1), $\{T_1, T_2, T_3\}$ the generators of the SU(2) factor. The Lie algebra of these operators is given by (3.101). As G has four generators, the theory contains four gauge fields $A_\alpha^{(\mu)}(x)$,

$$A_\alpha(x) = \mathrm{i}e \sum_{\mu=0}^{3} T_\mu A_\alpha^{(\mu)}(x), \tag{3.130}$$

where $A_\alpha^{(0)}$ is a singlet, while the fields $\{A_\alpha^{(i)},\ i = 1.2.3\}$ form a triplet with respect to SU(2), i.e. the adjoint representation. The two W-bosons which are charged and which are conjugates of each other, must be linear combinations of two of the triplet fields, cf. (3.72),

$$\mathrm{W}_\alpha^{\pm}(x) := \mp \frac{1}{\sqrt{2}} (A_\alpha^{(1)}(x) \pm \mathrm{i} A_\alpha^{(2)}(x)). \tag{3.131}$$

The photon field A_α and the Z^0-boson field Z_α must be linear combinations of the third triplet field $A_\alpha^{(3)}$ and the singlet field $A_\alpha^{(0)}$, as indicated in (3.77, 3.78).

The GSW Lagrangian is constructed on the basis of the following assumptions:

(I) There is no direct coupling between different lepton families, or between quark and lepton families.

(IIa) The photon couplings must conserve parity and must have the form $Q_\mathrm{f}\, \overline{\Psi^{(\mathrm{f})}}\, \gamma^\alpha \Psi^{(\mathrm{f})} A_\alpha$ with Q_f the electric charge of fermion f.

(IIb) In particular, neutrinos must not couple to the photon.

(III) The neutrinos that couple to weak interaction vertices are fully left-handed. In particular, all CC interaction vertices (f, ν_f, W) are of the form V–A. In other words, only the left-handed part of the massive fermion field $\Psi^{(\mathrm{f})}$ couples to the W-bosons.

(IV) The theory shall exhibit lepton universality in the sense that the Lagrangians describing the interaction of (μ, ν_μ) and (τ, ν_τ) with the gauge bosons $\mathrm{W}^\pm$, Z^0 and γ simply are copies of the interaction Lagrangian for the $(\mathrm{e}, \nu_\mathrm{e})$ family.

It is customary, for the sake of simplifying the notation to write the particle symbol instead of the field operator, i.e.

$$f(x) \equiv \Psi^{(\mathrm{f})}(x), \qquad q(x) \equiv \Psi^{(\mathrm{q})}(x)$$

for leptons f ($\mathrm{f} = \mathrm{e}, \mu, \tau$) or quarks q. Furthermore, it is useful to define the left-handed and right-handed parts of massive fields, viz.

$$f_{\mathrm{L}}(x) := \frac{1}{2}(\mathbb{1} - \gamma_5) f(x) \equiv P_- f(x), \tag{3.132a}$$

$$f_R(x) := \frac{1}{2}(\mathbb{1} + \gamma_5) f(x) \equiv P_+ f(x), \tag{3.132b}$$

and analogously for the quark fields $q_{\mathrm{L}}(x)$ and $q_{\mathrm{R}}(x)$. As we saw in Sect. 1.3 P_+ projects onto spinors of the first kind, P_- onto spinors of the second kind, cf. (1.76).

Assumption (I) implies that the electroweak interaction Lagrangian can be constructed for each of the lepton families (3.7) and for each of the quark generations separately. Since the case of the quark doublets is complicated by the mixing of d, s and b states, we start with the simpler case of one lepton family ($\mathrm{f}, \nu_{\mathrm{f}}$).

3.4.1 GSW Lagrangian for One Lepton Family

The matter fields $f(x) \equiv \Psi^{(\mathrm{f})}(x)$ and $\nu(x) \equiv \Psi^{(\nu)}(x)$ [we suppress the index f on ν_{f}, for simplicity], appear in three forms: $f_{\mathrm{L}}(x)$, $f_{\mathrm{R}}(x)$, and $\nu_{\mathrm{L}}(x)$. This means that the massive, charged lepton f is described by a spinor of the first kind and a spinor of the second kind. The neutrino, by virtue of the dynamic properties of the theory, appears only in the form of a spinor of the second kind.

The simplest possibility of classifying these fields with respect to G is to group them into a triplet

$$\Psi(x) := \begin{pmatrix} \nu_{\mathrm{L}}(x) \\ f_{\mathrm{L}}(x) \\ f_{\mathrm{R}}(x) \end{pmatrix}. \tag{3.133}$$

As f_{R} does not couple to $\mathrm{W}^{\pm}$, it must be a singlet with respect to the SU(2) factor of G. The pair

$$L(x) := \begin{pmatrix} \nu_{\mathrm{L}}(x) \\ f_{\mathrm{L}}(x) \end{pmatrix} \tag{3.134}$$

on the other hand, forms a doublet of SU(2). Thus, the triplet (3.133) is a reducible multiplet of SU(2). In the space of the triplet (3.133) the generators T_μ are represented by the 3×3 matrices [the factor $\frac{1}{2}$ in $V(T_0)$ is introduced for convenience],

$$V(T_0) = \frac{1}{2}\begin{pmatrix} \lambda_\mathrm{d} & 0 & 0 \\ 0 & \lambda_\mathrm{d} & 0 \\ 0 & 0 & \lambda_\mathrm{S} \end{pmatrix}, \quad V(T_i) = \frac{1}{2}\begin{pmatrix} \sigma^{(i)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.135}$$

Two of these are diagonal, $V(T_0)$ and $V(T_3)$, and have the eigenvalues

$$\left\{\begin{array}{c} \frac{1}{2}\lambda_\mathrm{d} \\ \frac{1}{2}\lambda_\mathrm{d} \\ \frac{1}{2}\lambda_\mathrm{s} \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{array}\right\},$$

respectively. The eigenvalues of $V(T_0)$ which pertain to the doublet partners ν_L and f_L must be the same, while the eigenvalue λ_s for f_R can be different from λ_d. Note that the trace (3.127) is $\frac{1}{2}$ for the generators T_1, T_2, T_3, and is $\frac{1}{4}(2\lambda_\mathrm{d}^2 + \lambda_s^2)$ for T_0.

We analyze first the interaction terms which follow from the term

$$\frac{\mathrm{i}}{2}(\overline{\Psi(x)},\ \gamma^\alpha \overleftrightarrow{D}_\alpha \Psi(x))$$

in the Lagrangian (3.128). With the definitions (3.117) and (3.112) these are

$$\begin{aligned} \mathscr{L}_1^{(\mathrm{f})} &= -e\left(\overline{\Psi(x)}\sum_{\mu=0}^{3} V(T_\mu)\gamma^\alpha \Psi(x)\right) A_\alpha^{(\mu)}(x) \\ &= -e\sum_{i=1}^{3}(\overline{L(x)}\frac{1}{2}\sigma^{(i)}\gamma^\alpha L(x))A_\alpha^{(i)}(x) - e\frac{1}{2}\lambda_\mathrm{d}(\overline{L(x)}\,\mathbb{1}\gamma^\alpha L(x))A_\alpha^{(0)}(x) \\ &\quad -e\frac{1}{2}\lambda_\mathrm{s}(\overline{f_\mathrm{R}(x)}\gamma^\alpha f_\mathrm{R}(x))A_\alpha^{(0)}(x). \end{aligned} \tag{3.136}$$

As it stands, this interaction contains three free parameters: e, λ_d, and λ_s. Our aim is now to work out the restrictions on these parameters that follow from the conditions (IIa) and (IIb), and to identify them with the phenomenological coupling constants α and G of (3.75).[4]

(a) CC weak interactions

Rewriting the couplings to the gauge fields $A_\alpha^{(1)}$ and $A_\alpha^{(2)}$ in terms of the W-fields (3.131) we have

$$\bar{L}\frac{1}{2}(\sigma^{(1)}A_\alpha^{(1)} + \sigma^{(2)}A_\alpha^{(2)})\gamma^\alpha L = \frac{1}{\sqrt{2}}(\bar{L}s_+\gamma^\alpha L)W_\alpha^- - \frac{1}{\sqrt{2}}(\bar{L}s_-\gamma^\alpha L)W_\alpha^+,$$

[4]This discussion follows closely the analysis of O'Raifeartaigh (1979), [O'R 86].

where $s_\pm = \frac{1}{2}(\sigma^{(1)} \pm \mathrm{i}\sigma^{(2)})$ are the usual step operators in the space of the doublet fields L. Inserting the definitions (3.134) and (3.132a) the CC interaction in $\mathscr{L}_\mathrm{I}$ is then found to be

$$\mathscr{L}^{(\mathrm{f})}_{\mathrm{CC}} = -\frac{e}{\sqrt{2}}\overline{\nu_\mathrm{f}(x)}\frac{1}{2}\gamma^\alpha(\mathbb{1}-\gamma_5)f(x)W^{(-)}_\alpha(x) + \mathrm{h.c.} \tag{3.137}$$

Comparing this to formulae (3.74, 3.75) of our phenomenological discussion in Sect. 3.2.3 we can identify $-e$ with the constant g, and relate its square to the Fermi constant G

$$-e \equiv g, \qquad g^2/8m_\mathrm{w}^2 = G/\sqrt{2}. \tag{3.138}$$

(b) Neutral couplings

From (3.136) we see that the couplings of the individual lepton fields to $A^{(0)}_\alpha$ and $A^{(3)}_\alpha$ are as follows:

$$\nu_\mathrm{L}(x) : \frac{1}{2}g\{A^{(3)}_\alpha + \lambda_\mathrm{d}A^{(0)}_\alpha\}\,\overline{\nu_\mathrm{L}(x)}\,\gamma^\alpha\nu_\mathrm{L}(x), \tag{3.139a}$$

$$f_\mathrm{L}(x) : \frac{1}{2}\,g\,\{-A^{(3)}_\alpha + \lambda_\mathrm{d}\,A^{(0)}_\alpha\}\,\overline{f_\mathrm{L}(x)}\;\gamma^\alpha\;f_\mathrm{L}\,(x), \tag{3.139b}$$

$$f_\mathrm{R}(x) : \frac{1}{2}\,g\,\lambda_\mathrm{s}\;A^{(0)}_\alpha\,(x)\;\overline{f_\mathrm{R}(x)}\;\gamma^\alpha\;f_\mathrm{R}\,(x). \tag{3.139c}$$

Let us work out the consequences of the assumption (IIa) and (IIb) on these couplings. It follows from the condition (IIb) that the linear combination of $A^{(3)}_\alpha$ and $A^{(0)}_\alpha$ which appears in the curly brackets of (3.139a) must be proportional to the Z^0-field. Comparing this to the ansatz (3.77b) we find that g' of (3.77) and λ_d must be related by

$$\lambda_\mathrm{d} = -g'/g, \tag{3.140}$$

and that the NC coupling of the neutrino (3.139a) is indeed given by (3.79').

Condition (IIa), in turn, implies that f_L and f_R couple to the photon field (3.77a) with the same strength. This means that the photon components of (3.139b, c) must be the same. With the transformation (3.78) we obtain the condition

$$-g' + g\lambda_\mathrm{d} = g\lambda_\mathrm{s},$$

hence

$$\lambda_\mathrm{s} = 2\lambda_\mathrm{d} = -2g'/g. \tag{3.141}$$

Inserting the results (3.140, 3.141) and also (3.139b) into the equation (3.136) for $\mathscr{L}_\mathrm{I}$, the coupling of f to the photon field is

$$\mathscr{L}^{(\mathrm{f})}_\gamma = -\frac{gg'}{\sqrt{g^2+g'^2}}\,\overline{f(x)}\,\gamma^\alpha\,f(x)A_\alpha(x),$$

so that the electric charge of the lepton f is to be identified as indicated in (3.81). Similarly, one easily verifies that the neutral coupling of the Z^0 to the fields f and v are indeed those of (3.80', 79') respectively, or, after introducing the parametrization (3.82) in terms of the Weinberg angle, by (3.84).

(c) Weak isospin, weak hypercharge and the electric charge

Let us return to the general form (3.136) of the interaction and let us extract from it the couplings to the photon field by means of (3.78)

$$\mathscr{L}_{\gamma}^{(\mathrm{f})} = -\frac{gg'}{\sqrt{g^2+g'^2}}\overline{\Psi(x)}\left\{V(T_3)+\frac{g}{g'}V(T_0)\right\}\gamma^{\alpha}\Psi(x)A_{\alpha}(x). \tag{3.142}$$

From (1.192) we know that the interaction of a charged lepton with the photon field has the general form $-Q\overline{\Psi}\gamma^{\alpha}\Psi A_{\alpha}$. The factor in front of (3.142) is $-e$. Therefore, the eigenvalues of the diagonal matrix $V(T_3)+V((g/g')T_0)$ are the electric charges of the members of the multiplet (3.133), in units of the elementary charge e.

Because of the close analogy to the Gell-Mann-Nishijima formula relating the electric charge of a hadron to its strong isospin and hypercharge, the SU(2) factor of G is called the *weak isospin* group, and the operator

$$Y := 2\frac{g}{g'}T_0 \tag{3.143}$$

is called the *weak hypercharge*. In the space of the triplet we have

$$V(T_3)=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}, V(Y)=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

Denoting the eigenvalues by t_3 and y, respectively, the electric charge of the member m of the triplet (3.133) is

$$Q(m)/e = t_3(m)+\frac{1}{2}y(m). \tag{3.145}$$

This indeed gives 0 for the neutrino and -1 for the two chiral states of the charged lepton field.

(d) Some remarks and open problems

We will not write down the kinetic energy and interaction Lagrangian of the vector boson fields $W_{\alpha}^{\pm}$ and Z_{α} because we do not need them for our discussion. It should be clear, however, how to construct these terms from our general discussion in Sects. 3.3.2–3.3.5. We note, in particular, that the electromagnetic properties and interactions of $W^{\pm}$ -bosons are completely fixed, including their anomalous magnetic moment [this used to be a problem in the older theories with W-bosons].

The extension to all leptons families is simply effected by taking the sum of $\mathscr{L}_{\mathrm{I}}^{(\mathrm{f})}$, (3.136), over all leptons,

$$\sum_{\mathrm{f}=\mathrm{e},\mu,\tau} \mathscr{L}_{\mathrm{I}}^{(\mathrm{f})} = \mathscr{L}_{\mathrm{I}}\,(\text{leptons}). \tag{3.146}$$

The generalization to the quark families is slightly more complicated and will be dealt with in the next section.

The unified theory that we developed thus far, is invariant under the entire group (3.129), $G = \mathrm{U}(2)$. As it stands, it is still far from a realistic theory for the electromagnetic and the weak interactions because the weak bosons $\mathrm{W}^{\pm}$ and Z^0 remain massless, like the photon. Furthermore, the charged leptons f also remain massless because it is not possible to construct an invariant mass term on the basis of the fields $f_{\mathrm{L}}(x)$ and $f_{\mathrm{R}}(x)$ as they appear in the triplet (3.133).

Indeed, we know from our discussion in Sect. 1.8.4 that a particle which carries a conserved charge can only have a Dirac mass term. Such a mass term is of the form $\{\overline{f_{\mathrm{L}}(x)}\, f_{\mathrm{R}}(x) + \overline{f_{\mathrm{R}}(x)}\, f_{\mathrm{L}}(x)\}$. However, as f_{L} belongs to a doublet of SU(2) whilst f_{R} belongs to a singlet, this term cannot be invariant.

As a consequence of leptons being massless, all particle currents of this theory are conserved ones. While this conservation law is welcome for the diagonal vector currents, it cannot hold for the axial currents and for the nondiagonal (CC) vector currents. This discussion shows that the symmetry group (3.129) must be broken very strongly and following a specific pattern: One of the gauge bosons, the photon, must remain massless to all orders, while $\mathrm{W}^{\pm}$ and Z^0 must become massive and, in fact, very heavy. Thus G must be broken down to the residual U(1) symmetry of electrodynamics. A way to do this is provided by the mechanism of spontaneous symmetry breaking, at the price of introducing further degrees of freedom into the theory. At the same time this extension allows to give the charged fermions finite masses, so that the model becomes realistic and can be compared to experiment. Before we turn to a discussion of symmetry breaking (Sect. 3.5) we conclude the present topic by extending our results to the quark families.

3.4.2 GSW Lagrangian for the Quark Families

For the sake of convenience let us introduce the following notation for the quarks with charge $Q = \frac{2}{3}$ (in units of e),

$$\{u_f;\ f = 1,2,3\} \quad \text{for} \quad \mathrm{u},\mathrm{c},\text{and}\,\mathrm{t} \tag{3.147a}$$

and similarly for the quarks with charge $Q = -\frac{1}{3}$,

$$\{b_f;\ f = 1,2,3\} \quad \text{for} \quad \mathrm{d},\mathrm{s},\text{and}\,\mathrm{b}. \tag{3.147b}$$

These are the quark states with the quantum numbers (relevant to strong interactions) that we discussed in Sect 3.1.1b. They are the constituents of the physical meson and baryon states.

The weak interactions conserve neither the strong isospin nor the additive quantum numbers introduced in Sect. 3.1.1b. In fact, they couple to new states d_f, with electric charge $-\frac{1}{3}$, which are related to the states (3.147b) by a unitary transformation, viz.

$$d_f = \sum_{f'} V_{ff'}\, b_{f'}, \text{with } V V^\dagger = \mathbb{1}. \tag{3.148}$$

These new states are referred to as the *weak interaction eigenstates*. A way to visualize this relation is by assuming that the weak gauge bosons $\mathrm{W}^\pm$ and Z^0 couple to quark currents which contain the fields $u_f(x)$ and $d_f(x)$, but that the strong interaction Lagrangian contains quark mass terms which are not diagonal in the basis of the states d_f, i.e.

$$-\mathscr{L}_{\text{mass}} = \sum_f m(f)\overline{u_f(x)}u_f(x) + \sum_{f,f'} M_{ff'}\,\overline{d_f(x)}d_{f'}(x). \tag{3.149}$$

The mass matrix $M_{ff'}$ can be diagonalized by means of the unitary transformation V,

$$\sum_{a,b} V^\dagger_{f\,a}\, M_{ab}\, V_{bf'} = m_f\, \delta_{ff'}. \tag{3.150}$$

These states $b_{f'}$, (3.147b), are then the *mass eigenstates* whilst the states d_f are the *weak eigenstates*.

In the case of two quark families, V can be taken to be the rotation matrix $D^{(1/2)}(2\psi, 2\theta, 2\phi)$, cf. (1.141). The phases $\mathrm{e}^{\pm \mathrm{i}\psi}$ and $\mathrm{e}^{\pm \mathrm{i}\phi}$ are irrelevant for any observable because they can be absorbed into the field operators. After this redefinition of the fields we have

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} b_1 \equiv \mathrm{d} \\ b_2 \equiv \mathrm{s} \end{pmatrix} \tag{3.151}$$

The remaining mixing angle θ (which is measurable) is called the Cabibbo angle.

In the case of three quark families the most general transformation matrix V can be constructed, for example, by taking the product of three successive, two-dimensional transformations of the same kind as above (see exercise 3.7). Some of the resulting phases can be absorbed in the field operators, as before. The remainder is a unitary matrix that depends on three real mixing angles $\theta_1, \theta_2, \theta_3$ and one phase $\mathrm{e}^{i\delta}$ (Kobayashi et al. 1973),

$$V_{\text{CKM}} = \begin{pmatrix} c_1 & s_1\, c_3 & s_1\, s_3 \\ -s_1\, c_2 & c_1\, c_2\, c_3 - s_2\, s_3\, \mathrm{e}^{\mathrm{i}\delta} & c_1\, c_2 s_3 + s_2\, c_3\, \mathrm{e}^{\mathrm{i}\delta} \\ -s_1\, s_2 & c_1\, s_2\, c_3 + c_2\, s_3\, \mathrm{e}^{\mathrm{i}\delta} & c_1\, s_2 s_3 - c_2\, c_3\, \mathrm{e}^{\mathrm{i}\delta} \end{pmatrix}, \tag{3.152}$$

where $c_i := \cos\theta_i, s_i := \sin\theta_i$. Clearly, if $\theta_2 = \theta_3 = 0$ this matrix reduces to the previous case (3.151) with the b-quark decoupling from the (d,s) sector. The real angle θ_1, in particular, is seen to take over the role of the Cabibbo angle of the previous case with only two families of quarks.

Before moving on we note the following

Remarks: (i) We have followed the convention that all the mixing is in the sector of down-type quarks, i.e. of the quarks with electric charge -1/3. This is reflected in the form of the mass Lagrangian (3.149) above where the mass matrix $M^{(\mathrm{u})}$ of up-type quarks (electric charge +2/3) is assumed diagonal. Furthermore, we have taken the mass matrix $M^{(\mathrm{d})}$ of down-type quarks to be hermitean. In fact, in the context of a specific theory, it may well be that neither $M^{(\mathrm{u})}$ nor $M^{(\mathrm{d})}$ is diagonal in the basis of the states which couple at weak CC vertices. Also, as we know from the analysis of fermionic mass terms in Chap. 1, these matrices need not be hermitean. We shall return to this more general situation and the necessary modifications in Sect. 3.4.6. below.

(ii) The parametrization (3.152) of the Cabibbo-Kobayashi-Maskawa (CKM) matrix is but one of many possibilities. One often writes, more generally,

$$\begin{pmatrix} d_1 \equiv \mathrm{d}' \\ d_2 \equiv \mathrm{s}' \\ d_3 \equiv \mathrm{b}' \end{pmatrix} = \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} \begin{pmatrix} d_1 \equiv \mathrm{d} \\ d_2 \equiv \mathrm{s} \\ d_3 \equiv \mathrm{b} \end{pmatrix}, \tag{3.152'}$$

the subscripts {u-type, d-type} indicating the CC-vertex that appears multiplied with the corresponding matrix element. For example, the quark vertex $(\bar{\mathrm{u}}\mathrm{d})\mathrm{W}$ responsible for the decay of the neutron $\mathrm{n} \to \mathrm{p} + \mathrm{e} + \bar{\nu}_{\mathrm{e}}$ is multiplied by V_{ud}, the vertex $(\bar{\mathrm{s}}\mathrm{u})\mathrm{W}$ which appears in the decay $\mathrm{K}^+ \to \mu^+ \nu_\mu$ is multiplied by V_{us}. If there were only two generations, we would have $V_{\mathrm{ud}} = \cos\theta$, cf. (3.151).

(iii) Note that CKM-mixing, in fact, describes a rotation of the weak CC interaction relative to the strong, electromagnetic and weak NC interactions. Therefore, in the absence of CC interactions, the quark states d, s, b carry additively conserved quantum numbers. Being separated from each other through the corresponding selection rules, their relative phases are unobservable and, hence, can be chosen at will. This statement applies to the basis of mass eigenstates. The weak interaction states $(d_i \equiv \mathrm{d}', \mathrm{s}', \mathrm{b}')$, too, are defined up to phases, one for each state. Therefore, if V is a mixing matrix, the product

$$V' = \mathrm{diag}\,(\mathrm{e}^{-\mathrm{i}\beta_1}, \mathrm{e}^{-\mathrm{i}\beta_2}, \mathrm{e}^{-\mathrm{i}\beta_3}) V \,\mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha_1}, \mathrm{e}^{\mathrm{i}\alpha_2}, \mathrm{e}^{\mathrm{i}\alpha_3}), \quad \alpha_i, \beta_i \in [0, 2\pi]$$

is also a mixing matrix, equivalent to the first. As a consequence, there is the freedom to choose six phases, or, more precisely, *five* linearly independent phase differences. The matrix V, being a unitary 3×3 matrix, depends on nine real parameters. Taking into account the freedom in the choice of the phases, this shows that V depends on four, physically meaningful, parameters. Another way to state this result is to say that it needs at least four independent measurements to determine V completely.

(iv) One may ask the following question: Suppose one measures the absolute values $|V_{u_i \mathrm{d}_j}|$ of all elements of the CKM matrix. What freedom is left in reconstructing the full, unitary (hence complex) matrix V, taking due account of the freedom to choose certain phases, as discussed above? The answer for two generations is trivial. The case of three generations is also dealt with without too much effort: Except for the freedom in choosing the phases α_i, β_k above, the solution is unique. Matters become a lot more complicated for the case of four or more generations (Auberson et al. 1991).

We return now to the construction of the GSW-model for quarks. In analogy to the case of the leptons, and on the basis of the phenomenological information on hadronic weak interactions, the quark fields $u_f(x)$ and $d_f(x)$ of a given family are classified in the following reducible multiplet of G,

$$\Psi_{q_f} = \begin{pmatrix} u_f(x)_{\mathrm{L}} \\ d_f(x)_{\mathrm{L}} \\ u_f(x)_{\mathrm{R}} \\ d_f(x)_{\mathrm{R}} \end{pmatrix}, \tag{3.153}$$

where

$$L_f := \begin{pmatrix} (u_f)_{\mathrm{L}} \\ (d_f)_{\mathrm{L}} \end{pmatrix}$$

forms a doublet with respect to the SU(2) factor, while $(u_f)_{\mathrm{R}}$ and $(d_f)_{\mathrm{R}}$ are singlets. In order to obtain the correct charges from (3.145), the weak hypercharges must be chosen as follows:

$$V(y) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix} \tag{3.154}$$

The choice guarantees the correct coupling of the quark currents to the photon

$$\begin{aligned} \mathscr{L}_{\gamma}^{(\mathrm{q})} &= \frac{g g'}{\sqrt{g^2 + g'^2}} \sum_f \overline{\Psi_{q_f}(x)} \{ V(T_3) + \frac{1}{2} V(Y) \} \gamma^{\alpha} \Psi_{q_f}(x) A_{\alpha}(x) \\ &= -e j^{\alpha}_{\mathrm{e.m.}}(x) A_{\alpha}(x) \end{aligned} \tag{3.155}$$

with

$$j^{\alpha}_{\mathrm{e.m.}} = \sum_{f=1}^{3} \left\{ \frac{2}{3} \overline{u}_f \gamma^{\alpha} u_f - \frac{1}{3} \overline{b}_f \gamma^{\alpha} b_f \right\}. \tag{3.156}$$

The weak CC interactions are given by the analogue of (3.137)

$$\begin{aligned}\mathscr{L}_{\mathrm{CC}}^{(q)} &= \frac{g}{\sqrt{2}} \sum_f \overline{L_f(x)} s_+ \gamma^\alpha L_f(x) W_\alpha^-(x) + \text{h.c.} \\ &= \frac{g}{2\sqrt{2}} \sum_f \overline{u_f(x)}\, \gamma^\alpha (1-\gamma_5) d_f(x) W_\alpha^-(x) + \text{h.c.} \end{aligned} \tag{3.157}$$

The neutral weak interactions, finally, are found by isolating the coupling to the Z^0-boson in the neutral interaction Lagrangian

$$\sum_f \overline{\Psi_{q_f}} \{ g V(T_3)\, \gamma^\alpha A_\alpha^{(3)} + g' \frac{1}{2} V(Y)\, \gamma^\alpha A_\alpha^{(0)} \} \Psi_{q_f}.$$

Using the decomposition (3.78) this gives

$$\begin{aligned}\mathscr{L}_{\mathrm{NC}}^{(\mathrm{q})} &= \sum_f \frac{1}{\sqrt{g^2+g'^2}} \overline{\Psi_{q_f}} \{ g^2\, V(T_3) - \frac{1}{2} g'^2\, V(Y) \}\, \gamma^\alpha\, \Psi_{q_f}\, Z_\alpha(x) \\ &= \sqrt{g^2+g'^2} \sum_f \{ \overline{\Psi_{q_f}}\, V(T_3)\, \gamma^\alpha\, \Psi_{q_f} - \frac{g'^2}{g^2+g'^2} \overline{\Psi_{q_f}} V\, ((T_3) \\ &\quad + \frac{1}{2} V(Y)) \gamma^\alpha \psi_{q_f}\, Z_\alpha(x) \\ &= \frac{g}{4 \cos\, \theta_{\mathrm{w}}} \Bigg\{ \sum_f [\overline{u_f(x)}\, \gamma^\alpha\, (\mathbb{1}-\gamma_5) u_f(x) - \overline{b_f(x)}\, \gamma^\alpha (\mathbb{1}-\gamma_5)\, b_f(x)] \\ &\quad -4\, \sin^2\, \theta_{\mathrm{w}}\, j_{\mathrm{e.m.}}^\alpha(x) \Bigg\} \mathrm{Z}_\alpha(x). \end{aligned} \tag{3.158}$$

We have rewritten the neutral current which couples to the Z^0 in such a way that it appears as a linear combination of the neutral partner of the CC current in (3.157) and of the electromagnetic current. This form stresses the analogy to the leptonic neutral current, (3.86). It is particularly useful when we wish to compute matrix elements of these currents between physical hadron states: In this case these matrix elements are "dressed" or renormalized by the strong interactions so that the bare vertices $\left\langle \overline{q(x)}\, \gamma^\alpha\, q(x) \right\rangle$ and $\left\langle \overline{q(x)}\, \gamma^\alpha \gamma_5\, q(x) \right\rangle$ are replaced by vertex functions containing the corresponding covariants and a set of invariant form factors which parametrize these dressing effects.

The neutral current

$$\begin{aligned} K_\alpha^{(q)}(x) :&= \sum_f [\overline{u_f(x)}\, \gamma_\alpha\, (\mathbb{1}-\gamma_5) u_f\, (x) - \overline{b_f(x)}\, \gamma_\alpha\, (\mathbb{1}-\gamma_5)\, b_f(x)] \\ &\quad -4 \sin^2\, \theta_{\mathrm{w}} j_\alpha^{\mathrm{e.m.}}\, (x), \end{aligned} \tag{3.159}$$

which appears in the interaction (3.158), contains only terms which are diagonal in flavour, i.e. which are of the form $\bar{u}u$, $\bar{d}d$, $\bar{c}c$, $\bar{s}s$, $\bar{t}t$, and $\bar{b}b$. This is because the fields (3.148) appear in diagonal form in the neutral currents K_α and $j_\alpha^{\text{e.m.}}$, viz.

$$\sum_f \overline{d}_f \, \Gamma_\alpha \, d_f = \sum_f \sum_{f',f''} (V^\dagger_{\text{CKM}})_{f'f} \, (V_{\text{CKM}})_{ff''} \, \overline{b}_{f'} \, \Gamma_\alpha \, b_{f''} = \sum_f \overline{b}_f \, \Gamma_\alpha \, b_f \,. \tag{3.160}$$

Thus, the interaction (3.158) does not contain neutral couplings of the type $\bar{d}\Gamma_\alpha s$ or $\bar{s}\Gamma_\alpha d$ that would change the strangeness (Glashow et al. 1970). NC processes with $\Delta S \neq 0$ can only come about in higher orders of perturbation theory. This result is in accord with the empirical findings, see e.g. (3.45).

As in the case of the leptons, the quarks remain massless in this version of the theory which still possesses the full internal symmetry $G = \mathrm{U}(2)$. In particular, all vector and axial-vector currents are exactly conserved at this stage.

3.4.3 *Spontaneous Symmetry Breaking*

The GSW unified theory as developed up to this point is gauge invariant with respect to the group G, (3.129), of local gauge transformations. Very much like in the case of QED (which is Abelian) gauge invariance is essential for the theory to be renormalizable. At the same time, however, the theory is also invariant under G, considered as a *global* symmetry. As a consequence, the vector bosons of the theory are mass-degenerate, i.e. the W and the Z^0 are massless like the photon.

As this is in conflict with observation it is clear that at least part of the internal symmetry must be broken. More precisely, the symmetry should be broken in such a way that the gauge invariance of the theory is preserved (in view of its renormalizability) but that the mass degeneracy of the multiplets of gauge fields is lifted. This can only be achieved if one succeeds in modifying the underlying Lagrangian, while still preserving its full local gauge symmetry, in such a way that the ground state of the theory, i.e. its physical realization, exhibits less symmetry than the Lagrangian itself. The phenomenon which is well-known from the physics of condensed matter (superconductivity, ferromagnetism, etc.) is called *spontaneous symmetry breaking* or *hidden symmetry*.

Regarding the electroweak interactions there is the possibility that this spontaneous breakdown of the symmetry comes about in a dynamical way (as it does in the theory of superconductivity, for instance), from elements which are integral part the theory. As yet, there is no convincing scheme or theoretical proof for this, and symmetry breaking is introduced by hand through what is called the Higgs mechanism: One adds to the Lagrangian of the GSW model an appropriate set of scalar fields $\phi = \{\phi_1(x), \ldots, \phi_M(x)\}$, so-called Higgs fields, with a self-interaction $V(\phi)$ that is chosen such as to induce spontaneous breakdown of the symmetry.

The phenomenon in itself is very interesting and deserves further detailed study. However, in order not to leave the scope of this book, we concentrate on those results which are relevant for gauge theories. We refer the reader to the excellent reviews

in the literature (Bernstein 1974, O'Raifeartaigh 1979, [O'R 86]) for more detailed presentations.

(a) Definition of spontaneous symmetry breaking
Let ϕ be a set of scalar fields which form a representation of the symmetry group G, and let $V(\phi)$ be a potential term, which satisfies the following conditions:

(i) $V(\phi)$ is invariant under the whole group G, $V(U(g)\phi) = V(\phi)$ for all $g \in G$,
(ii) $V(\phi)$ has an absolute minimum at $\phi^0 = \{\phi_1^0, \ldots, \phi_M^0\}$,
(iii) this minimum of $V(\phi)$ is degenerate, i.e. ϕ^0 is *not* invariant under G.

The degeneracy of the absolute minimum, condition (iii), implies that there is at least one $g \in G$ for which

$$\sum_k U_{ik}(g)\phi_k^0(x) \neq \phi_i^0, \tag{3.161}$$

If we write $U(g)$ as an exponential series, $\exp\{\mathrm{i}\Sigma_{k=1}^N \Lambda_k(x)\, U(T_k)\}$, the condition (3.161) says that there is at least one generator T_i of the Lie algebra of G for which $U(T_i)\phi^0$ is not zero. Therefore, it is useful to form independent linear combinations $S_i = \Sigma_{k=1}^N C_{ik}T_k$ of the generators such that S_i fall in either of the two classes

$$\text{(A) } \{S_1, \ldots, S_P\} \text{ for which } U(S_i)\phi^0 = 0, \tag{3.162a}$$

$$\text{(B) } \{S_{P+1}, \ldots, S_N\} \text{ for which } U(S_k)\phi^0 \neq 0. \tag{3.162b}$$

It is not difficult to convince oneself that class (A) generates a subgroup of G. All transformations $g \in G$ which contain only generators of class (A), viz.

$$g = \exp\left\{ i \sum_{k=1}^{P} \Lambda_k S_k \right\}, \tag{3.163}$$

leave the minimum ϕ^0 invariant. In other words, these transformations (3.163) form the *little group* H of ϕ^0 with

$$\dim H = P. \tag{3.164}$$

Expanding the potential $V(\phi)$ around ϕ^0, we have

$$V(\phi) = V(\phi^0) + \frac{1}{2}\sum_{m,n} M_{mn}\,\delta\phi_m\delta\phi_n + O((\delta\phi)^3),$$

where the matrix

$$M_{mn} := \frac{\partial^2 V}{\partial\phi_m\partial\phi_n}\Big|_{\phi^0}$$

is positive semi-definite, and where the infinitesimal variation of the field ϕ_n is given by

$$\delta\phi_n = i \sum_{m=1}^{M} \sum_{k=1}^{N} U_{nm}(S_k)\, \phi_m^0\, \delta\, \Lambda_k .$$

Let us define the following vectors

$$\upsilon_n^{(k)} := \sum_{m=1}^{M} U_{nm}(S_k)\, \phi_m^0 . \tag{3.165}$$

As the variations $\delta\Lambda_k$ of the group parameters are linearly independent, the condition for $V(\phi)$ to have a minimum in $\phi = \phi^0$ reads

$$\sum_m \upsilon_n^{(i)}\, M_{nm}\, \upsilon_m^{(k)} = 0.$$

Furthermore, as $M \geqslant 0$, this is fulfilled provided

$$\sum_m M_{nm}\, \upsilon_m^{(k)} = 0. \tag{3.166}$$

The matrix M takes on the role of a mass matrix for the scalar fields ϕ. A number P of its eigenvectors are identically zero. These are the ones for which $k \in \{1, \ldots, P\}$, cf. (3.162a), i.e. those which are formed with a generator of the little group H of ϕ^0. For all other values of $k \in \{P+1, \ldots, N\}$, $\upsilon^{(k)}$ does not vanish. From this we conclude that M has

$$N_G := \dim G - \dim H = N - P \tag{3.167}$$

eigenvalues which are zero. This result is a consequence of a theorem by Goldstone[5]: A manifestly Lorentz invariant theory which has a hidden symmetry (i.e. whose internal symmetry is spontaneously broken) contains a set of massless scalar fields, the so-called Goldstone bosons. The number of Goldstone fields N_G is given by (3.167). It is the difference of the dimension of Lie algebras of the full symmetry group G and of the little group $H(\phi^0)$ of ϕ^0, respectively. Clearly, N_G cannot be zero because, by the very definition of spontaneous symmetry breaking, the class (B), (3.162b), is not empty. As is clear from our analysis above, N_G, the number of Goldstone fields, does not depend on the representation of the scalar fields. It depends solely on the symmetry group G and on $H(\phi^0)$, i.e. on the residual symmetry of the physical realization of the theory, after it has been spontaneously broken.

There is no restriction on the remaining P eigenvalues of the mass matrix M [corresponding to those $\upsilon^{(k)}$, (3.162a), which vanish identically]. These eigenvalues

[5]See Bernstein (1974) for a complete list of references and a detailed exposition of this theorem and its consequences.

pertain to a set of $(M - N_G)$ massive scalar fields. In contrast to the case of the Goldstone fields their number depends on the representation spanned by the fields ϕ.

The massless Goldstone particles have a simple geometrical interpretation. By assumption, the minimum of $V(\phi)$ in ϕ^0 is degenerate. Thus there exist transformations $g \in G$ for which $\phi'^0 = U(g)\phi^0$ is not identical with ϕ^0, $\phi'^0 \neq \phi^0$. The set of all ϕ'^0 which can be reached by applying all possible group transformations to ϕ^0, form the *group orbit* of ϕ^0. The Goldstone fields are proportional to those vectors $\upsilon^{(k)} = U(S_k)\phi^0$ which do not vanish identically. On the other hand, an infinitesimal transformation of ϕ^0 is given by

$$\phi'^0 = U(g \simeq \mathbb{1} + \mathrm{i}\Lambda_k\, S_k)\phi^0 = \{\mathbb{1} + \mathrm{i}\Lambda_k U(S_k)\}\phi^0,\ \Lambda_k \ll 1.$$

This shows that $U(S_k)\phi^0$ is a tangent to the orbital of ϕ^0. Thus, a Goldstone field can be understood as an excitation of the system along the orbit of ϕ^0. As this does not lead out of the (degenerate) minimum of $V(\phi)$, a Goldstone excitation can have arbitrarily small frequency. In other words, a Goldstone field describes a massless particle.

(b) Spontaneous symmetry breakdown in the frame of a local gauge theory
The discussion of the previous section is incomplete insofar as the scalar fields ϕ cannot be discussed in isolation from the rest of the theory. For instance, the question of whether or not the Goldstone fields correspond to physical, massless particles, cannot be answered without knowing how these fields couple to other fields such as spinor or vector fields. It is particularly interesting to investigate the role of the Goldstone fields in the case where G is not only a global symmetry but is also a local gauge symmetry of the Lagrangian. In this case the Goldstone fields do not describe observable massless scalars. Instead, as we shall see, they decouple from the other particles in the theory. At the same time some of the gauge bosons which formerly were all massless, acquire finite masses. This happens for as many of them as there are Goldstone fields.

Let us return to the Lagrangian (3.128) which describes a set of spinor fields Ψ as well as set of scalar fields ϕ, besides the gauge fields A_α. Let us assume that the potential term $V(\phi)$ is constructed such that it leads to spontaneous breakdown of the symmetry of the Lagrangian (3.128). Thus, $V(\phi)$ fulfills the conditions (i) to (iii) of Sect. 3.4.3a, i.e. it has a degenerate minimum at ϕ^0. As ϕ^0 is the position of the absolute minimum of $V(\phi)$ we introduce new scalar fields by subtracting $\phi(x)$ and ϕ^0,

$$\Theta_n(x) := \phi_n(x) - \phi_n^0, n = 1 \ldots, M. \tag{3.168}$$

Rewriting (3.128) in terms of these new dynamical fields we obtain

$$\begin{aligned}\mathscr{L}' = &-\frac{1}{4k\mathrm{e}^2}(F_{\alpha\beta}, F^{\alpha\beta}) + \frac{1}{2}(U(A_a)\phi^0, U(A^\alpha)\phi^0) + \frac{1}{2}(D_\alpha\Theta(x), D^\alpha\Theta(x))\\ &+ \mathrm{Re}(D_\alpha\Theta(x), U(A^\alpha)\phi^0) - V(\Theta(x) + \phi^0) - g(\overline{\Psi(x)}, \phi^0\Psi(x))\\ &+ \frac{\mathrm{i}}{2}\overline{(\Psi(x)}, \gamma^\alpha \overleftrightarrow{D}_\alpha \Psi(x)) - (\overline{\Psi}, (m + g\Theta(x))\Psi).\end{aligned} \tag{3.169}$$

Note that in this Lagrangian the fields ϕ^0 are constants whilst the role of the dynamical scalar fields is taken over by the fields $\Theta(x)$. Although its structure is similar to the structure of $\mathscr{L}$, (3.128), the modified Lagrangian (3.169) has two remarkable new properties:

(i) The term $-g(\overline{\Psi}, \phi^0\Psi)$ yields a finite mass for the fermion fields, provided the multiplets Ψ and ϕ are chosen such that $\overline{\Psi}, \phi^0$, and Ψ can be coupled to an invariant with respect to G. The mass term which is due to the spontaneous symmetry breakdown is particularly relevant for the GSW model for which it was not possible to construct an invariant mass term $(\overline{\Psi}, m\Psi)$ on the basis of the multiplet assignments (3.133) and (3.153).

(ii) Second, and perhaps more importantly, the term

$$\frac{1}{2}(U(A_\alpha)\phi^0, U(A^\alpha)\phi^0) \tag{3.170}$$

provides mass terms for at least some of the gauge fields without destroying the gauge invariance of the theory. This is what we now wish to analyze in more detail.

In the expression (3.170) only the generators of class (B), (3.162b), give nonvanishing contributions

$$\frac{1}{2}(U(A_\alpha)\phi^0, U(A^\alpha)\phi^0) = \frac{1}{2}\sum_{i,k=P+1}^{N} m_{ik}A_\alpha^{(k)}(x)A^{(k)\alpha}(x), \tag{3.170'}$$

where

$$m_{ik} := \mathrm{e}^2(U(S_i)\phi^0, U(S_k)\phi^0). \tag{3.171}$$

The quadratic mass matrix has the dimension $N_G \times N_G$, with N_G as given by (3.167). It has N_G positive eigenvalues. The eigenvectors of this matrix are orthogonal linear combinations of the original fields $A_\alpha^{(i)}$. These new vector fields now describe massive gauge bosons. Thus we obtain the following important result: If G is the original full symmetry, H the residual symmetry (after the symmetry is spontaneously broken), of the locally gauge invariant Lagrangian then a number $N_G = \dim G - \dim H$ of the gauge bosons of the theory acquire finite masses. The remaining $P = \dim H$ gauge bosons which correspond to the generators of the residual symmetry group H remain massless. The number of vector particles which become massive is equal to the number of Goldstone scalar bosons. In fact, one can show that these massless scalars decouple from the rest of the theory. Their role is merely to provide the third independent polarization component that distinguishes the massive vector particle from the massless one (which has only two components).

(c) Application to the GSW theory

The analysis of the preceding section provides a constructive principle of how to give the $W^\pm$ and Z^0 finite masses while leaving the photon massless. The original symmetry (3.129) whose Lie algebra has dimension $N = \dim G = 4$, must be spontaneously broken such that the residual symmetry is $H = \mathrm{U}(1)$ with dimension

$P = \dim H = 1$. If this is achieved, three of the gauge fields become massive and one remains massless. In practice, this means that we add the following Higgs sector to the Lagrangian of the GSW model:

$$\frac{1}{2}(D_\alpha\phi, D^\alpha\phi) - V(\phi).$$

The potential $V(\phi)$ shall have a degenerate minimum at $\phi = \phi^0$ such that the residual symmetry is U(1). In fact, it is not necessary to write down an explicit form for the potential $V(\phi)$. Indeed, it turns out that it is enough to arrange the eigenvalues of T_3 (weak isospin) and of Y (weak hypercharge) for the Higgs fields such as to make one of them decouple from the photon field. This is seen as follows.

Let $U(T_\mu)$ be the representation of the generators T_μ in the space of the Higgs fields ϕ. If ϕ^0 denotes the position of the degenerate, absolute minimum of $V(\phi)$, the mass matrix for the vector fields is given by

$$\frac{1}{2}(D_\alpha\phi^0, D^\alpha\phi^0) = \frac{1}{2}(U(A_a)\phi^0, U(A^\alpha)\phi^0), \tag{3.172}$$

where

$$U(A_\alpha)\phi^0 = -\mathrm{i}\left\{g\sum_{k=1}^{3} U(T_k)A_\alpha^{(k)}(x) + \frac{1}{2}g'U(Y)A_\alpha^{(0)}(x)\right\}\phi^0. \tag{3.173}$$

Here we have used the identification $-e = g$, (3.138), and the definition (3.143) for Y in terms of T_0. Replacing the fields $A_\alpha^{(\mu)}$ by the physical charged fields (3.72) and neutral fields (3.78) we obtain

$$\begin{aligned} U(A_a)\phi^0 = -i\Bigg\{ & \frac{1}{\sqrt{2}}g[U(T_+)W_\alpha^{(-)}(x) - U(T_-)W_\alpha^{(+)}(x)] \\ & + \sqrt{g^2+g'^2}U\left(\frac{g^2}{g^2+g'^2}T_3 - \frac{1}{2}\frac{g'^2}{g^2+g'^2}Y\right)Z_\alpha^0(x) \\ & + \frac{gg'}{\sqrt{g^2+g'^2}}U(T_3+\frac{1}{2}Y)A_\alpha(x)\Bigg\}\phi^0, \end{aligned} \tag{3.173'}$$

where $T_\pm := T_1 \pm \mathrm{i}T_2$. Let t_3 and y be the eigenvalues of T_3 and Y for the fields $\phi(x)$, or $\theta(x) = \phi(x) - \phi^0(x)$, respectively. Equation (3.173') shows that the electric charge of the field ϕ^0 is proportional to

$$Q_\gamma := t_3 + \frac{1}{2}y.$$

The photon field remains massless if and only if $Q_\gamma \phi^0$ vanishes. Thus, if $\phi^0_m \neq 0$, we impose the condition $y = -2t_3$.[6] With this condition we see from (3.173') that the coupling to the Z^0 field is proportional to

$$\frac{g^2}{g^2 + g'^2} t_3 - \frac{g'^2}{2(g^2 + g'^2)} y = t_3.$$

On the other hand, in order to make the Z^0 massive we must require t_3 to be different from zero. Therefore, let us assume that $\phi(x)$ belongs to an irreducible representation of SU(2) with total weak isospin t (different from zero) and projection quantum number t_3,

$$U(\boldsymbol{T}^2)\phi = t(t+1)\phi, \quad U(T_3)\phi = t_3\phi. \tag{3.174}$$

Equations (3.172, 173') then yield the following expressions for the gauge boson masses:

$$\begin{aligned} m_{\mathrm{W}}^2 &= \frac{1}{2} g^2 \frac{1}{2} (\phi^0, \{U(T_+)U(T_-) + U(T_-)U(T_+)\}\phi^0) \\ &= \frac{1}{2} g^2 [t(t+1) - t_3^2](\phi^0, \phi^0), \end{aligned} \tag{3.175}$$

$$\begin{aligned} m_{\mathrm{Z}}^2 &= (g^2 + g'^2)(\phi^0, U(T_3)U(T_3)\phi^0) \\ &= (g^2 + g'^2) t_3^2 (\phi^0, \phi^0). \end{aligned} \tag{3.176}$$

In (3.175) we have made use of the relation

$$\frac{1}{2}(T_+T_- + T_-T_+) = \boldsymbol{T}^2 - T_3^2.$$

Equations (3.175, 3.176) immediately give the mass relation

$$\rho \equiv \frac{m_{\mathrm{w}}^2}{m_{\mathrm{Z}}^2 \cos^2 \theta_{\mathrm{w}}} = \frac{t(t+1) - t_3^2}{2t_3^2}, \tag{3.177}$$

where we have used the relation $\cos^2 \theta_{\mathrm{w}} = g^2/(g^2 + g'^2)$. Note that (3.177) is precisely the quantity ρ as defined in (3.88). The experimental information that ρ is 1 with a rather small error bar [RPP10] is compatible with the assignment

$$t = \frac{1}{2} \tag{3.178}$$

for the Higgs fields. This means that these fields form a doublet with respect to SU(2). One of the fields is electrically neutral and has $y = -2t_3 = 1$. This is the

[6] In other words, all components of ϕ^0 vanish except the one for which $y = -2t_3$.

one that has a nonvanishing vacuum expectation value ϕ^0 and which gives rise to the vector meson masses (3.175, 3.176). Its partner in the doublet has the quantum numbers $(y = 1, t_3 = +\frac{1}{2})$ and electric charge $Q = 1$.

Finally, we can introduce a Yukawa coupling of the fermion fields to the Higgs doublet without destroying the local gauge invariance of the theory,

$$g_f\{(\overline{L_f}\phi)\, f_R + \overline{f_R}(\tilde{\phi} L_f)\}, \tag{3.179}$$

with $\tilde{\phi}$ the charge conjugate of ϕ.

This extra term gives rise to a genuine interaction of the fermion with the neutral Higgs field. Its vacuum expectation value gives rise to a fermion mass term of the form (we consider the example of a lepton doublet),

$$g_f \upsilon\{\overline{f_R} f_L + \overline{f_L} f_R\} = g_f \upsilon \overline{f(x)} f(x) \tag{3.180}$$

with

$$\upsilon = \sqrt{(\phi^0, \phi^0)}.$$

It is gratifying that in the spontaneously broken version of the theory it is possible to construct fermion mass terms which do not violate the gauge symmetry of the theory. On the other hand, (3.180) neither predicts the scale of the fermion masses nor does it yield relations between the masses of different lepton families. In fact, for every fermion family, the coupling constant g_f must be adjusted such that (3.180) yields the correct mass term, viz $g_f = m_f/(\phi^0, \phi^0)^{\frac{1}{2}}$. At the same time this implies that the fermion coupling to the physical, neutral Higgs field, (3.179), is proportional to m_f and hence very small. This makes it difficult to subject such an interaction to experimental verification.

Finally, we note that (3.175, 3.176) predict the W- and Z-masses in terms of the unknown quantity (ϕ^0, ϕ^0). The absolute values of m_W and m_Z, as yet, are derived from the empirical coupling constants α, G and from the Weinberg angle θ_W, as given in (3.89). These expressions are modified somewhat by corrections of higher order. As the theory is renormalizable, these corrections are finite and can be calculated in a unique way from perturbation theory.

The experimental values of the masses of $\mathrm{W}^{\pm}$ and Z are given in (3.90a) above. Without radiative corrections the parameters $\sin\theta_W$, m_W and m_Z are related by (3.88) and (3.89a). Furthermore, by (3.175, 3.176) m_W and m_Z are proportional to (ϕ^0, ϕ^0), the square of the value ϕ^0 where $V(\phi)$ assumes its minimum, viz.

$$m_{\mathrm{w}}^2 = \frac{g^2}{4}(\phi^0, \phi^0) \qquad m_{\mathrm{Z}}^2 = \frac{g^2}{4\cos^2\theta_{\mathrm{w}}}(\phi^0, \phi^0).$$

Inserting (3.75) in the first of these formulae one finds

$$\upsilon^2 = (\phi^0, \phi^0) = \frac{1}{G\sqrt{2}} = (246.2\,\mathrm{GeV})^2. \tag{3.181}$$

The number υ which in the quantized form of the theory is the vacuum expectation value of the neutral Higgs field, may be regarded as the energy scale typical for the weak interactions.

When radiative corrections are taken into account, some or all of these simple, "tree-level" relations are modified to some extent. (They are called tree level because they correspond to the lowest order in the interactions between the particles of the theory. Pictorially they are described by diagrams which look like the trees we drew when we were children. In contrast, any higher correction involves at least one closed loop.) Unlike the case of quantum electrodynamics (QED) of electrons and photons there is no natural, physically preferred, renormalization scheme and different choices are possible, (for a thorough discussion and a guide to the literature see the review 10 in [RPP10]). For example, one may adopt a so-called *on-shell scheme* where the tree-level formula $\sin^2\theta_\mathrm{w} = 1 - m_\mathrm{W}^2/m_\mathrm{Z}^2$, i.e. the formula (3.88) with $\rho = 1$, holds for the physical values of these parameters, that is, for the renormalized quantities to all orders of perturbation theory. To get an impression of the magnitude of the corrections let us make use of the fact that m_Z, α, and G are known very precisely, in contrast to m_w which is not. The quantity $\sin\theta_\mathrm{w}$ is then obtained from the formulae

$$m_\mathrm{W}^2 = \frac{\pi\alpha}{G\sqrt{2}}\frac{1}{\sin^2\theta_\mathrm{w}}\frac{1}{(1-\Delta r)}, \quad m_\mathrm{Z}^2 = \frac{m_\mathrm{W}^2}{1-\sin^2\theta_\mathrm{w}}, \tag{3.182}$$

the first of which is (3.89) corrected by terms of higher order. The quantity Δr, which is typical for the size of radiative corrections, is found to be of the order of 0.04, depending on the values of the masses of the t-quark and the Higgs particle which appear in loop diagrams.

3.4.4 Summary of CC and NC Interactions in the GSW Theory

In this section we collect and summarize the leptonic and hadronic charged current and neutral current interactions as they follow from the results in Sects. 3.2.3, 3.4.2 and 3.4.3. For convenience we express all coupling constants in terms of the elementary charge in natural units $e = \sqrt{4\pi\alpha}$. The interaction Lagrangian of leptons and quarks with the gauge bosons γ, W and Z^0 is given by

$$\begin{aligned}\mathscr{L}_{int} = -e\Big\{ & j^\alpha_{e.m}\,(x)A_\alpha(x) + \frac{1}{2\sqrt{2}\sin\theta_\mathrm{w}}[J^\alpha(x)W^-_\alpha(x) + \mathrm{h.c.}] \\ & + \frac{1}{4\sin\theta_\mathrm{w}\cos\theta_\mathrm{w}}K^\alpha(x)Z^0_\alpha(x)\Big\}\,.\end{aligned} \tag{3.183}$$

In this expression $j^\alpha_{\mathrm{e.m.}}$ is the electromagnetic current, J^α is the charged current,

$$J^{\alpha}(x) = \sum_{\mathrm{f=e},\mu,\tau} \overline{\nu}_{\mathrm{f}} \gamma^{\alpha}(\mathbb{1} - \gamma_5) f(x) + \sum_{f=1}^{3} \overline{u_f(x)} \gamma^{\alpha}(\mathbb{1} - \gamma_5) d_f(x), \tag{3.184a}$$

where u_f denotes the quarks with electric charge $+\frac{2}{3}$,

$$\{u_f\} = \{\mathrm{u}, \mathrm{c}, \mathrm{t}\}$$

while d_f stands for the weak eigenstates

$$d_f = \sum_{f'=1}^{3} V_{ff'} b_{f'},$$

given in terms of the strong interaction eigenstates d, s and b with electric charge $-\frac{1}{3}$. We recall that $V_{ff'}$ is the mixing matrix (3.152). The neutral current K_{α} is given by (3.86, 3.156, 3.159) as

$$\begin{aligned} K_{\alpha}(x) = & \sum_{\mathrm{f=e},\mu,\tau} \overline{\nu_f(x)} \gamma_{\alpha}(\mathbb{1} - \gamma_5) \nu_f(x) \\ & + \sum_{\mathrm{f=e},\mu,\tau} \overline{f(x)} \{-\gamma_{\alpha}(\mathbb{1} - \gamma_5) + 4 \sin^2 \theta_{\mathrm{W}} \gamma_{\alpha}\} f(x) \\ & + \sum_{\mathrm{f}=1}^{3} \overline{u_f(x)} \{\gamma_{\alpha}(\mathbb{1} - \gamma_5) - \frac{8}{3} \sin^2 \theta_{\mathrm{W}} \gamma_{\alpha}\} u_f(x) \\ & + \sum_{\mathrm{f}=1}^{3} \overline{b_f(x)} \{-\gamma_{\alpha}(\mathbb{1} - \gamma_5) + \frac{4}{3} \sin^2 \theta_{\mathrm{W}} \gamma_{\alpha}\} b_f(x). \end{aligned} \tag{3.184b}$$

Note, in particular, that this current has the general form

$$\overline{\Psi^{(i)}(x)} \Gamma_{\alpha}^{(i)} \Psi^{(i)}(x),$$

where the matrix $\Gamma_{\alpha}^{(i)}$ depends on the charge and weak isospin of the lepton or quark i:

$$\Gamma_{\alpha}^{(i)} = 2t_3(i) \gamma_{\alpha}(\mathbb{1} - \gamma_5) - Q(i) \gamma_{\alpha} 4 \sin^2 \theta_{\mathrm{W}}. \tag{3.185}$$

The exchange of $\mathrm{W}^{\pm}$ and Z^0 bosons, in processes where the momentum transfer is small as compared to the gauge boson masses, gives rises to the effective four-fermion interaction [cf. (3.74, 3.85)],

$$-\mathscr{L}_{\text{int}}^{\text{eff}} = \frac{G}{\sqrt{2}}\left\{ J_\alpha^\dagger(x) J^\alpha(x) + \frac{1}{2}\rho K_\alpha^\dagger(x) K^\alpha(x)\right\}, \tag{3.186}$$

where

$$\frac{G}{\sqrt{2}} = \frac{e^2}{8 m_{\text{W}}^2 \sin^2\theta_{\text{W}}} = \frac{\pi\alpha}{2 m_{\text{W}}^2 \sin^2\theta_{\text{W}}} \tag{3.187a}$$

and

$$\rho = \frac{m_{\text{W}}^2}{m_{\text{Z}}^2 \cos^2\theta_{\text{W}}}. \tag{3.187b}$$

Note that nondiagonal terms in the product $K_\alpha^\dagger K^\alpha$ must be counted only once because they stem from the exchange of a Z^0 between a vertex with one type of lepton (or quark) and a vertex with another type. Alternatively, one may introduce an extra factor 1/2 as we did in (3.85), in order to compensate for double counting.

[Equation (3.187a) holds for the uncorrected value of m_{w}, to be distinguished from the corrected value (3.182).] These formulae summarize the weak interactions of leptons and quarks in the GSW theory.

It is instructive to recapitulate the number and the nature of the parameters of the theory, none of which is predicted. These are

(i) Coupling constants and gauge boson masses: $\alpha, \sin\theta_{\text{w}}, m_{\text{W}}, m_{\text{Z}}$. The Fermi constant G is determined in terms of the first three parameters by (3.187a), m_Z is fixed by (3.177) if the weak isospin of the Higgs field is given.
(ii) Quark masses and mixing matrix (3.152): $m_{\text{u}}, m_{\text{d}}, m_{\text{c}}, m_{\text{s}}, m_{\text{t}}, m_{\text{b}}, \theta_1, \theta_2, \theta_3, \delta$.
(iii) Lepton masses: $m_{\text{e}}, m_\mu, m_\tau$.

This list does not include the Higgs sector of the theory. Furthermore, if the neutrinos are massive, then there are three more mass values as well as another set of mixing angles because the weak neutrino states could appear to be mixtures of their mass eigenstates in a way analogous to the quark mixing (3.148, 3.152).

3.4.5 *The Higgs Sector of the GSW Model*

Let us return for a moment to spontaneous symmetry breaking and the Higgs sector. For the construction described in Sec. 3.4.3 it was not necessary to specify the explicit form of the potential $V(\Phi)$. It was sufficient to require that $V(\Phi)$ have a degenerate minimum at a value of the field not equal to zero and, for the specific needs of the GSW model, that Φ be a doublet with respect to that SU(2) of weak isospin, with $t = \frac{1}{2}$, $t_3 = \pm\frac{1}{2}$, and weak hypercharge $y = +1$, cf. (3.178). In fact, there is one further restriction related to quantization: In order not to destroy renormalizability of the theory the potential $V(\Phi)$ should be polynomial in the field, at most of order four. Taking account of these conditions the *form* of the potential is fixed uniquely. Indeed, the most general SU(2)-invariant potential which

is compatible with renormalizability must have the functional form

$$V(\Phi) = \frac{1}{2}\kappa\Phi^\dagger(x)\Phi(x) + \frac{1}{4}\lambda(\Phi^\dagger(x)\Phi(x))^2 + C, \qquad \text{with} \quad \kappa \equiv -\mu^2 < 0, \lambda > 0. \tag{3.188}$$

C is a constant and $\Phi(x)$ is the doublet field with $y = +1$

$$\Phi(x) = \begin{pmatrix} \phi^{(+)}(x) \\ \phi^{(0)}(x) \end{pmatrix} \quad (t = \frac{1}{2}, y = 1). \tag{3.189}$$

Note that if the constant κ were *positive*, say $\kappa = m_0^2$, the term $\kappa\Phi^\dagger\Phi/2$ would be the mass term in the free Lagrangian for the field ϕ,

$$\mathscr{L}^{(\text{Higgs}}_{\text{noSSB}} = \frac{1}{2}(\partial_a\Phi^\dagger\partial^\alpha\Phi) - \frac{1}{2}m_0^2(\Phi^\dagger\Phi) - \frac{1}{4}\lambda(\Phi^\dagger\Phi)2 - C. \tag{3.190}$$

The potential term would then have its minimum at $\Phi = 0$ and there would be no spontaneous symmetry breaking.

In turn, if κ is *negative*, $\kappa = -\mu^2$, the potential has an absolute minimum for all Φ which fulfill the equation

$$\Phi^\dagger\Phi \equiv (\phi_0, \phi_0) = \upsilon^2 = \frac{\mu^2}{\lambda}. \tag{3.191}$$

It can then be written in the form

$$V(\Phi) = \frac{\lambda}{4}[(\Phi^\dagger\Phi)^2 - 2(\phi_0, \phi_0)(\Phi^\dagger\Phi)] + C = \frac{\lambda}{4}[(\Phi^\dagger\Phi) - \upsilon^2]^2 - \frac{\lambda}{4}\upsilon^4 + C. \tag{3.192}$$

With this example at hand, it is instructive to verify, explicitly and step by step, the phenomenon of spontaneous symmetry breaking whose general pattern was described in Sect. 3.4.3a, b. As long as Φ is the dynamical field describing the Higgs doublet, the potential $V(\Phi)$,(3.192), enjoys the full global *and* local invariance with respect to SU(2) × $U(1)$. However, the configuration $\Phi = 0$ is not the ground state, i.e. the state of lowest energy of the theory, because $V(\Phi)$ has a local *maximum* at $\Phi = 0$. The true ground state(s) is (are) characterized by the value (3.191) for which $V(\Phi)$ is *minimal*. If we think of the theory as a quantized field theory this means that the ground state is such that Φ develops a nonvanishing vacuum expectation value. Because of charge conservation only the electrically neutral component $\phi^{(0)}(x)$ of the doublet (3.189) can have an expectation value $\langle\Omega|\phi^{(0)}(x)|\Omega\rangle$ in the vacuum state Ω which does not vanish. The *dynamical* field describing the physical neutral Higgs boson is defined as in (3.168), i.e.

$$\Theta^{(0)}(x) := \phi^{(0)} - \langle\Omega|\phi^{(0)}(x)|\Omega\rangle.$$

It is not difficult to derive the following features: The original, full, gauge invariance of the theory can be used to redefine the Higgs doublet by means of a local SU(2) transformation $\exp\{\mathrm{i}\sigma^{(k)}\alpha^k(x)/2\}$, such that the transformed field is

$$\Phi'(x) = \begin{pmatrix} 0 \\ \upsilon + \Theta^{(0)}(x) \end{pmatrix}, \tag{3.193}$$

with $\Theta^{(0)}(x)$ a real field. [A gauge in which the Higgs field has this form (3.193) is called a *unitary gauge*.] Thereby the potential (3.192) becomes

$$V(\Phi) \to V(\Theta^{(0)}) = \frac{1}{2}m^2\Theta^{(0)2}(x)\left\{1 + \frac{\Theta^{(0)}(x)}{\upsilon} + \frac{\Theta^{(0)2}(x)}{4\upsilon^2}\right\} - \frac{m^4}{16\lambda} + C, \tag{3.194}$$

where we have set $2\lambda\upsilon^2 = m^2$. Obviously, m is the mass of the particle described by the real field $\Theta^{(0)}(x)$. (The additive constant C is arbitrary and can be chosen such as to compensate the term - $m^4/16\lambda$.)

The result (3.193) shows that the positively charged partner of the doublet (3.189) does not couple explicitly to any other particle in the theory. The gauge can be chosen such that the field $\phi^{(+)}$ disappears from the theory altogether. The real field $\phi^{(0)}(x)$ was originally complex, $\phi^{(0)}(x) = \phi_1^{(0)}(x) + \mathrm{i}\phi_2^{(0)}(x)$, and, hence, represented *two* degrees of freedom. Equation (3.193) shows that only one of them, the real part, is needed to describe the Higgs boson.

Of course, one need not adopt the unitary gauge for which (3.193) holds true. It turns out, in fact, that other gauges are useful in practical calculations within the quantized version of the GSW model. The Higgs doublet is then, more generally,

$$\Theta(x) = \begin{pmatrix} \Theta^{(+)}(x) \\ \Theta^{(0)}(x) \end{pmatrix} = \Phi(x) + \upsilon = \begin{pmatrix} \phi^{(+)} \\ \phi^{(0)}(x) + \upsilon \end{pmatrix}, \tag{3.195}$$

where υ is the vector $(0, \upsilon)^{\mathrm{T}}$. The first term on the right-hand side of (3.192) becomes

$$[(\Theta^\dagger + \upsilon)(\Theta + \upsilon) - \upsilon^2]^2. \tag{3.196}$$

Comparing this form to the original one $[(\Phi^\dagger\Phi) - \upsilon^2]$ shows clearly that the primordial symmetry SU(2) × U(1) of the theory that acts on Φ by the affine transformation (3.195) becomes a hidden symmetry: The expression (3.196) is invariant under the U(1) transformations of electromagnetism which are generated by the charge operator $\widehat{Q} = T_3 + Y/2$, cf. (3.145), but is not invariant if the field $\Theta(x)$ is subject to an SU(2) transformation or a U(1) transformation generated by Y alone. The little group of ϕ_0 which leaves the position of the potential minimum invariant, is the U(1) known from electrodynamics. It is the gauge group of electrodynamics and is intimately related to the conservation of electric charge.

3.4.6 Note on Quark Masses and CKM Mixing

The Cabibbo–Kobayashi–Maskawa mixing of quark states, in the minimal standard model, is intimately related to the structure of the mass matrices in the sectors of *up*- and *down*-type quarks. Neither these mass matrices nor the CKM mixing matrix are predicted by the model. Hence their understanding will need a new framework, not known as yet, which goes beyond the minimal model. Nevertheless, if the mass matrices are given in their most general form it is possible to derive the mixing matrix from them. This is what we set out to show in this section.

Quarks carry electric charge, as well as further, additively conserved, quantum numbers. Therefore they can only have Dirac mass terms but no Majorana mass terms, as discussed in Sect. 1.8.4. Dirac mass terms connect spinors of the first kind to spinors of the second kind, cf. (1.163), or, in the language of Dirac four-component spinors, left-chiral to right-chiral fields. Recall that these are defined by means of the projection operators (3.32) as

$$\psi_{\mathrm{R}}(x) := P_+\psi(x) = \frac{1}{2}(\mathbb{1} + \gamma_5)\psi(x), \quad \psi_L(x) := P_-\psi(x) = \frac{1}{2}(\mathbb{1} - \gamma_5)\psi(x).$$

We again use the convention of writing the particle symbol for the corresponding quantized field, i.e., for example, $\upsilon(x)$ for $\psi^{(\mathrm{u})}(x)$. The *weak interaction states* (i.e. the states which couple to $\mathrm{W}^{\pm}$ at CC vertices) are denoted by

$u^{(i)}(x)$ for states of charge $+\frac{2}{3}$, $d^{(i)}(x)$ for states of charge $-\frac{1}{3}(i = 1, 2, 3)$.

The *mass eigenstates* are denoted temporarily by

$t^{(i)}(x)$ for states of charge $+\frac{2}{3}$, $b^{(i)}(x)$ for states of charge $-\frac{1}{3}(i = 1, 2, 3)$.

The general mass term in the Lagrangian, written in the basis of weak interaction states, reads

$$\mathscr{L}_{\mathrm{mass}} = \frac{1}{2}\left(\sum_{i,k=1}^{3} \overline{u_L^{(i)}}\boldsymbol{M}_{ik}^{(\mathrm{u})}u_{\mathrm{R}}^{(k)} + \overline{d_{\mathrm{L}}^{(i)}}\boldsymbol{M}_{ik}^{(\mathrm{d})}d_{\mathrm{R}}^{(\mathrm{k})}\right) + \mathrm{h.c.} \tag{3.197}$$

As we know from Sect. 1.8.4 the matrices $\boldsymbol{M}^{(\mathrm{u})}$ and $\boldsymbol{M}^{(\mathrm{d})}$ need not be hermitean. Therefore they are diagonalized by two independent unitary transformations, $\boldsymbol{V}_{\mathrm{L}}^{(\mathrm{u})}$ acting on left-chiral fields, $\boldsymbol{V}_{\mathrm{R}}^{(\mathrm{u})}$ acting on right-chiral fields (and likewise for the *down* sector), i.e.

$$\boldsymbol{V}_L\boldsymbol{M}\boldsymbol{V}_R^{\dagger} = \overset{\circ}{\boldsymbol{M}}, \quad \overset{\circ}{\boldsymbol{M}} = \mathrm{diag}(\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3),$$

where we have written the symbol $\boldsymbol{M}$ for either $\boldsymbol{M}^{(\mathrm{u})}$ or $M^{(\mathrm{d})}$, and, likewise, $\boldsymbol{V}_{\mathrm{L/R}}$ for either charge sector. A transformation of this kind is called a *bi- unitary transformation*. If $\boldsymbol{M}$ is not hermitean then $\boldsymbol{V}_{\mathrm{L}}$ is the (ordinary) unitary transformation that diagonalizes the hermitean product $\boldsymbol{M}\boldsymbol{M}^{\dagger}$ while $\boldsymbol{V}_R$ is the unitary transformation that diagonalizes $\boldsymbol{M}^{\dagger}\boldsymbol{M}$, viz.

$$\boldsymbol{V}_{\mathrm{R}}\boldsymbol{M}^{\dagger}\boldsymbol{M}\boldsymbol{V}_{R}^{\dagger} = \overset{\mathrm{o}}{\boldsymbol{M}}{}^{2} = \boldsymbol{V}_{L}\boldsymbol{M}\boldsymbol{M}^{\dagger}\boldsymbol{V}_{L}^{\dagger}.$$

Clearly, the bi-unitary transformations which diagonalize $\boldsymbol{M}^{(\mathrm{u})}$ and $\boldsymbol{M}^{(\mathrm{d})}$, respectively, take us to the basis of mass eigenstates,

$$t_{\mathrm{L}}^{(i)} = \sum_k (\boldsymbol{V}_{\mathrm{L}}^{(u)})_{ik} u_{\mathrm{L}}^{k}, \quad b_{\mathrm{L}}^{(i)} = \sum_k (\boldsymbol{V}_{\mathrm{L}}^{(d)})_{ik} b_{\mathrm{L}}^{k},$$

for the left-chiral fields of *up*- and *down*-quarks, with analogous formulae for the right-chiral fields. The weak interaction states follow by inverting these relations,

$$u_{\mathrm{L}}^{(i)} = \sum_k (\boldsymbol{V}_{\mathrm{L}}^{(u)\dagger})_{ik} t_{\mathrm{L}}^{k}, \quad d_{\mathrm{L}}^{(i)} = \sum_k (\boldsymbol{V}_{\mathrm{L}}^{(d)\dagger})_{ik} b_{\mathrm{L}}^{k},$$

and analogous formulae for the right-chiral fields. The CC vertices of weak interactions which are read off from (3.184a), involve only left-chiral fields. They are proportional to $\overline{u_{\mathrm{L}}^{(i)}} d_{\mathrm{L}}^{(k)}$ and hence are multiplied by $\boldsymbol{V}_{\mathrm{L}}^{(\mathrm{u})}\boldsymbol{V}_{\mathrm{L}}^{(\mathrm{d})\dagger}$ when transformed to the basis of mass eigenstates. Thus, the CKM mixing matrix is given by the formula

$$\boldsymbol{V}_{\mathrm{L}}^{(\mathrm{u})}\boldsymbol{V}_{\mathrm{L}}^{(\mathrm{d})\dagger} = \boldsymbol{V}_{\mathrm{CKM}}. \tag{3.198}$$

In this minimal version of the model all the complexity of the mixing matrix stems from the mass matrices. In particular, the *up*- and *down*-matrices should not be proportional to each other since in that case the CKM matrix, by (3.198), would be unity, there would be no mixing. Another most remarkable feature of this analysis is the fact that the right-chiral fields are not observable and, hence, may be chosen at will. This is due to the empirical observation that CC weak interactions couple exclusively to left-chiral fields.

Remarks:

1. The most general transformation of the mass matrices which leaves the mass eigenvalues and the mixing matrix V_{CKM}, eq. (3.198), invariant is

$$\mathbf{U}^{\dagger}\mathbf{M}^{(u)}\mathbf{W}^{(u)} \quad \text{and} \quad \mathbf{U}^{\dagger}\mathbf{M}^{(d)}\mathbf{W}^{(d)},$$

   where $\mathbf{U}$, $\mathbf{W}^{(u)}$, and $\mathbf{W}^{(d)}$ are arbitrary unitary 3×3-matrices. Note, that $\mathbf{U}$ which acts on *left*-chiral fields must be the same for *up*- and *down*-quark states, whereas $\mathbf{W}^{(u)}$ and $\mathbf{W}^{(d)}$ which act on the (unobservable) *right*-chiral fields, are independent. As a consequence, the state mixing can be shifted entirely to one of the two charge sectors, leaving the other one diagonal. This is conventionally done by assuming the weak eigenstates of *up*-quarks to coincide with their mass eigenstates, and to shift mixing to the *down*-quark sector only. Clearly, one may as well shift all mixing to the *up*-sector, or choose bases such that mixing is distributed over the two charge sectors.

2. Once the mass matrices in (3.197) are given, the procedure outlined above yields the CKM matrix (3.198) in a straightforward manner, up to the freedom in choosing bases for right-chiral fields. This freedom explains why there are many, in fact infinitely many, equivalent parametrisations for V_{CKM}. The inverse problem is more complicated: Suppose the four observables in V_{CKM} and the six mass eigenvalues are given. Can one reconstruct the mass matrices in the two charge sectors and, if so, what is the remaining freedom? It turns out that one can control the space of mass matrices which are admissible in the sense that they all yield the same eigenvalues and mixing pattern[7].

3.4.7 *A Comment About Fermion Multiplets in a Unified Gauge Theory of Electroweak Interactions*

(a) Fermion multiplets in the language of SL (2,$\mathbb{C}$*) spinors*
As we have noted in the introduction to Sect. 3.4 and in Sect.3.4.1, the GSW theory is built on the assumptions that CC interactions (mediated by $\mathrm{W}^{\pm}$ bosons) are fully left-handed and that physical neutrinos carry negative helicity $h(\nu) = -1$. As a consequence the basic fermion fields of the theory are van der Waerden spinor fields. This is seen, for instance, in the leptonic field operator (3.133) which contains two spinor fields of the second kind, one spinor field of the first kind. Thus, the natural language in formulating unified gauge theories that contain the weak interactions is the one of SL(2, $\mathbb{C}$) spinors developed in Chap. 1.

For the example of one lepton family (3.133) we should introduce a doublet of spinors of the second kind

$$\chi^{(m)A}(x), \qquad m = +\frac{1}{2}, -\frac{1}{2},$$

describing the neutrino $m = +\frac{1}{2}$ and the left-handed part of the charged fermion field $m = -\frac{1}{2}$, as well as a spinor of the first kind, $\phi_a(x)$, which is a singlet with respect to SU(2) and which describes the right-handed part of the charged fermion field. In terms of these spinors the generalized kinetic term reads

$$\frac{\mathrm{i}}{2}(\overline{\Psi(x)}, \gamma^{\alpha} \overleftrightarrow{D}_{\alpha} \Psi(x)) = \frac{\mathrm{i}}{2} \left\{ \sum_{m} \chi^{(m)*a} (\sigma^{\alpha})_{aB} \overleftrightarrow{D}_{\alpha} \chi^{(m)B} + \phi^*{}_A (\hat{\sigma}^{\alpha})^{Ab} \overleftrightarrow{D}_{\alpha} \phi_b \right\},$$

which contains, in fact, the genuine kinetic energy and the interaction with the gauge fields. D_{α} is the covariant derivative (3.117), as before.

The more standard formulation in terms of Dirac fields that we used in Sects. 3.4.1–3.4.4, is completely equivalent, of course. However, the formula-

[7] An efficient solution is given in S. Falk et al., Phys. Rev. **D65** (2002) 093011-1, including reference to the earlier literature on this topic.

tion in terms of van der Waerden spinors is very useful for the extension to supersymmetries.

(b) Triangle anomaly and renormalizability of the GSW theory
We mention briefly a theoretical and somewhat technical point which is crucial for the renormalizability of the GSW theory and which sheds some light on the empirical symmetry between the leptons and quark families: the so-called triangle anomaly of the axial current (Adler 1969). One can show that the axial vector part

$$a_\alpha^{(0)}(x) := \sum_{\mathrm{f}=\mathrm{e},\mu,\tau} \overline{\Psi_{\mathrm{f}}(x)}\gamma_\alpha\gamma_5 V(Y)\Psi_{\mathrm{f}}(x) + \sum_{q=1}^{3}\sum_{c=1}^{3} \overline{\Psi_{q,c}(x)}\gamma_\alpha\gamma_5 V(Y)\Psi_{q,c}(x) \tag{3.199}$$

of the current that couples to the gauge field $A_\alpha^{(0)}(x)$ has an anomalous term in its divergence $\partial^\alpha a_\alpha^{(0)}(x)$ which is proportional to $S = S_{\mathrm{leptons}} + S_{\mathrm{quarks}}$ with

$$S_{\mathrm{leptons}} = \sum_{\mathrm{e},\mu,\tau} \mathrm{tr}\{V(T_m T_m Y)\},\ S_{\mathrm{quarks}} = \sum_{q,c} \mathrm{tr}\{V(T_m T_m Y)\}.$$

m is a component of weak isospin. [It is to be contracted with bilinear and trilinear products of vector fields $A_\alpha^{(m)}(x)$.] The sums in (3.199) run over the three lepton families and over the three, threefold degenerate quark families. However, as the traces contain the weak isospin, only the *doublets* within the multiplets (3.133) and (3.153) contribute to the anomaly. It suffices to consider the case $m = 3$. From (3.144, 3.154) we find

$$S_{\mathrm{leptons}} = \frac{1}{4}\cdot 3\times(-2) = -\frac{3}{2}, \quad S_{\mathrm{quarks}} = \frac{1}{4}\cdot 3\times 3\cdot\frac{2}{3} = +\frac{3}{2},$$

so that their sum vanishes. In other words, the anomaly due to the quark fields cancels the anomaly due to the leptons provided the quarks have the additional colour degree of freedom.

The renormalizability of a local gauge theory rests on its internal symmetry, on current conservation and on its specific Ward identities. An anomaly in the divergence of a current that couples to a gauge boson would destroy renormalizability. The minimal model avoids this disaster by arranging its fermion multiplets such as to cancel the anomaly.

There are further chiral anomalies which may appear in local gauge theories, depending on the structure group on which these are built. Such anomalies are called chiral because they emerge in one-loop diagrams which involve fermionic states of definite chirality. The anomaly mentioned above, historically the first that was discovered, is particularly interesting because its hadronic part, i.e. the second term on the right-hand side of (3.199), is responsible for the decay $\pi^0 \to \gamma\gamma$. Indeed, up

to corrections of higher order, the axial quark current of (3.199) yields the correct amplitude for this decay. We return to this application in Sect. 4.2.5b.

3.5 Quantum Chromodynamics

Quantum chromodynamics is the local gauge theory which is obtained by gauging the colour group $SU(3)_c$ that we introduced in (3.16). Taking $SU(3)_c$ to be the structure group we construct a fully gauge invariant Lagrangian following the prescriptions developed in Sect. 3.3. Curiously enough the construction of this theory is considerably simpler than the construction of the electroweak Lagrangian. Yet, its interpretation in terms of observable physics is considerably more difficult because it is a highly nontrivial matter to make contact between the elementary fields in terms of which the Lagrangian is written (i.e. quarks and gluons), on one hand, and the physical hadrons and their interactions, on the other. In this section we derive and describe the Lagrangian of quantum chromodynamics (QCD) and discuss its use in describing the physics of strong interactions.

3.5.1 Construction of the Lagrangian

In the defining, three-dimensional representation an element u of SU(3) is written as an exponential series (3.102), i.e. in terms of eight linearly independent, traceless, and hermitean elements of the Lie algebra *Lie*(SU(3)),

$$h_k = \frac{1}{2}\lambda_k, \quad k = 1, 2, \ldots, 8.$$

The matrices λ_k are called Gell-Mann matrices and are given in Appendix D. The dimension of the Lie algebra and the rank of SU(3) are, respectively,

$$\dim \mathrm{SU}(3) = 8 \quad \mathrm{rank}\, \mathrm{SU}(3) = 2.$$

This is obvious from the explicit choice (D.2) of the matrices λ_k: Every 3×3 matrix which is traceless and hermitean can be written as a linear combination of $\lambda_1, \ldots, \lambda_8$. Two of these, λ_3 and λ_8, are simultaneously diagonal. Using the defining representation one verifies the trace formula

$$\mathrm{tr}(T_i T_k) = \frac{1}{4}\mathrm{tr}(\lambda_i \lambda_k) = \frac{1}{2}\delta_{ik}. \tag{3.200}$$

The fundamental representation, its conjugate, the adjoint representation as well as the invariant tensors of SU(3) are found in Appendix D.

There are four major reasons why quarks should carry a colour degree of freedom, cf. (3.16), the first of which was already discussed in Sect. 3.1. They are

(A) *The problem of spin and statistics for baryons*. As shown in Sect. 3.1, by requesting all baryons to be in singlet states with respect to the structure group $SU(3)_c$ of colour, it becomes possible, in the most natural way, to construct wave functions for baryons from quarks obeying ordinary Fermi-Dirac statistics.

(B) *The decay amplitude* $\pi^0 \to \gamma\gamma$ is enhanced by a factor of 3 if every quark state comes in three colours. The decay rate as observed in experiment supports that factor, cf. Sect. 4.2.5b.

(C) *The ratio R of hadron to muon production in* e^+e^- *collisions*,

$$R := \frac{\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})}{\sigma_{\text{tot}}(e^+e^- \to \mu^+\mu^-)}, \tag{3.201}$$

at any given energy, provides a direct measurement of the number and squared charges of quark fields which can be produced up to that energy. In the approximation of one-photon exchange, the total cross section for a fermionic particle and its antiparticle is easily calculated. For instance, for the production of a muon pair one finds

$$\sigma_{\text{tot}}(e^+e^- \to \mu^+\mu^-) \approx \frac{4\pi\alpha^2}{3}\frac{Q^2(\mu)}{s}$$

with $Q(\mu) = -1$ the charge of the muon, s the square of the total energy in the centre-of-mass system. (The approximate sign refers to the fact that we consider electron-positron collisions at energies large compared to their rest mass.) This formula is obtained from the expression (3.240) below for the differential cross section $d\sigma/d\Omega^*$, by omitting all terms proportional to k, and by integrating over the solid angle, viz.

$$\sigma_{\text{tot}} = 2\pi \int_{-1}^{+1} d(\cos\theta^*)\frac{\alpha^2}{4s}(1 + \cos^2\theta^*) = \frac{4\pi\alpha^2}{3s}.$$

The production of hadronic final states is thought to occur in two stages: first, production of a pair of quark and antiquark by one-photon exchange, $e^+e^- \to (\gamma) \to q\bar{q}$, then formation of hadronic, colour-neutral, final states through "dressing" of the quark and antiquark lines by a sufficient number of additional $q\bar{q}$ states. Independently of this hadronic final state interaction the energy dependence of the total cross section is expected to be given by $1/s$, in the same way as for muons. Its magnitude should be proportional to the sum of (primordial) quark-antiquark pairs which can be produced at the given energy, each term multiplied with the square of the quark's charge $Q(q)$. Although this model is rough because it neglects higher-order corrections and ignores what happens in the final state for physical hadrons to emerge in the end, it gives a valid indication for the ratio (3.201). For fixed value of s the ratio is approximately

$$R(s) \approx \sum_{A<A(s)} \sum_{k=1}^{3} Q^2(A) = 3 \sum_{A<A(s)} Q^2(A), \tag{3.202}$$

where the sum over the *flavours* A, (u, d, s, c, b, t) in the order of increasing mass, extends only over those species of quarks which can be produced at the given value of the energy. For example, above the threshold for $s\bar{s}$ production but below $c\bar{c}$ threshold, it should be

$$3\left[\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2\right]=2$$

for A = u, d, s, while beyond the $c\bar{c}$ threshold it should be $10/3$. Indeed, both the stepwise increase from below the kinematic threshold for production of a given quark–antiquark pair to just above that threshold, and the colour factor 3 in (3.202) are confirmed by electron–positron annihilation into hadrons [RPP10].

Note that this is a refinement of the analysis described in Subsect. 2.8.4, eq. (2.212), by the colour quantum number.

The electric charge of each quark (in terms of the elementary charge e) is related to its eigenvalues of the 3-component of isospin and the other additive quantum numbers by the formula

$$Q = I_3 + \frac{1}{2}(B + S + C + Bo + To).$$

This is akin to the formula (3.145) and, historically, goes back to the relation $Q = I_3 + Y/2$ with $Y = B + S$ relating the *strong interaction* isospin and hypercharge to the electric charge. In the context of SU(3), the "eightfold way", it was called Gell-Mann Nishina formula.

(D) *Asymptotic freedom and scaling*. Perhaps the most interesting consequence of the introduction of colour as a new degree of freedom and of gauging the structure group $SU(3)_c$ is the following prediction: The effective strong interaction coupling constant $\alpha_s(\mu)$ which is characteristic for the energy scale μ, tends to zero as the energy scale μ becomes large. This implies that strong interaction processes, at large energy scales, are characterized by a small effective coupling constant and, hence, may be amenable to a treatment within perturbation theory. This asymptotic behaviour, in turn, explains certain scaling properties of nucleonic structure functions which are observed experimentally.

Experiment seems to tell us that all physical hadron states are singlets with respect to colour $SU(3)_c$. While the rich spectrum of mesons and baryons reflects in a beautifully simple pattern the *flavours* of quarks, no open colour has ever been seen. Thus, with $A, B, C, \ldots$ denoting *flavour*, $i, j, k, \ldots$ denoting *colour* degrees of freedom, meson and baryon states must be linear combinations of the colour singlets, respectively,

$$\sum_{i=1}^{3}\overline{\mathrm{q}^{A,i}}\mathrm{q}^{B,i}, \qquad \sum_{i,j,k=1}^{3}\varepsilon_{ijk}\mathrm{q}^{A,i}\mathrm{q}^{B,j}\mathrm{q}^{C,k}, \tag{3.203}$$

the flavour quantum numbers being combined to their flavour wave functions. For example, in the good old world of light quarks u, d, s these are multiplets of another $SU(3)_f$, the flavour group postulated long ago by Gell-Mann and Ne'eman. In this case, more specifically, they are nonets $\bar{3} \times 3 = 1 + 8$ for mesons, and singlets, octets, and decuplets $3 \times 3 \times 3 = 1 + 8 + 8 + 10$ for baryons as observed in the spectroscopy of light mesons and baryons. With the c-quark included, the flavour group can be $SU(4)_f$, while the b-quark would extend it to $SU(5)_f$. At the level of the quark interactions the flavour group is no good symmetry and, therefore, its details are of no importance for the dynamics. It merely serves to classify meson and baryon states at rest. Knowing that (most) mesons are quark–antiquark states and that (most) baryons are three-quark states, one can construct these states, in a rather elementary way, from quarks and/or antiquarks such as to obtain the correct spin and internal quantum numbers.

For the sake of reference we summarize the additive flavour quantum numbers of the six known quarks in the following table

	u	d	c	s	t	b
Q	2/3	−1/3	2/3	−1/3	2/3	−1/3
I_3	1/2	−1/2	0	0	0	0
S	0	0	0	−1	0	0
C	0	0	+1	0	0	0
Bo	0	0	0	0	0	−1
To	0	0	0	0	+1	0

All quarks have baryon number $B = 1/3$. Q denotes their electric charge, I_3 the 3-component of (strong) isospin, S the strangeness, C the charm, *Bo* the bottomness (or beauty), *To* the topness (also called truth).

While the flavour quantum numbers are relevant for the spectroscopy of mesons and baryons, the colour group $SU(3)_c$ is the basis of the dynamics of strong interactions. Indeed, strong interactions are described by the local gauge theory which is built on the structure group $SU(3)_c$. Unlike electroweak interactions this gauge theory remains unbroken. Furthermore, it acts on quark fields only, not on the lepton fields which, therefore, are said to be "colour-blind".

It is not difficult to write down this gauge theory by following the general principles developed in Sect. 3.3.5. The building blocks are

(i) the vector potential (3.112), written in abstract notation

$$A_x(x) = \mathrm{i}g_s \sum_{k=1}^{8} T_k A_\alpha^{(k)}(x).$$

When this potential acts in the representation space of the quark triplet the generators are given by $U_{(3\times3)}(T_k) = \lambda_k/2$. The bosonic sector of the

Lagrangian, on the other hand, needs the adjoint representation $U_{(8\times 8)}(T_k) =: L_k$ which is given in terms of the structure constants: $(L_k)_{mn} = -\mathrm{i} f_{kmn}$;

(ii) The covariant derivative (3.117) expressed in the representation space of the quark triplet, which reads

$$\begin{aligned}[D_x(A)]_{Bm,Cn} &= [\mathbb{1}\partial_\alpha + U_{(3\times 3)}(A_\alpha)]_{Bm,Cn} \\ &= \left(\delta_{mn}\partial_\alpha + \mathrm{i}g_s \sum_{k=1}^{8} (\lambda_k/2)_{mn} A_\alpha^{(k)}(x)\right) \delta_{BC},\end{aligned}$$

(iii) The field strength tensor (3.119) in abstract notation, which is given by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] \equiv \mathrm{i}g_s \sum_{k=1}^{8} T_k F_{\alpha\beta}^{(k)}(x),$$

with the tensor fields on the right-hand side given by

$$F_{\alpha\beta}^{(k)}(x) = \partial_\alpha A_\beta^{(k)}(x) - \partial_\beta A_\alpha^{(k)}(x) - g_s \sum_{m,n=1}^{8} f_{kmn} A_\alpha^{(m)}(x) A_\beta^{(n)}(x). \quad (3.204)$$

Here we denote the coupling constant by g_s instead of e.

With the experience of Sect. 3.3.5 we can write down at once a gauge invariant Lagrangian based on the colour group $\mathrm{SU}(3)_\mathrm{c}$. From (3.200), with g_s instead of e, and with $k = 1/2$ we find

$$\mathscr{L}_\mathrm{QCD} = -\frac{1}{4} \sum_{k=1}^{8} F_{\alpha\beta}^{(k)}(x) F^{(k)\alpha\beta}(x) + \frac{\mathrm{i}}{2} \sum_B \sum_{m,n} \overline{q^{B,m}(x)} \gamma^\alpha (\overleftrightarrow{D}_\alpha(A))_{mn} q^{B,n}(x), \quad (3.205)$$

with $D_\alpha(A)$ as given above, the left-right arrow reminding us that this operator acts on the right and on the left (consult list of symbols if you have forgotten the precise definition.)

3.5.2 Discussion of QCD Lagrangian

Quantum chromodynamics, which was proposed for the description of strong interactions in the early nineteen-seventies, has become a wide field of research in elementary particle physics. A comprehensive discussion of this topic would fill a monograph of its own and, therefore, goes far beyond the scope of this book. At first sight, looking back at the simplicity of the underlying Lagrangian (3.205) and comparing to the complexity of electroweak interactions, this may seem somewhat surprising. The following comments are meant to explain why this is so and, at the same

time, to awaken the reader's curiosity to learn more about the intricacies in interpreting the QCD Lagrangian and about the difficulties in applying it to the observable physics of hadrons. We group these comments into a number of remarks as follows.

(i) *Gluons*: As it stands the Lagrangian (3.205) describes eight massless vector bosons, called *gluons*, and a number of massless quarks $B = u, d \ldots$ each of which comes in three different states $m = 1, 2, 3$ called *colours*. The gluons are photon-like particles which means that if it were possible to produce them as *free* particles they would be massless and, like the photon, would come in two, not three, helicity states. They are flavour-neutral but, in contrast to the photon, they carry nonvanishing colour $k = 1, 2, \ldots, 8$. The masslessness of gluons might lead one to conclude that the interaction they mediate should be of infinite range (recall the $1/r$ dependence of the Coulomb potential which follows from the exchange of photons)–in sharp contrast to the empirical finding that strong interactions are very short-ranged, typical ranges being of the order of $\lambda\!\!\!^{-}_{\pi} = \hbar/m_\pi c \approx 1.41\,\text{fm}$, the Compton wavelength of the pion, or less. This paradox is resolved by observing that QCD is a theory which shows the phenomenon of *confinement*: The elementary excitations of the theory described by the Lagrangian (3.205) are very different from the fields it contains. All physical states are such that all colour degrees of freedom are coupled to the singlet, the trivial representation of $\mathrm{SU}(3)_\mathrm{c}$. Even in a world containing only gluons, the physical hadron states are composite, two or more gluon states, called *glueballs*, in which the colour quantum numbers are saturated to form singlets. The observable interactions between such glueball states may be thought of as analogues of Van der Waals forces between electrically neutral atoms or molecules. Van der Waals forces which are due to mutual polarization of electrically neutral, composite systems, are known to be of much shorter range than the Coulomb force. Therefore, even though the dynamics of the underlying gauge theory is different in the two cases, this analogy renders plausible the short range nature of the effective force that acts between colour- neutral, composite systems.

(ii) *Quark sector and chiral symmetry*: A free one-particle state of any quark field $q^{B,m}(x)$ would carry not only the flavour quantum numbers B of that species, hence, third-integral electric charge, but necessarily also non-vanishing colour m. Again, like in the case of gluons, no states with open colour seem to exist. By the mechanism of *confinement* quarks and antiquarks are bound to the colour-neutral states (3.203) of mesons and baryons. Therefore, it is at once plausible that the interactions of physical hadrons are very different from the interactions between isolated quarks as described by the simple gauge theory (3.205).

The theory described by the Lagrangian (3.205) still possesses a high degree of (global) symmetry. In order to simplify the discussion let us consider the first two flavours u and d only. The extension to other flavours will be evident at the end. It is easy to verify that the following vector and axial vector currents are conserved,

$$\begin{aligned} \upsilon_\alpha^{(k)} &:= \sum_{B,C} \sum_m \overline{q^{B,m}(x)} \gamma_\alpha (\tau_k)_{BC} q^{C,m}(x), \\ a_\alpha^{(k)} &:= \sum_{B,C} \sum_m \overline{q^{B,m}(x)} \gamma_\alpha \gamma_5 (\tau_k)_{BC} q^{C,m}(x), \end{aligned} \tag{3.206}$$

or, equivalently, that

$$\upsilon_\alpha^{(k)} \pm a_\alpha^{(k)} = \sum_{B,C} \sum_m \overline{q_{R/L}^{B,m}(x)} \gamma_\alpha(\tau_k)_{BC} q_{R/L}^{C,m}(x), \tag{3.206'}$$

are conserved. If this were a genuine global symmetry of QCD the charges corresponding to the currents (3.206) or (3.206') would be conserved. Starting from the currents in the helicity basis (3.206'), it is easy to verify that the symmetry group would be $\mathrm{SU}(2)\times\mathrm{SU}(2)$, the first of these corresponding to $\upsilon_\alpha^{(k)} + a_\alpha^{(k)}$, the second to $\upsilon_\alpha^{(k)} - a_\alpha^{(k)}$. (Had we included the next flavour s this would be $\mathrm{SU}(3)\times\mathrm{SU}(3)$, and so on). This symmetry is called *chiral symmetry*. In reality quarks are not really massless and chiral symmetry is not an exact symmetry of the theory. In fact, it is not so easy to give a precise meaning to a quark mass term because quarks cannot be produced as free particles and, hence, their mass cannot be measured by kinematics or by way of a propagator. Perhaps the best answer to this question is to say that quark mass terms, which are thought to be due to electroweak interactions, are merely symmetry breaking parameters: their magnitude is a measure for the extent to which the chiral symmetry is broken. The numbers quoted in the literature, say, typically, $m_\mathrm{u} = 2.5\,\mathrm{MeV}, m_\mathrm{d} = 5\,\mathrm{MeV}$, are also called *current quark masses*, for historical reasons, and are very different from what are called *constituent quark masses*. The latter would be the masses in a naive bound state model of hadrons where a proton or neutron is made up of three quarks with weak binding forces so that the constituent mass would be approximately $m_\mathrm{p}/3$, hence of the order of 300 MeV.

(iii) *The QCD ground state*: While it is not difficult to formally introduce quark mass matrices into the Lagrangian (3.205), cf. Sect. 3.4.6. above, a most important observation is that the ground state of QCD is very different from the vacuum of perturbation theory and that perturbation theory based on (3.205) is not of much use for the analysis of the confinement regime. All indications are that the true ground state is a highly correlated state which contains interesting internal structure and that this state is close to a limit of exact chiral symmetry. In fact, the closeness of strong interactions at low energy to a chirally symmetric situation was known long before the development of QCD, one of the most prominent indications for it being the existence of the pion which is so much lighter than all other hadrons. Therefore, nonperturbative methods which can deal with QCD in the confinement regime must meet a formidable challenge: they must cope with the spectrum of hadronic resonances and the approximate chiral symmetry which are the salient features of strong interactions at low energies.

(iv) *Asymptotics, or how to make quarks and gluons visible*: At large energy scales, i.e. in processes involving large momentum transfers, it becomes possible to factorize strong interactions into short-range and long-range contributions, and thereby to separate the regime of large momentum transfer from the regime of small momentum transfer. Due to the decrease of the effective coupling constant $\alpha_\mathrm{s}(\mu)$ with increasing energy scale μ, the former can be calculated by means of covariant perturbation theory as applied to the Lagrangian (3.205), to one-loop, two-loop, or

higher orders. The latter concerns the fragmentation of the hadrons in the initial state into quarks and gluons as well as the reconstruction of final hadronic states, called hadronization, from quark and gluon lines. This part of the physical reaction amplitude is of non-perturbative nature and must be dealt with by parametrization and fixing of parameters by experiment. With this caveat in mind, QCD processes at high energies and large momentum transfers do indeed test the physics of quarks and gluons as described by the simple Lagrangian (3.205). It is in this sense that the existence of three and four-gluon vertices which are predicted by the first term on the right-hand side of (3.205), can be tested experimentally. For example, a vertex with four gluons, one of which is incoming and three of which are outgoing, will give rise to an event with three well-defined, outgoing *jets* of hadronic particles. This means that the three gluons produced in the final state will be "dressed" to bunches of physical hadrons through addition of quark-antiquark pairs as well as further gluons. The "soft", long-range part of the processes cannot be calculated as easily. It introduces a certain amount of model dependence and, hence, puts a limit on the accuracy to which higher- order perturbation theory applied to the short-range part can be tested. In spite of this limitation of principle, quantum chromodynamics has convincingly passed all experimental tests at high energies and, after many years of intense investigation we have a good phenomenological basis for our conviction that it is indeed the correct theory of strong interactions.

3.6 Simple Applications of the GSW Model at Energies Below the Vector Boson Masses

In this section we study three simple examples which illustrate the specific predictions of the GSW theory for reactions due to Z^0 exchange and, in particular, to the interference between the exchange of a virtual Z^0-boson and of a photon. In the terminology introduced earlier these examples concern NC interactions for which the GSW theory gives new and specific predictions. The CC interactions of this model are the same as in the older, effective, theory. They are dealt with, in a more general framework, in Chap. 4.

3.6.1 Scattering of Longitudinally Polarized Electrons from a Nucleus with Spin Zero

In Sect. 2.2 we have learnt that the two helicity states of a fast electron (2.17) are scattered in exactly the same way by a target with spin zero, provided the interaction is invariant under rotations *and* under space reflection. This is indeed the case when we consider the electromagnetic interaction only (as we did in Chap. 2).

The interaction Lagrangian (3.183) of the GSW model contains neutral interactions which are not invariant under parity. This is due to the simultaneous presence of vector and axial vector couplings of the fermion fields to the Z^0 -boson. Thus, very much like for the case of the CC weak interactions, one expects to find

manifestations of parity violation in purely weak NC reactions, such as neutrino scattering on leptonic or hadronic targets. In addition, amplitudes due to exchange of a virtual Z^0 can interfere with amplitudes due to exchange of a photon. Z^0 -exchange gives rise to effective VV, VA, AV, and AA couplings whilst one-photon exchange gives rise to VV couplings only. As the VA and AV terms are pseudoscalars while the VV and AA terms are scalars, the γ–Z^0 interference must also lead to observable, parity violating effects.

Taking into account the interference with the weak NC interactions, the scattering amplitudes f and g for the two helicity states of the incident electron are not equal anymore. They differ precisely by the new parity-odd VA and AV terms. Thus, the *difference* of the differential cross sections $d\sigma_+$ and $d\sigma_-$ for electrons with positive and negative helicity, respectively, is a direct measure of the interference of vector and axial vector amplitudes. This is what we now wish to work out in a simple case: scattering on a nucleus with spin zero, using the first Born approximation.

The quantity of interest is the asymmetry in the cross sections

$$A := \frac{d\sigma_+/d\Omega - d\sigma_-/d\Omega}{d\sigma_+/d\Omega + d\sigma_-/d\Omega} = \frac{|f|^2 - |g|^2}{|f|^2 + |g|^2}. \tag{3.207}$$

In computing the scattering amplitudes due to one-photon and to one-Z^0 exchange we need the nucleonic and nuclear matrix elements of the weak neutral current K_α, (3.184b). This is complicated by the fact that nucleons are composite states and that hadronic matrix elements are modified by the strong interactions. Let us denote the vector currents by υ_α, the axial vector currents by a_α. The proton contains two u-quarks and one d-quark. Thus by simply counting the quarks, the matrix element of K_α, (3.184b), between two proton states with momenta p and p' is given by

$$\langle p'|K_\alpha(0)|p\rangle = (1 - 4\sin^2\theta_w)\langle p'|\upsilon_\alpha(0)|p\rangle - \langle p'|a_\alpha(0)|p\rangle. \tag{3.208a}$$

Similarly, a neutron contains one u-quark and two d-quarks and the matrix element of K_α between two neutron states with momenta n and n' is given by

$$\langle n'|K_\alpha(0)|n\rangle = -\langle n'|\upsilon_\alpha(0)|n\rangle + \langle n'|a_\alpha(0)|n\rangle. \tag{3.208b}$$

The vector matrix elements in (3.208a,b) have the same general decomposition in terms of Lorentz invariant form factors as the electromagnetic current (2.46). In particular, if υ_α is conserved (this is indeed the case from (3.184a), the third form factor F_3 must vanish identically. The matrix elements of a_α may be decomposed, in a similar fashion, in terms of axial vector covariants, viz.

$$\langle p'|a_\alpha(0)|p\rangle = \frac{1}{(2\pi)^3}\overline{u(p')}\left\{\gamma_\alpha\gamma_5 F_A(q^2) + \frac{1}{2m}q_\alpha\gamma_5 F_p(q^2) + \frac{i}{2m}\sigma_{\alpha\beta}q^\beta\gamma_5 F_T(q^2)\right\}u(p). \tag{3.209}$$

[The third term in the curly brackets can be shown to vanish if the strong interactions are invariant under G-parity, i.e. $F_{\mathrm{T}}(q^2) \equiv 0$.]

Let us now consider scattering of fast electrons on a nucleus with spin zero. Let k, k' be the four-momenta of the electron before and after the scattering, let p, p' be the corresponding momenta of the nucleus and let

$$q : k - k' = p' - p$$

be the momentum transfer. The one-photon exchange amplitude is given by

$$T_\gamma = \frac{e^2}{(2\pi)^3} \overline{u_e(k')} \gamma^\alpha u_{\mathrm{e}}(k) \frac{-g_{\alpha\beta}}{q^2} \langle p' | j^\beta_{\mathrm{e.m.}}(0) | p \rangle, \tag{3.210}$$

and the one Z^0-exchange amplitude is

$$T_Z = -\frac{e^2}{(2\pi)^3} \frac{1}{16 \sin^2\theta_w \cos^2\theta_w} \overline{u_e(k')} \{-\gamma^\alpha (1 - 4\sin^2\theta_w) + \gamma^\alpha \gamma_5\} u_e(k) \times \frac{-g_{\alpha\beta} + q_\alpha q_\beta / m_Z^2}{q^2 - m_Z^2} \langle p' | K^\beta(0) | p \rangle. \tag{3.211}$$

The nuclear matrix element of the electromagnetic current has the form

$$\langle p' | j^\beta_{\mathrm{e.m.}}(0) | p \rangle = \frac{1}{(2\pi)^3} (p + p')^\beta Z F(q^2), \tag{3.212}$$

where F is the electric form factor of the nucleus and is normalized to $F(0) = 1$. Regarding the nuclear matrix element of K_α we note the following: Due to angular momentum and parity conservation in the hadronic vertex, the nucleonic axial currents a_α do not contribute to elastic scattering, $\langle p' | a_\alpha | p \rangle = 0$. The vector current of the protons is proportional to the electromagnetic current. Therefore, its *nuclear* matrix element is proportional to the electric form factor $F(q^2)$. If we assume the neutron density to be the same as the proton density, $\rho_{\mathrm{n}}(r) = \rho_{\mathrm{p}}(r)$, then the matrix element of the vector current due to neutrons is also proportional to that same form factor. In this case we have

$$\langle p' | K_\beta(0) | p \rangle = \frac{1}{(2\pi)^3} (p + p')^\beta \{Z(1 - 4\sin^2\theta_w) - N\} F(q^2), \tag{3.213}$$

where N is the neutron number.

It is easy to see that the terms $q_\alpha q_\beta$ of the Z^0-propagator in (3.211) do not contribute. Thus both the photon and the Z^0 exchange give an effective four-fermion coupling, the first multiplied with the photon propagator $1/q^2$, the second with the factor $1/(q^2 - m_{\mathrm{Z}}^2)$.

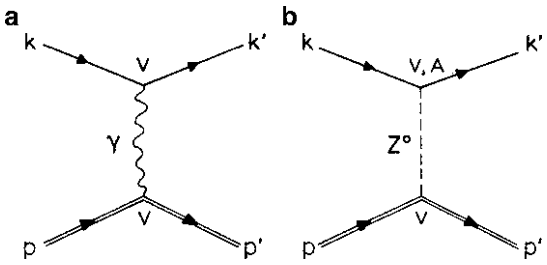

Fig. 3.6 (**a**) One-photon exchange in electron-nucleus scattering. (**b**) Exchange of a Z^0-boson between electron and nucleus

The two amplitudes (3.210, 3.211) are depicted in Figs. 3.6a and b. The leptonic vector coupling in Fig. 3.6b gives a term which is analogous to the one-photon exchange and whose magnitude depends on the squared momentum transfer. Clearly, at intermediate energies $|q^2|$ is very small as compared to m_Z^2 and this term is then negligible.

Using the following abbreviations:

$$\upsilon(q^2) := 1 + \frac{(1 - 4\sin^2\theta_w)[(1 - 4\sin^2\theta_w) - N/Z]}{16\sin^2\theta_w \cos^2\theta_w} \frac{q^2}{q^2 - m_Z^2}, \tag{3.214a}$$

$$a(q^2) := -\frac{[(1 - 4\sin^2\theta_w) - N/Z]q^2}{16\sin^2\theta_w \cos^2\theta_w (q^2 - m_Z^2)} \Big/ \upsilon(q^2), \tag{3.214b}$$

and writing the leptonic vectors and axial vectors in terms of $\gamma^\alpha P_+$ and $\gamma^\alpha P_-$, with $P_\pm$ as defined in (1.76), we then have

$$T_\gamma + T_Z = -\frac{e^2}{(2\pi)^6} \frac{Ze^2}{q^2} F(q^2)\upsilon(q^2)(p + p')_\alpha \overline{u(k')}\, \gamma^\alpha \{(1 + a(q^2))\, P_+ + (1 - a(q^2))P_-\} u(k). \tag{3.215}$$

It is easy to calculate the differential cross section from the expression (3.215) and, in fact, on the basis of experience in Chap. 2 we guess that we will obtain the Mott cross section (2.27), multiplied by $(1 \pm a(q^2))^2$ for positive and negative helicity, respectively. For the calculation of the asymmetry (3.207), equation (3.215) is entirely sufficient as it shows that

$$d\sigma_+ \propto (1 + a(q^2))^2 \qquad \text{and} \qquad d\sigma_- \propto (1 - a(q^2))^2.$$

Thus we find

$$A = \frac{2a(q^2)}{1 + a^2(q^2)}. \tag{3.216}$$

At intermediate energies of a few hundred MeV q^2 is small as compared to m_Z^2, so that

$$A \simeq 2a \simeq \frac{1 - 4\sin^2\theta_w - N/Z}{8\sin^2\theta_w \cos^2\theta_w} \frac{q^2}{m_Z^2}, \tag{3.217}$$

or, with $q^2 \simeq -4E^2 \sin^2\theta/2$,

$$A \simeq \frac{N/Z - (1 - 4\sin^2\theta_w)}{2\sin^2\theta_w \cos^2\theta_w} \frac{E^2}{m_Z^2} \sin^2\theta/2. \tag{3.217'}$$

Assume $E = 500\,\text{MeV}$, $m_Z \simeq 92\,\text{GeV}$, $\sin^2\theta_w = 0.232$, then $A \simeq (N/Z)8.4 \times 10^{-5} \sin^2\theta/2$. This asymmetry is small but may well be measurable. Obviously, it is of interest to choose a target with a large neutron excess.

As it is evident from our derivation this asymmetry is a direct measure for the leptonic axial vector times hadronic vector couplings to the Z^0. Also noteworthy is the fact that the nuclear form factor $F(q^2)$ drops out of the ratio (3.207). This is a special feature of first Born approximation.

The same asymmetry (3.207) may also be calculated for the case of scattering on a proton or a neutron, along the same lines as in the example above. In this case there is not only electric (charge) scattering but also magnetic dipole scattering. For small and intermediate scattering angles the asymmetry is dominated by charge scattering, near the backward direction it is given predominantly by M1 scattering.

3.6.2 Neutrino and Antineutrino Scattering on Electrons

As another example of a clean test of NC interactions we consider the following elastic reactions

$$\nu_\mu + e \to e + \nu_\mu, \tag{3.218a}$$

$$\overline{\nu}_\mu + e \to e + \overline{\nu}_\mu, \tag{3.218b}$$

$$\overline{\nu}_e + e \to e + \overline{\nu}_e. \tag{3.218c}$$

The first two of these are pure NC reactions. They are observable with the neutrino beams of high-energy accelerators. Reaction (3.218c) has contributions from both NC and CC interactions. It is observable in experiments at nuclear reactors which produce intense $\overline{\nu}_e$ beams.

The kinematics is the same as in reactions (3.21) and (3.91) that we discussed in Sects. 3.1.2a and 3.2.4, respectively. Denoting the neutrino momenta before and after the scattering by p and p', those of the electron by q and q', we have in the c.m. system

$$s = (p + q)^2 \simeq 4E^{*2}, \tag{3.219a}$$

$$t = (p - p')^2 = -2E^{*2}(1 - z^*) \simeq -\frac{1}{2}s(1 - z^*), \tag{3.219b}$$

$$u = (p - q')^2 = 2m_e^2 - s - t \simeq -\frac{1}{2}s(1 + z^*). \tag{3.219c}$$

Here E^* is the neutrino energy, z^* is the cosine of the scattering angle in the c.m. system; the $\simeq$ sign refers to our choosing E^* very much larger than m_e so that m_e can be set equal to zero in (3.219). In the laboratory system s is given by (cf. (3.93')

$$s = m_e^2(1 + 2\omega), \qquad \omega := E_\nu^{\text{lab}}/m_e. \tag{3.220}$$

So, for $E_\nu^{\text{lab}} \gg m_e$, $s \simeq 1.022 \times 10^{-3} \cdot E_\nu^{\text{lab}}$ with E_ν^{lab} expressed in GeV. Of relevance is the comparison of s to the W and Z^0 masses:

$$\frac{s}{m_W^2} \simeq (1.48 \times 10^{-7}\,\text{GeV}^{-1})E_\nu^{\text{lab}},$$

$$\frac{s}{m_Z^2} \simeq (1.16 \times 10^{-7}\,\text{GeV}^{-1})E_\nu^{\text{lab}}.$$

The NC contributions to reactions (3.218) contain a denominator $(t - m_Z^2)$ due to the Z^0-propagator, the CC contributions contain a denominator $(s - m_W^2)$ due to the W-propagator. In either case, s and t are very small as compared to m_W^2 and to m_Z^2, for neutrino energies of the order of 10^2–10^3 GeV. Therefore, it is a very good approximation to use the effective contact interaction (3.186) instead of the full Lagrangian (3.183).

The differential cross section in the c.m. system is given by the same expression as for reaction (3.91) viz.

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{32s4\pi^2}(2\pi)^{12}\sum |T|^2. \tag{3.221}$$

In calculating reactions (3.218a,b) let us write the effective contact interaction in the general form

$$-\mathscr{L}_{\text{NC}}^{\text{eff}} = -\frac{G}{2\sqrt{2}}(\overline{e(x)}(c_V\gamma_\alpha - c_A\gamma_\alpha\gamma_5)e(x))(\overline{\nu_\mu(x)}(\gamma^\alpha - \lambda\gamma^\alpha\gamma_5)\nu_\mu(x)). \tag{3.222}$$

In the case of the GSW model, this is identical to the expression (3.85), so that we must identify the parameters of (3.222) as follows:

$$c_V = 1 - 4\sin^2\theta_w, \tag{3.223a}$$

$$c_A = 1, \tag{3.223b}$$

$$\lambda = 1. \tag{3.223c}$$

Furthermore, let us assume that the incident ν_μ in reaction (3.218a) carries helicity h, the incident $\overline{\nu}_\mu$ in (3.218b) carries helicity $\overline{h}$. In laboratory experiments these neutrinos stem primarily from pion and kaon decays, $\pi \to \mu\nu_\mu$, $\mathrm{K} \to \mu\nu_\mu$. As we saw in Sect. 3.1.2c (i), these helicities are known to be maximal, so that one may assume

$$h = -1 \qquad \text{and} \qquad \overline{h} = +1. \tag{3.224}$$

The T-matrix element and the traces in $\Sigma|T|^2$ are worked out along the same lines as in the example of sec. 2.4, except that the incident $\nu_\mu(\overline{\nu}_\mu)$ is polarized along its momentum with polarization $h(\overline{h})$. In calculating the traces this means that we must set

$$u_\nu(p)\,\overline{u_\nu(p)} = \frac{1}{2}(\mathbb{1} + h\gamma_5)\not{p},$$
$$\upsilon_\nu(p)\overline{\upsilon_\nu(p)} = \frac{1}{2}(\mathbb{1} - \overline{h}\gamma_5)\not{p}.$$

These expressions are obtained from (1.148, 1.149) with $\boldsymbol{n} = h\hat{\boldsymbol{p}}$ (or $\bar{h}\hat{\boldsymbol{p}}$), in the limit $m \to 0$, or, equivalently, from (1.160), with the identifications $h \equiv \lambda$, $\overline{h} = -\lambda$, respectively.

The differential cross section for reaction (3.218a) in the c.m. system is found to be

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}(\nu_\mu \mathrm{e} \to \mathrm{e}\nu_\mu) = \frac{G^2}{64s \cdot 4\pi^2}\{(s^2+u^2)(1+|\lambda|^2 - 2h\,\mathrm{Re}\lambda)(|c_\mathrm{V}|^2 + |c_A|^2)$$
$$-2(s^2-u^2)(h + h|\lambda|^2 - 2\,\mathrm{Re}\lambda)\,\mathrm{Re}(c_\mathrm{V}c_\mathrm{A}^*)\}. \tag{3.225}$$

The analogous cross section for reaction (3.218b) is obtained from (3.225) by interchanging s and u, and by replacing h with $-\bar{h}$. From these results it is now easy to calculate the integrated elastic cross section. Integrating over $\mathrm{d}\Omega^*$ and using (3.219c) one finds

$$\int (s^2+u^2)\mathrm{d}\Omega^* = \frac{16\pi}{3}s^2, \qquad \int (s^2-u^2)\mathrm{d}\Omega^* = \frac{8\pi}{3}s^2,$$

so that

$$\sigma(\nu_\mu \mathrm{e}) = \frac{G^2 s}{12\pi}\frac{1}{4}\{(1+|\lambda|^2 - 2h\,\mathrm{Re}\lambda)(|c_\mathrm{V}|^2 + |c_\mathrm{A}|^2) \tag{3.226a}$$
$$-(h + h|\lambda|^2 - 2\,\mathrm{Re}\lambda)\,\mathrm{Re}(c_\mathrm{V}c_\mathrm{A}^*)\},$$
$$\sigma(\overline{\nu}_\mu \mathrm{e}) = \frac{G^2 s}{12\pi}\frac{1}{4}\{(1+|\lambda|^2 + 2\overline{h}\,\mathrm{Re}\lambda)(|c_\mathrm{V}|^2 + |c_\mathrm{A}|^2) \tag{3.226b}$$
$$-(\overline{h} + \overline{h}|\lambda|^2 + 2\,\mathrm{Re}\lambda)\,\mathrm{Re}(c_\mathrm{V}c_\mathrm{A}^*)\}.$$

Let us now make the following assumptions:

(i) the incident neutrino ν_μ is fully left-handed, so that $h = -1$ and, correspondingly, $\bar{h} = +1$,
(ii) c_V and c_A are both real.

With these assumptions (3.226) simplify to the following expressions:

$$\sigma\begin{pmatrix} \nu_\mu & \mathrm{e} \\ \overline{\nu}_\mu & \mathrm{e} \end{pmatrix} = \frac{G^2 s}{12\pi}\frac{|1+\lambda|^2}{4}\{c_\mathrm{V}^2 + c_\mathrm{A}^2 \pm c_\mathrm{V} c_\mathrm{A}\}. \tag{3.227}$$

The integrated cross sections (3.227) show the increase with the square of the neutrino energy in the c.m. system (or with the first power of its laboratory energy), which is typical for the contact interaction. (As we noted in Sect. 3.2.4 this increase cannot hold at arbitrarily large energies.)

Suppose that both $\sigma(\nu_\mu \mathrm{e})$ and $\sigma(\overline{\nu}_\mu \mathrm{e})$ are measured and that we know that $\lambda = 1$. We may then use (3.227) to determine c_v and c_A: Writing $c_\mathrm{V} = (1/\sqrt{2})(x+y)$ and $c_\mathrm{A} = (1/\sqrt{2})(x-y)$ we see that (3.227) defines two ellipses whose symmetry axes are rotated by $\pi/4$ with respect to a coordinate frame (c_v,c_A), as shown in Fig. 3.7. If the two ellipses intersect, c_v and c_A are determined up to a fourfold ambiguity. To this we add the information which comes from a measurement of reaction (3.218c). The integrated cross section $\sigma(\overline{\nu}_\mu \mathrm{e})$ is given by an expression analogous to (3.226b) supplemented by the contribution due to the CC interactions.

Assuming $\overline{h}(\overline{\nu}_\mathrm{e}) = +1, \lambda = 1$ and assuming that CC couplings are precisely the ones of (3.74), one finds after some calculation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}(\overline{\nu}_\mathrm{e}\mathrm{e} \to \mathrm{e}\overline{\nu}_\mathrm{e}) = \frac{G^2}{64 s\pi^2}\{s^2(c_\mathrm{V}-c_\mathrm{A})^2 + u^2((c_\mathrm{V}+c_\mathrm{A})^2 + 8(2-c_\mathrm{V}-c_\mathrm{A}))\}.$$

Intergration over $\mathrm{d}\Omega^*$ yields the following expression for the total elastic cross section:

$$\sigma(\overline{\nu}_\mathrm{e}\mathrm{e}) = \frac{G^2 s}{12\pi}(c_\mathrm{V}^2 + c_\mathrm{A}^2 - c_\mathrm{V}c_\mathrm{A} - 2c_\mathrm{V} - 2c_\mathrm{A} + 4\}. \tag{3.228}$$

Expressed in terms of the neutrino energy in the laboratory system we have $s = 2m_\mathrm{e}E_\nu^\mathrm{lab}$. The cross sections (3.227) have the order of magnitude

$$\sigma_0 = \frac{2m_\mathrm{e}G^2}{12\pi}E_\nu^\mathrm{lab} = 1.44 \times 10^{-42}\mathrm{cm}^2\mathrm{GeV}^{-1}E_\nu^\mathrm{lab}.$$

Suppose that we had found, in these units,

$$\sigma(\nu_\mu e)/\sigma_0 \simeq \sigma(\overline{\nu}_\mu \mathrm{e})/\sigma_0 \simeq 1,$$
$$\sigma(\overline{\nu}_\mu \mathrm{e})/\sigma_0 \simeq 3.$$

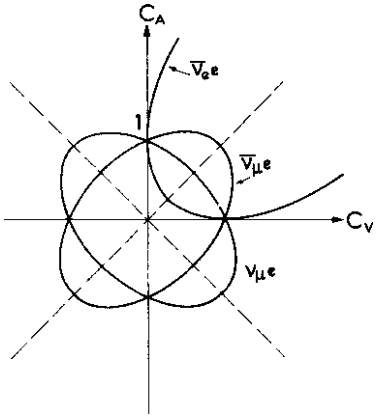

Fig. 3.7 The elastic neutrino reactions (3.218) determine three ellipses in the plane of the effective coupling constants c_v and c_A of (3.222)

The three ellipses defined by (3.227, 3.228) would then be the ones shown in Fig. 3.7. They intersect in the points ($c_V = 0$, $c_A = 1$) and ($c_V = 1$, $c_A = 0$), the first of which would be compatible with (3.223a,b) and would imply $\sin^2\theta_w = \frac{1}{4}$.

All three reactions have indeed been measured in the laboratory, with error bars of the order of 30 to 50%. The results are found to be consistent with the predictions of the GSW theory but seem to favour a somewhat larger value of $\sin^2\theta_w$. Note that for $\sin^2\theta_w = \frac{1}{4}$, the two cross sections (3.227) are exactly equal. For $\sin^2\theta_w$ smaller (larger) than $\frac{1}{4}$, $\sigma(\nu_\mu e)$ is larger (smaller) than $\sigma(\overline{\nu}_\mu e)$. The comparison of these two cross sections measures $\sin^2\theta_w$ relative to the value $\frac{1}{4}$, at which they are equal.

One would think that relative phases of parity-even to parity-odd interactions can only be obtained from a spin-momentum correlation or some other pseudoscalar observable. Therefore, one might wonder why and how a measurement of total cross sections can determine the relative strength and relative sign of vector to axial vector NC couplings. The answer to this question can be derived from (3.226). If we restrict our considerations to NC couplings only, we may assume c_v to be real, without loss of generality, and take this constant out of the curly brackets in (3.226a,b). In fact, by redefining c_A, c_v may be taken to be unity. If indeed, $h = -1$ and $\overline{h} = +1$ the cross sections are proportional to

$$\sigma\begin{pmatrix}\nu_\mu e\\ \overline{\nu}_\mu e\end{pmatrix} \propto |1+\lambda|^2\{1+|c_A|^2 \pm \mathrm{Re}(c_A)\}. \tag{3.229}$$

On the other hand, if the incident ν_μ and $\overline{\nu}_\mu$ were unpolarized, i.e. if the beam contained an equal amount of neutrinos of either helicity, we would obtain from (3.226)

$$\sigma\begin{pmatrix}\nu_\mu e\\ \overline{\nu}_\mu e\end{pmatrix}_{\mathrm{unpol}} \propto \{(1+|\lambda|^2)(1+|c_A|^2 \pm 2\mathrm{Re}(\lambda)\mathrm{Re}(c_A)\}. \tag{3.230}$$

Clearly, (3.230) does not allow to determine $\mathrm{Re}\, c_\mathrm{A}$, unless λ is known and is different from zero. Therefore, it contains less information than (3.229). (The two are the same if and only if $\lambda = 1$.) This analysis shows that in order to extract $\mathrm{Re} c_\mathrm{A}$, some information on neutrino helicities must be given: In the first case (3.229), we have assumed $h(\nu_\mu) = -1$, and therefore $h(\overline{\nu}_\mu) = +1$. In the second, the knowledge of $\lambda \neq 0$ implies that we know the longitudinal polarization of the outgoing neutrino. In either of these cases, we make use of input information on a *pseudoscalar* quantity, the neutrino helicity. (See also exercises 3.10 and 3.11.)

Finally, the assumption that both c_V and c_A are real was essential in determining both of them (up to a twofold ambiguity) from the three cross sections discussed above. [It is not sufficient that they only be relatively real because of reaction (3.218c) for whose calculation we need to know the phases of c_V, c_A relative to the CC couplings.]

3.6.3 *Angular Asymmetry in* $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$ *and* $\mathrm{e}^+\mathrm{e}^- \to \tau^+\tau^-$

In the two previous examples one could not discriminate between the full GSW theory (3.183), with large but finite masses of the gauge bosons, and the effective contact interaction (3.186) (corresponding to the former with $m_\mathrm{Z}, m_\mathrm{W} \to \infty$). This was so because these examples concerned reactions on fixed targets in which case a major fraction of the projectile energy resides in the kinetic energy in the centre-of-mass and only a small fraction goes into the momentum variable that appears in the gauge boson propagators. The third example we choose is a colliding beam reaction,

$$\mathrm{e}^+ + \mathrm{e}^- \to \mathrm{F}^+ + \mathrm{F}^-, \qquad \mathrm{F} = \mu \quad \text{or} \quad \tau, \tag{3.231}$$

in which the squared momentum of the gauge boson is equal to $s = 4E^{*2}$, i.e. where the virtual boson carries the full energy that is available in the reaction.

To lowest order in e, the amplitude for reaction (3.231) is given by the sum of the one-photon and the one-Z^0 diagrams of Fig. 3.8. The kinematics is as indicated in Fig. 3.8a. For simplicity, we consider the case where s is large as compared to both m_e^2 and m_F^2. (For the more general case with $m_\mathrm{F}^2 \neq 0$ see exercise 3.12.) With

$$s = (p_+ + p_-)^2 = (q_+ + q_-)^2 \simeq 4k^{*2}, \tag{3.232a}$$

where k^* is the magnitude of the c.m. three-momentum, we can express t and u in terms of s and of θ^*, the scattering angle in the c.m. system:

$$t = (p_- - q_-)^2 \simeq -\frac{s}{2}(1 - \cos\theta^*), \tag{3.232b}$$

$$u = (p_- - q_+)^2 \simeq -\frac{s}{2}(1 + \cos\theta^*), \tag{3.232c}$$

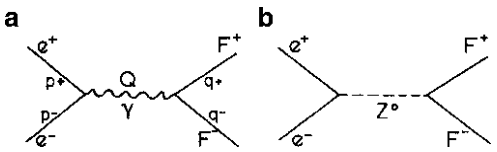

Fig. 3.8 One-photon (**a**) and one Z^0-exchange (**b**) in electron–positron annihilation and pair creation of muons or τ-leptons

The incident, colliding beams are taken to be unpolarized. Therefore, as the spins are not discriminated, one might be tempted to conclude from the symmetry of Fig. 3.8 that the differential cross section is left invariant if we interchange q_+ and q_-. Interchanging the final state momenta means interchanging t and u, or from (3.232b,c), means replacing θ^* by $\pi - \theta^*$. Thus, we would conclude that the cross section $\mathrm{d}\sigma/\mathrm{d}\Omega^*$ is a function only of $\cos^2\theta^*$ and is symmetric about $\theta^* = 90°$. However, a closer examination of the diagrams, taking into account the helicity transfer at the two vertices, shows that this is not true in general.

We know from the general analysis in Sects. 3.1.2c, d that vector and axial vector vertices connect the helicities of massless particles as indicated in Figs. 3.4a,b. Two examples of the helicity transfer at the photon and Z^0 vertices are shown in Figs. 3.9a, b. (Since the incident particles are unpolarized, we must add incoherently the contribution of these two amplitudes and of the ones with all four helicities reversed.) As can be seen from Fig. 3.9, the interchange of q^+ and q^- effectively means that diagram 3.9a goes over into diagram 3.9b, and vice versa. Therefore, the cross section is $t - u$ symmetric only if the two amplitudes have the same relative weight. This happens if the sum of the diagrams, Fig. 3.9, contains either pure VV or pure AA couplings, but not both. Let us calculate this in more detail.

The differential cross section in the c.m. system (this is the reference system in the laboratory if the colliding beams have equal and opposite momenta), in the limit $s \gg m_\mathrm{e}^2, m_\mathrm{F}^2$, is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} \simeq \frac{1}{16s}(2\pi)^{10}\frac{1}{4}\sum |T|^2. \tag{3.233}$$

In writing down the T-matrix element for the one-Z^0 exchange we note that the term $Q_\alpha Q_\beta / m_\mathrm{Z}^2$ in the propogator gives a negligibly small contribution: Q_α is equal to $(p_+ + p_-)_\alpha$ and to $(q_+ + q_-)_\alpha$ so that by virtue of the Dirac equations (1.84,85')

$$\overline{\upsilon(p_+)}\gamma^\alpha u(p_-)Q_\alpha = 0,$$
$$\overline{\upsilon(p_+)}\gamma^\alpha\gamma_5 u(p_-)\,Q_\alpha = -2m_\mathrm{e}\overline{\upsilon(p_+)}\gamma_5 u(p_-),$$

and analogously for Q_β contracted with the $(\mathrm{F}^+\mathrm{F}^-)$ vertex. Thus, the axial-couplings yield a negligible contribution of the order $m_\mathrm{e}m_\mathrm{F}/m_\mathrm{Z}^2$, relative to the term $g_{\alpha\beta}$. With this simplification the amplitudes in Fig. 3.9 are

$$T_\gamma = \frac{1}{(2\pi)^6}\frac{e^2}{s}\overline{(\upsilon(p_+)\gamma^\alpha u(p_-))}\,\overline{(u(q_-)\gamma_\alpha \upsilon(q_+))}, \tag{3.234}$$

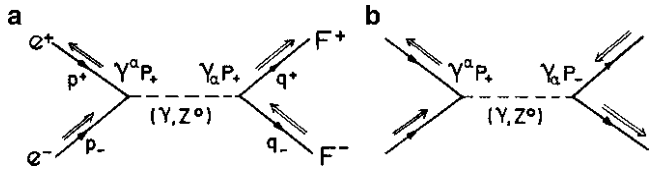

Fig. 3.9 Two examples of helicity transfers in reaction (3.231) as described by the diagrams of Fig. 3.8. (**a**) (V + A)(V + A) coupling, (**b**) (V + A)(V − A) coupling

$$T_Z \simeq \frac{1}{(2\pi)^6}\frac{e^2}{16\sin^2\theta_\mathrm{w}\cos^2\theta_\mathrm{w}}\frac{1}{s-m_Z^2} \times\overline{\upsilon(p_+)}\{c_\mathrm{V}^{(\mathrm{e})}\gamma^\alpha - c_\mathrm{A}^{(\mathrm{e})}\gamma^\alpha\gamma_5\}u(p_-))\overline{u(q_-)}\{c_\mathrm{V}^{(\mathrm{F})}\gamma^\alpha - c_\mathrm{A}^{(\mathrm{F})}\gamma^\alpha\gamma_5\}\upsilon(q_+)), \tag{3.235}$$

where $c_\mathrm{V}^{(\mathrm{e/F})}$ and $c_\mathrm{A}^{(\mathrm{e/F})}$ are defined in (3.223a, b). [For electrons (e) and muons (F = μ) they are the same. If F is a neutrino or a quark, the term 4 $\sin^2\theta_\mathrm{w}$ is multiplied by the electric charge of these particles, see below.] Using the relation $m_Z^2\cos^2\theta_\mathrm{w} = m_\mathrm{W}^2$ and (3.187a), one can write the factor on the r.h.s. of (3.235) as

$$\frac{e^2}{s}\kappa(s) \text{ with } \kappa(\mathrm{s}) = \frac{Gm_Z^2}{8\pi\alpha\sqrt{2}}\frac{s}{s-m_Z^2} \approx 0.374\frac{s}{s-m_Z^2}. \tag{3.236}$$

Decomposing the V and A vertex factors in terms of γ^α multiplied by helicity projection operators (3.32), that is

$$\gamma^\alpha = \gamma^\alpha P_+ + \gamma^\alpha P_-, \quad \gamma^\alpha\gamma_5 = \gamma^\alpha P_+ - \gamma^\alpha P_-,$$

and introducing the shorthand notation

$$A_\pm^\alpha := \overline{\upsilon(p_+)}\gamma^\alpha P_\pm u(p_-),$$
$$B_\pm^\alpha := \overline{u(q_-)}\gamma^\alpha P_\pm \upsilon(q_+),$$

the sum of the amplitudes (3.234) and (3.235) is

$$\begin{aligned} T_\gamma + T_Z = \frac{1}{(2\pi)^6}\frac{e^2}{s}\{&(A_+\cdot B_+)[1+\kappa(s)(c_\mathrm{V}^{(\mathrm{e})}-c_\mathrm{A}^{(\mathrm{e})})(c_\mathrm{V}^{(\mathrm{F})}-c_\mathrm{A}^{(\mathrm{F})})] \\ &+(A_-\cdot B_-)[1+\kappa(s)(c_\mathrm{V}^{(\mathrm{e})}+c_\mathrm{A}^{(\mathrm{e})})(c_\mathrm{V}^{(\mathrm{F})}+c_\mathrm{A}^{(\mathrm{F})})] \\ &+(A_+\cdot B_-)[1+\kappa(s)(c_\mathrm{V}^{(\mathrm{e})}-c_\mathrm{A}^{(\mathrm{e})})(c_\mathrm{V}^{(\mathrm{F})}+c_\mathrm{A}^{(\mathrm{F})})] \\ &+(A_-\cdot B_+)[1+\kappa(s)(c_\mathrm{V}^{(\mathrm{e})}+c_\mathrm{A}^{(\mathrm{e})})(c_\mathrm{V}^{(\mathrm{F})}-c_\mathrm{A}^{(\mathrm{F})})]\}. \end{aligned} \tag{3.237}$$

The calculation of $\Sigma_{\text{spins}}|T_\gamma + T_Z|^2$ is greatly simplified by observing that an amplitude that contains A_+ cannot interfere with an amplitude that contains A_-. Likewise B_+ cannot interfere with B_-. Therefore, in the sum over the spins all interference terms vanish. [Note, however, that this holds only in the limit $m_e = m_F = 0$. See exercise 3.12]. Furthermore, it is easy to see [from the explicit expression (3.239a) below] that

$$\sum |(A_+ \cdot B_+)|^2 = \sum |(A_- \cdot B_-)|^2, \tag{3.238a}$$

$$\sum |(A_+ \cdot B_-)|^2 = \sum |(A_- \cdot B_+)|^2. \tag{3.238b}$$

Finally, as the amplitude $(A_+ \cdot B_+)$ corresponds to the diagram in Fig. 3.9a the amplitude $(A_+ \cdot B_-)$ to that in Fig. 3.9b, the expression (3.238b) is obtained from (3.238a) by the transformation $t \leftrightarrow u$. Therefore, we only need to calculate the term (3.238a). Neglecting the masses we have

$$\begin{aligned}
\sum |(A_+ \cdot B_+)|^2 &= \frac{1}{16} \operatorname{tr}\{\gamma^\alpha(1+\gamma_5)\not{p}_-\gamma^\beta(1+\gamma_5)\not{p}_+\} \\
&\quad \times \operatorname{tr}\{\gamma_\alpha(1+\gamma_5)\not{q}_+\gamma_\beta(1+\gamma_5)\not{q}_-\} \\
&= \frac{1}{4} \operatorname{tr}\{(1-\gamma_5)\gamma^\alpha \not{p}_-\gamma^\beta \not{p}_+\} \operatorname{tr}\{(1-\gamma_5)\gamma_\alpha \not{q}_+\gamma_\beta \not{q}_-\} \\
&= 4\{p_-^\alpha p_+^\beta - (p_-p_+)g^{\alpha\beta} + p_+^\alpha p_-^\beta - \mathrm{i}\varepsilon^{\alpha\mu\beta\nu} p_{-\mu} p_{+\nu}\} \\
&\quad \times \{q_{+\alpha} q_{-\beta} - (q_+q_-)g_{\alpha\beta} + q_{-\alpha} q_{+\beta} - \mathrm{i}\varepsilon_{\alpha\sigma\beta\tau} q_+^\sigma q_-^\tau\} \\
&= 16(p_+q_-)(p_-q_+) = 4u^2.
\end{aligned} \tag{3.239a}$$

By the $t - u$ symmetry noted above we conclude

$$\sum |(A_+ \cdot B_-)|^2 = 4t^2. \tag{3.239b}$$

With the results (3.239) and keeping in mind the remarks made above, we obtain

$$\begin{aligned}
(2\pi)^{12} \sum |T_\gamma + T_Z|^2 = \frac{8e^4}{s}\{&u^2[1 + 2\kappa(c_V^{(e)} c_V^{(F)} + c_A^{(e)} c_A^{(F)} \\
&+ \kappa^2((c_V^{(e)} c_V^{(F)} + c_A^{(e)} c_A^{(F)})^2 + (c_V^{(e)} c_A^{(F)} + c_A^{(e)} c_V^{(F)})^2)] \\
&+ t^2[1 + 2\kappa(c_V^{(e)} c_V^{(F)} - c_A^{(e)} c_A^{F}) \\
&+ \kappa^2((c_V^{(e)} c_V^{(F)} - c_A^{(e)} c_A^{(F)})^2 + (c_V^{(e)} c_A^{(F)} - c_A^{(e)} c_V^{(F)})^2]\}.
\end{aligned}$$

Finally, inserting (3.232) and replacing e^4 by $(4\pi\alpha)^2$, the differential cross section (3.233) is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega*} = \frac{\alpha^2}{4s}\{(1+\cos^2\theta^*)[(1+\kappa(s)c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})})^2 + \kappa^2(s)(c_{\mathrm{V}}^{(\mathrm{e})2}c_{\mathrm{A}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{e})2}c_{\mathrm{V}}^{(\mathrm{F})^2} + c_{\mathrm{A}}^{(\mathrm{e})2}c_{\mathrm{A}}^{(\mathrm{F})2})] + 4\kappa(s)c_{\mathrm{A}}^{(\mathrm{e})}c_{\mathrm{A}}^{(\mathrm{F})}(1+2\kappa(s)c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})})\cos\theta^*\}. \tag{3.240}$$

The first term in the curly brackets of (3.240) is symmetric about the angle 90°, the second is not. The asymmetric term is present only if there is both a VV and an AA interaction in the sum of the diagrams in Fig. 3.8. The forward–backward asymmetry in the cross section

$$A := \left\{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega*}(0) - \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega*}(\pi)\right\} \Big/ \left\{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega*}(0) + \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega*}(\pi)\right\} \tag{3.241}$$

is found to be

$$A = 2\kappa(s)c_{\mathrm{A}}^{(\mathrm{e})}c_{\mathrm{A}}^{(\mathrm{F})}\frac{1+2\kappa(s)c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})}}{(1+\kappa(s)c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})})^2 + \kappa^2(s)(c_{\mathrm{V}}^{(\mathrm{e})2}c_{\mathrm{A}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{e})2}c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{e})2}c_{\mathrm{A}}^{(\mathrm{F})2})} \tag{3.242}$$

with $\kappa(s)$ as defined in (3.236).

Let us now estimate the magnitude of the asymmetry as it is predicted by the GSW model. With $\sin^2\theta_{\mathrm{w}} = 0.232$ and m_{Z} as given in (3.90a) we have

$$c_{\mathrm{V}}^2 \approx 0.0052 \qquad c_{\mathrm{A}} = 1$$
$$\kappa(s) \approx 0.374\frac{s}{s-m_{\mathrm{Z}}^2}$$

At an energy of 18 GeV per beam, $\sqrt{s} = 36\,\mathrm{GeV}$, the propogator effect is about 17%,

$$\frac{1}{1-s/m_{\mathrm{Z}}^2} \approx 1.17 \tag{3.243}$$

and $\kappa = -0.069$. At this energy both $\kappa^2 c_{\mathrm{A}}^{(\mathrm{e})}c_A^{(\mu)}$ and $\kappa^2 c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mu)}$ are very small so that the asymmetry is

$$A(\sqrt{s} = 36\ \mathrm{GeV}) \approx 2\kappa = -13.8\%.$$

Historically, long before the Z^0 was discovered and later on produced in large numbers, this forward–backward asymmetry was measured at DESY (Hamburg) at this energy. This was an important test of the Z^0–γ interference in $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$, confirming the sign and magnitude predicted by the minimal standard model. At the same time this was a first announcement of the Z^0 resonance and, one could say, the experiment had "climbed" the first seventeen percent of the resonance peak. The example also shows that at an energy of 36 GeV there is already a marked difference

between the prediction of the standard model and the result one would have obtained from the effective contact interaction ($m_Z \to \infty$), see (3.243).

There are additional contributions to the asymmetry (3.241) which stem from digrams with *two* virtual photons and from radiative corrections. These were calculated and were taken into account in a quantitative comparison with experiment. Finally, as the energy in the center-of-mass system approaches m_Z one can no longer neglect the finite width of the Z^0 resonance. We return to this topic in the next section.

3.7 Electroweak Physics at the Z Pole

Experimental tests of electroweak interactions changed qualitatively and quantitatively when the first large electron-positron colliders came into operation: The Large Electron Positron collider (LEP) at CERN which was operated from 1989 until 2000, and the Stanford Linear Collider (SLC) which came into operation in 1989, too. The energy in the center-of-mass system at these colliders was large enough to allow for production of large numbers of Z^0 bosons whose creation and whose decays were studied in a number of highly sophisticated detectors. In fact, this was an era of precision measurements at and around the Z^0 pole in processes such as $e^+e^- \to F\overline{F}$ with F a lepton or quark.

3.7.1 Cross Sections Near the Z^0 Resonance

Let us consider the total cross section for the production of a fermion–antifermion pair $F\overline{F}$ in e^+e^- collisions. Although radiation effects in the initial state and radiative corrections to the production amplitude must be taken into account in a quantitative analysis of the experiments, it is instructive to calculate the process in the Born approximation, where it is given by the one-photon diagram, the one-Z^0 diagram, sketched in Fig. 3.8 above, and their interference term. Let us choose kinematic variables as in Sect. 3.6.3, i.e. the four-momenta of the electron and the positron of the initial state are denoted by p_- and p_+, the four-momenta of the outgoing fermion F and its antiparticle $\overline{F}$ are denoted by q_- and q_+, respectively. For the calculation of the total cross section and the forward–backward asymmetry we can make use of our previous result (3.240) for the differential cross section in the centre-of-mass system. The only modification we have to make is the following: the discussion of Sect. 3.6.3 concerned an energy regime well below the mass of the Z^0 where the finite width of this resonance $\Gamma_Z \approx 2.5\,\text{GeV}$ could be neglected. Indeed, for energies well below the Z^0 resonance, the denominator of the Z^0 propagator is dominated by $(s - m_Z^2)^2 \gg \Gamma_Z$. In the immediate neighbourhood of the resonance position $s = m_Z^2$, this approximation is not admissible. We shall argue that this important effect is taken into account by the following replacement

in the denominator of (3.236)

$$(s - m_{\mathrm{Z}}^2) \to (s - m_{\mathrm{Z}}^2) + \mathrm{i}m_{\mathrm{Z}}\Gamma_{\mathrm{Z}}. \tag{3.244}$$

The argument goes as follows: s is the square of the energy in the center-of-mass system. Write m for the mass of the resonance, Γ for its width. In the vicinity of the resonance we may write s–$m^2 = (E + m)(E - m) \approx 2m(E - E_0)$ with $E_0 \equiv m$ and $E - E_0 = E - m$ the kinetic energy of the resonance. A resonance is an energy distribution (or "energy packet") which is centered at $E = E_0$ and which is described by a probability amplitude of the form

$$\frac{\Gamma/2}{E - E_0 + \mathrm{i}\Gamma/2}\mathrm{d}E.$$

The time dependence of this amplitude is given by

$$\int_{-\infty}^{+\infty} \mathrm{d}E \frac{e^{-\mathrm{i}(E-E_0)t}}{E - (E_0 - \mathrm{i}\Gamma/2)} = 2\pi\mathrm{i}\mathrm{e}^{-t\Gamma/2}\mathrm{e}^{-\mathrm{i}E_0 t}.$$

Therefore, the probability of finding the energy packet, in the center-of-mass system and at its central position, decreases exponentially like $\mathrm{e}^{-\Gamma t}$. Thus, the resonance behaves like an unstable state whose lifetime is the inverse of the width, $\tau = 1/\Gamma$. This is the essence of what is called a Breit–Wigner description of an unstable particle. (According to G. Breit and E. Wigner who developed the scattering theory including resonances. The reader is invited to consult a treatise on collision theory if she or he is not familiar with this topic.)

Thus, in the neighbourhood of the Z^0 pole the function $\kappa(s)$ of (3.236) is replaced by

$$\kappa(s) = \frac{Gm_{\mathrm{Z}}^2}{8\pi\alpha\sqrt{2}} \frac{s}{(s - m_{\mathrm{Z}}^2) + \mathrm{i}m_{\mathrm{Z}}\Gamma_{\mathrm{Z}}}. \tag{3.245}$$

Finally, going through the formulae (3.237–3.239) once more, it is clear that in the result (3.240) κ^2 is replaced by $|\kappa|^2$, and the linear terms obtain the factor $\mathrm{Re}(\kappa)$ instead of κ.

We return to the calculation of the total cross section $\sigma\left(\mathrm{e}^+\mathrm{e}^- \to \mathrm{F}\overline{\mathrm{F}}\right)$ which we write as the sum of the cross section due to the one-photon diagram, the Z^0 diagram, and the interference term,

$$\sigma_{\mathrm{F}} = \sigma_{\mathrm{F}}^{(\gamma)} + \sigma_{\mathrm{F}}^{(\mathrm{Z})} + \sigma_{\mathrm{F}}^{(\mathrm{int})}. \tag{3.246}$$

The first of these, $\sigma_{\mathrm{F}}^{(\gamma)}$, is obtained from the differential cross section (3.240) by dropping all terms in κ, and by integrating over Ω^*. If the fermions in the final state are quarks, the cross section must be multiplied with $Q^2(\mathrm{F})$, $Q(\mathrm{F})$ being the electric charge of the produced species of quark in units of the elementary charge, and by the number of colours $N_{\mathrm{c}} = 3$. The integration over Ω^* gives

$$\int \mathrm{d}\Omega^*(1+\cos^2\theta^*) = 2\pi\frac{8}{3},$$

so that the total photonic cross section is found to be

$$\sigma_{\mathrm{F}}^{(\gamma)} = \frac{4\pi\alpha^2}{3s}Q^2(\mathrm{F})N_{\mathrm{c}}. \tag{3.247}$$

[For F a charged lepton set $N_{\mathrm{c}} = 1$, $Q^2 = 1$. For neutrinos (3.247) vanishes.]

The cross section which is due to the Z^0 diagram of Fig. 3.8 is obtained from (3.240) by collecting all terms proportional to κ^2, or more precisely $|\kappa|^2$:

$$\frac{d\sigma_{\mathrm{F}}^{(\mathrm{Z})}}{d\Omega^*} = \frac{\alpha^2|\kappa|^2}{4s}\Big\{(1+\cos^2\theta^*)(c_{\mathrm{V}}^{(\mathrm{e})2}+c_{\mathrm{A}}^{(\mathrm{e})2})(c_{\mathrm{V}}^{(\mathrm{F})2}+c_{\mathrm{A}}^{(\mathrm{F})2}) + 8c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})}c_{\mathrm{A}}^{(\mathrm{e})}c_{\mathrm{A}}^{(\mathrm{F})}\cos\theta^*\Big\}.$$

Upon integration over the polar angles the first term gives $16\pi/3$, as above, while the second term gives zero. Thus we obtain

$$\sigma_{\mathrm{F}}^{(\mathrm{Z})} = \frac{G^2m_{\mathrm{Z}}^2}{96\pi}(c_{\mathrm{V}}^{(\mathrm{e})2}+c_{\mathrm{A}}^{(\mathrm{e})2})(c_{\mathrm{V}}^{(\mathrm{F})2}+c_{\mathrm{A}}^{(\mathrm{F})2})\frac{sm_{\mathrm{Z}}^2}{(s-m_{\mathrm{Z}}^2)^2+m_{\mathrm{Z}}^2\Gamma_{\mathrm{Z}}^2}. \tag{3.248}$$

Before we turn to the interference term let us rewrite the result (3.248) in terms of partial decay widths of the Z^0 into a given pair of a fermion and its antiparticle. Thus, we calculate the process

$$\mathrm{Z}^0 \to \mathrm{F}\overline{\mathrm{F}}.$$

From (3.183) the decay amplitude is given by

$$T(\mathrm{Z}^0\to\mathrm{F}\overline{\mathrm{F}})\frac{1}{(2\pi)^{9/2}}\Lambda\varepsilon^{\mu}(k,\lambda)\overline{u(q_-)}(c_{\mathrm{V}}^{(\mathrm{F})}\gamma_\mu - c_{\mathrm{A}}^{(\mathrm{F})}\gamma_\mu\gamma_5)\upsilon(q_+),$$

where k is the four-momentum of the decaying Z^0, $k = q_+ + q_-$, λ its polarization,

$$\Lambda = \frac{e}{4\sin\theta_{\mathrm{w}}\cos\theta_{\mathrm{w}}} = \frac{g}{4\cos\theta_{\mathrm{w}}}, \text{ from which } \Lambda^2 = \frac{Gm_{\mathrm{Z}}^2}{2\sqrt{2}},$$

and the coupling constants are given by the values in the standard model[8]

[8] In the literature on electroweak physics a different definition of coupling constants is often used, viz.

$$g_{\mathrm{V}}^{(\mathrm{F})} = -\frac{1}{2} + 2Q(\mathrm{F})\sin^2\theta_{\mathrm{w}} \qquad g_{\mathrm{A}}^{(\mathrm{F})} = -\frac{1}{2}.$$

$$c_{\mathrm{V}}^{(\mathrm{F})} = 1 - 4Q(\mathrm{F})\sin^2\theta_{\mathrm{w}},\ c_{\mathrm{A}}^{(\mathrm{F})} = 1. \tag{3.249}$$

Averaging over the three polarization states of the Z^0 gives 1/3 of the sum $\Sigma_\lambda \varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda) = (k^\mu k^\nu/m_{\mathrm{Z}}^2 - g^{\mu\nu})$. The decay width is given by equation (B.7), with

$$\frac{1}{3}\sum_\lambda \sum_{s_+ s_-} |T|^2 = \frac{\Lambda^2}{3(2\pi)^9} \sum_\lambda \sum_{s_+ s_-} \left(\frac{k^\mu k^\nu}{m_{\mathrm{Z}}^2} - g^{\mu\nu}\right)$$
$$\mathrm{tr}\{(c_{\mathrm{V}}^{(\mathrm{F})}\gamma_\mu - c_{\mathrm{A}}^{(\mathrm{F})}\gamma_\mu\gamma_5)(\not{q}_+ + m_{\mathrm{F}})(c_{\mathrm{V}}^{(\mathrm{F})}\gamma_\nu - c_{\mathrm{A}}^{(\mathrm{F})}\gamma_\nu\gamma_5)(\not{q}_- - m_{\mathrm{F}})\}.$$

If F is a neutrino, the mass m_{F} is zero. In all other cases we may safely neglect this mass as compared to m_{Z}. Evaluating the trace in this limit, the latter expression is found to be

$$\frac{4\Lambda^2(c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{F})2})}{3(2\pi)^9}\left\{\frac{2}{m_{\mathrm{Z}}^2}\{k\cdot q_+\}\{k\cdot q_-\} + \{q_+\cdot q_-\}\right\}.$$

Inserting the kinematics of the Z^0 rest frame, viz.

$$k = (m_{\mathrm{Z}}, \mathbf{0}),\ q_\pm = (E, \pm\boldsymbol{q}),\ \text{with } E \approx \frac{m_{\mathrm{Z}}}{2} \approx |\boldsymbol{q}|,$$

and inserting the result into (B.7), we obtain finally

$$\Gamma_{\mathrm{F}} \equiv \Gamma(\mathrm{Z}^0 \to \mathrm{F}\overline{\mathrm{F}}) \approx \frac{G m_{\mathrm{Z}}^3}{24\pi\sqrt{2}}(c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{F})2}). \tag{3.250}$$

Inserting the result (3.250) into (3.248) the cross section due to the Z^0 diagram can be expressed in terms of the partial decay widths Γ_{e} into $\mathrm{e}^+\,\mathrm{e}^-$ and Γ_{F} into $\mathrm{F}\overline{\mathrm{F}}$ as follows

$$\sigma_{\mathrm{F}}^{(\mathrm{Z})} = \frac{12\pi}{m_{\mathrm{Z}}^2}\frac{\Gamma_{\mathrm{e}}\Gamma_{\mathrm{F}}}{\Gamma_{\mathrm{Z}}^2}\frac{s\Gamma_{\mathrm{Z}}^2}{(s-m_{\mathrm{Z}}^2)^2 + m_{\mathrm{Z}}^2\Gamma_{\mathrm{Z}}^2}. \tag{3.251}$$

Finally, the interference term is obtained from the linear term in κ in the first bracket on the right-hand side of (3.240), by integration over Ω^*. With $Q(\mathrm{F})$ denoting the charge of the fermion F, N_c the colour factor as above, we find

$$\sigma_{\mathrm{F}}^{(\mathrm{int})} = -N_{\mathrm{c}}Q(\mathrm{F})\frac{\alpha G}{3\sqrt{2}}c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})}\frac{m_{\mathrm{Z}}^2(s-m_{\mathrm{Z}}^2)}{(s-m_{\mathrm{Z}}^2)^2 + m_{\mathrm{Z}}^2\Gamma_{\mathrm{Z}}^2}. \tag{3.252}$$

It is, of course, no problem to convert our formulae to this convention wherever they depend explicitly on the couplings.

The interference term (3.252) is found to be small compared to the cross sections (3.247) and (3.251), mainly due to the fact that the coefficients $c_{\mathrm{V}}^{(\mathrm{e})}$ and $c_{\mathrm{V}}^{(\mathrm{F})}$ of (3.249) are small. Obviously, for neutrinos which are electrically neutral, only the cross section $\sigma_{\mathrm{F}=\nu}^{(\mathrm{Z})}$, (3.251), is different from zero.

3.7.2 *Forward-Backward Asymmetries and Polarizations*

As we saw in the preceding section the differential cross section for $\mathrm{e}^+\,\mathrm{e}^- \to \mathrm{F}\overline{\mathrm{F}}$ due to one-photon and one-Z^0 exchange is given by an expression of the form

$$\frac{\mathrm{d}\sigma_{\mathrm{F}}}{\mathrm{d}\Omega^*} = \frac{\alpha^2}{2s}\{A_0(1+\cos^2\theta^*) + A_1\cos\theta^*\} \tag{3.253}$$

with

$$A_0 = 1 + 2c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})}\mathrm{Re}\,\kappa(s) + (c_{\mathrm{V}}^{(\mathrm{e})2} + c_{\mathrm{A}}^{(\mathrm{e})2})(c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{F})2})|\kappa(s)|^2, \tag{3.254a}$$

$$A_1 = 4c_{\mathrm{A}}^{(\mathrm{e})}c_{\mathrm{A}}^{(\mathrm{F})}[\mathrm{Re}\,\kappa(s) + 2c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})}|\kappa(s)|^2]. \tag{3.254b}$$

Here, the function $\kappa(s)$ is given by (3.245) and the coupling constants by (3.249). In practice, the angular distribution is measured for leptons in the final state, $\mathrm{F} = \mathrm{e}$, μ, or τ, for which $c_{\mathrm{V}}^{(\mathrm{F})} = 1 - 4\sin^2\theta_{\mathrm{w}}$, $c_{\mathrm{A}}^{(\mathrm{F})} = 1$, and, therefore, the distinction of the coupling constants in (3.254) is unnecessary.

In the neighbourhood of the Z-pole it is not exactly the forward–backward asymmetry (3.241) that is being measured. Rather, one measures the asymmetry between the integral over all particles scattered in the *forward* hemisphere ($0 \le \theta^* \le \pi/2$) and that over all particles scattered in the *backward* hemisphere ($\pi/2 \le \theta^* \le \pi$). This means that one compares the piecewise integrated cross sections

$$\sigma_{\mathrm{F}}^{(\mathrm{f})} = 2\pi\int_0^1 \frac{\mathrm{d}\sigma_{\mathrm{F}}}{\mathrm{d}\Omega^*}\mathrm{d}(\cos\theta^*), \text{ and } \sigma_{\mathrm{F}}^{(\mathrm{b})} = 2\pi\int_{-1}^0 \frac{\mathrm{d}\sigma_{\mathrm{F}}}{\mathrm{d}\Omega^*}\mathrm{d}(\cos\theta^*).$$

Using (3.253) this forward–backward asymmetry is given by

$$A_{\mathrm{fb}} = \frac{\sigma_{\mathrm{F}}^{(\mathrm{f})} - \sigma_{\mathrm{F}}^{(\mathrm{b})}}{\sigma_{\mathrm{F}}^{(\mathrm{f})} + \sigma_{\mathrm{F}}^{(\mathrm{b})}} = \frac{3A_1}{8A_0}. \tag{3.255}$$

It is interesting to evaluate the result (3.255) *at* the resonance and *away* from the resonance, and to compare the two situations. Right at the Z pole, $s = m_{\mathrm{Z}}^2$, $\kappa(s)$ is pure imaginary, cf. (3.245). The total cross section is dominated by $\sigma_{\mathrm{F}}^{(\mathrm{Z})}$ and the third term on the right-hand side of (3.254a) is much larger than 1. Likewise, in

the neighbourhood of the Z pole, the quadratic term in (3.254b) dominates over the one proportional to Re $\kappa(s)$. Hence, with $c_{\mathrm{V/A}}^{(\mathrm{F})} = c_{\mathrm{V/A}}^{(\mathrm{e})}$, the asymmetry (3.255) is approximately

$$A_{\mathrm{fb}}(s = m_{\mathrm{Z}}^2) \approx 3 \frac{c_{\mathrm{V}}^{(\mathrm{e})} c_{\mathrm{A}}^{(\mathrm{e})} c_{\mathrm{V}}^{(\mathrm{F})} c_{\mathrm{A}}^{(\mathrm{F})}}{(c_{\mathrm{V}}^{(\mathrm{e})2} + c_{\mathrm{A}}^{(\mathrm{e})2})(c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{F})2})} = \frac{3}{4} A_{\mathrm{e}} A_{\mathrm{F}}. \tag{3.256}$$

In the second part of this equation we have introduced the definitions

$$A_{\mathrm{e}} := \frac{-2 c_{\mathrm{V}}^{(\mathrm{e})} c_{\mathrm{A}}^{(\mathrm{e})}}{c_{\mathrm{V}}^{(\mathrm{e})2} + c_{\mathrm{A}}^{(\mathrm{e})2}}, \quad A_{\mathrm{F}} := \frac{-2 c_{\mathrm{V}}^{(\mathrm{F})} c_{\mathrm{A}}^{(\mathrm{F})}}{c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{F})2}}. \tag{3.257}$$

These definitions are convenient and appropriate because A_{e} and A_{F}, up to a sign, describe the longitudinal polarization of the electron and the lepton F, respectively, as they are produced by the weak neutral current.

Away from the Z pole we have $|s - m_{\mathrm{Z}}^2| \gg m_{\mathrm{Z}} \Gamma_{\mathrm{Z}}$ and $\kappa(s)$ is approximately real. The terms in $|\kappa(s)|^2$ can be neglected as compared to the linear terms and, therefore, the asymmetry (3.255) becomes

$$A_{\mathrm{fb}}(\Gamma_{\mathrm{Z}} \approx 0) \approx \frac{3 c_{\mathrm{A}}^{(\mathrm{e})} c_{\mathrm{A}}^{(\mathrm{F})} \mathrm{Re}\, \kappa(s)}{2(1 + 2 c_{\mathrm{V}}^{(\mathrm{e})2} \mathrm{Re}\, \kappa(s))}, \tag{3.258}$$

with

$$\mathrm{Re}\, \kappa(s) \approx \frac{G m_{\mathrm{Z}}^2 s}{8 \pi \alpha \sqrt{2} (s - m_{\mathrm{Z}}^2)}.$$

It is not difficult to calculate the polarization of the fermion in the final state, as well as further polarization dependent observables. For this purpose let us return to the expression (3.237) for the scattering amplitude: The amplitude contains contractions of the covariants

$$A_{\pm}^{\alpha} = \overline{v^{(\mathrm{e})}(p_+)} \gamma^{\alpha} P_{\pm} u^{(\mathrm{e})}(p_-), \quad \text{and} \quad B_{\pm}^{\alpha} = \overline{u^{(F)}(q_-)} \gamma^{\alpha} P_{\pm} v^{(\mathrm{F})}(q_+)$$

The operators $P_{\pm}$ project, respectively, onto positive and negative chirality states of the electron e^- and of the produced fermion F^-. Let us denote these chiralities by

$$h(\mathrm{e}^-) = \lambda, \; h(\mathrm{F}^-) = \tau, \; \text{with } \sigma, \tau = + \text{ or } -.$$

In (3.237) the product $(A_\lambda \cdot B_\tau), \lambda = +$ or $-, \tau = +$ or $-$, is multiplied by the amplitude

$$S_{\lambda\tau} = 1 + \kappa(s)(c_{\mathrm{V}}^{(e)} - \lambda c_{\mathrm{A}}^{(\mathrm{e})})(c_{\mathrm{V}}^{(\mathrm{F})} - \tau c_{\mathrm{A}}^{(\mathrm{F})}). \tag{3.259}$$

From (3.238, 3.239) we know that $|(A_\lambda \cdot B_\tau)|^2$ is proportional to $u^2 \approx s^2(1 + \cos\theta^*)^2/4$ if $\lambda = \tau = \pm 1$, and is proportional to $t^2 \approx s^2(1 - \cos\theta^*)^2/4$ if

$\lambda = -\tau$. Thus, if we calculate the differential cross section for an incoming electron with chirality λ, and an outgoing F^- with chirality τ, the differential cross section is obtained from the expressions

$$(2\pi)^{12}|T_\gamma + T_Z|^2 = \frac{8e^4}{2s^2}u^2|S_{\pm\pm}|^2, \qquad \text{or} \qquad \frac{8e^4}{2s^2}t^2|S_{\pm\mp}|^2,$$

It is given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}\right)_{\lambda\tau} = \frac{\alpha^2}{4s}(1 + \lambda\tau\cos\theta^*)^2|S_{\lambda\tau}|^2. \tag{3.260}$$

For example, if one averages over the chiralities in the initial state, the differential cross section for producing a lepton F^- with chirality τ is obtained from the result (3.260) as follows

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}\right)_\tau &= \frac{1}{4}\sum_\lambda \left(\frac{d\sigma}{d\Omega^*}\right)_{\lambda\tau} \\ &= \frac{\alpha^2}{16s}\{(1+\cos^2\theta^*)(|S_{\tau\tau}|^2 + |S_{-\tau\tau}|^2) \\ &\quad + 2\cos\theta^*(|S_{\tau\tau}|^2 - |S_{-\tau\tau}|^2)\}. \end{aligned} \tag{3.261}$$

The polarization of the lepton F^- follows from this expression, viz.

$$A_{\mathrm{Pol}}(\theta^*) = \frac{(\mathrm{d}\sigma/\mathrm{d}\Omega^*)_+ - (\mathrm{d}\sigma/\mathrm{d}\Omega^*)_-}{(\mathrm{d}\sigma/\mathrm{d}\Omega^*)_+ + (\mathrm{d}\sigma/\mathrm{d}\Omega^*)_-} \equiv \frac{a(1+\cos^2\theta^*) + b\cos\theta^*}{c(1+\cos^2\theta^*) + d\cos\theta^*}. \tag{3.262}$$

In the neighbourhood of the Z pole the terms in $|\kappa(s)|^2$ dominate. Neglecting all other contributions the real constants $a, \ldots, d$ are seen to be given by

$$\begin{aligned} a &= |S_{++}|^2 + |S_{-+}|^2 - |S_{--}|^2 - |S_{+-}|^2 = -8c_{\mathrm{V}}^{(\mathrm{F})}c_{\mathrm{A}}^{(\mathrm{F})}(c_{\mathrm{V}}^{(\mathrm{e})2} + c_{\mathrm{A}}^{(\mathrm{e})2})|\kappa(s)|^2 \\ b &= 2(|S_{++}|^2 - |S_{-+}|^2 - |S_{--}|^2 + |S_{+-}|^2) = -8c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{A}}^{(\mathrm{e})}(c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{F})2})|\kappa(s)|^2 \\ c &= |S_{++}|^2 + |S_{-+}|^2 + |S_{--}|^2 + |S_{+-}|^2 = 4(c_{\mathrm{V}}^{(\mathrm{e})2} + c_{\mathrm{A}}^{(\mathrm{e})2})(c_{\mathrm{V}}^{(\mathrm{F})2} + c_{\mathrm{A}}^{(\mathrm{F})2})|\kappa(s)|^2 \\ d &= 2(|S_{++}|^2 - |S_{-+}|^2 + |S_{--}|^2 - |S_{+-}|^2) = 16c_{\mathrm{V}}^{(\mathrm{e})}c_{\mathrm{A}}^{(\mathrm{e})}c_{\mathrm{V}}^{(\mathrm{F})}c_{\mathrm{A}}^{(\mathrm{F})}|\kappa(s)|^2. \end{aligned}$$

Finally, introducing the definitions (3.257) we obtain the simple result

$$A_{\mathrm{Pol}}(\theta^*) = \frac{A_{\mathrm{F}}(1+\cos^2\theta^*) + 2A_{\mathrm{e}}\cos\theta^*}{(1+\cos^2\theta^*) + 2A_{\mathrm{e}}A_{\mathrm{F}}\cos\theta^*}. \tag{3.263}$$

In practice, this quantity can and has been measured for τ-leptons, $\mathrm{F} = \tau$, in the final state. Two alternative observables can be derived from this result:

(i) The average polarization is obtained by integrating (3.261) over Ω^*,

$$\sigma_\tau = \int \mathrm{d}\Omega^* \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}\right)_\tau ,$$

and by calculating the ratio $(\sigma_+ - \sigma_-)/(\sigma_+ + \sigma_-)$. One finds

$$\overline{A_{\mathrm{Pol}}} = A_{\mathrm{F}} \equiv -P_l(\mathrm{F}), \tag{3.264}$$

with $P_l(\mathrm{F})$ the longitudinal polarization of the lepton F.

(ii) The asymmetry between the cross sections for $\tau = +$ and $\tau = -$, integrated piecewise over the forward hemisphere and over the backward hemisphere, is found to be

$$\frac{\int_0^1 \mathrm{d}(\cos\theta^*)(\mathrm{d}\sigma_+ - \mathrm{d}\sigma_-)/\mathrm{d}\Omega^* - \int_{-1}^0 \mathrm{d}(\cos\theta^*)(\mathrm{d}\sigma_+ - \mathrm{d}\sigma_-)/\mathrm{d}\Omega^*}{\int_{-1}^1 \mathrm{d}(\cos\theta^*)(\mathrm{d}\sigma_+ + \mathrm{d}\sigma_-)/\mathrm{d}\Omega^*} = \frac{3}{4}A_{\mathrm{e}}. \tag{3.265}$$

The examples we have chosen show that there is a set of observables which are sensitive to the detailed structure of the leptonic weak neutral current.

Radiative corrections, i.e. contributions of higher order than that of the tree-level diagrams, are important and must be added to these results before a quantitative comparison with the data is possible. One shows, however, that to a good approximation these corrections are taken care of by replacing the coupling constants by effective values without altering the form of the formulae obtained at tree level. With this caveat, the experiments which determined these quantities as well as further observables that follow from our formulae gave perfect agreement with the predictions that follow from the standard model with the expressions (3.249) for the coupling constants.

3.7.3 Precision Tests of Electroweak Interactions

Experiments at the large colliders near the Z pole have yielded impressive progress in the task of testing the standard model of electroweak interactions. In fact, one of the primary aims of LEP and SLC, which at the time were dubbed "Z^0 factories", was to test the gauge theory describing electroweak interactions as a renormalizable quantum field theory, i.e. to test it at the level of the radiative corrections that it predicts. In this section we illustrate the potential of these experiments by means of two characteristic examples: (i) determination of the Lorentz structure of the leptonic weak neutral current, i.e. the test of the formulae (3.249) for the vector and axial vector couplings; (ii) counting of lepton generations from the knowledge of the total width of the Z^0, and its partial decay widths into charged leptons and into hadrons.

(i) In Sect. 3.6.2. we calculated the cross sections for the neutrino scattering processes (3.218). As illustrated by Fig. 3.7, the simultaneous analysis of these

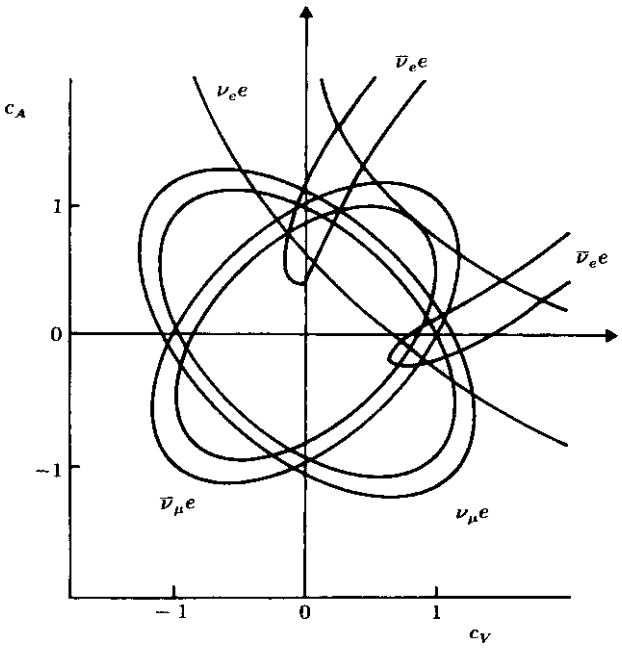

Fig. 3.10 Determination of effective coupling constants from elastic neutrino reactions, including experimental error bands

cross sections allows one to determine the couplings (3.249), at least in principle.[9] Detecting neutrinos is much more difficult than detecting charged leptons or hadrons. Therefore, the results are of limited accuracy regarding the determination of c_V and c_A. Figure 3.10 shows the best result that was obtained in this way from $\nu_\mathrm{e}\mathrm{e}$, $\bar{\nu}_\mathrm{e}\mathrm{e}$, $\nu_\mu\mathrm{e}$, and $\bar{\nu}_\mu\mathrm{e}$ elastic scattering. Figure 3.11 shows the combined results for leptons, as obtained by experiments at LEP. Here the leptonic branching ratios R_l, the forward–backward asymmetries (3.255), the average τ polarization (3.264), and its forward-backward asymmetry (3.265) were analyzed in terms of the couplings (3.249). Note the different scales in Figs. 3.10 and 3.11. The improvement in accuracy is indeed very impressive.

(ii) One of the most spectacular results of experiments at the Z pole was the determination of the number of leptonic generations. The analysis goes as follows. Suppose one has measured the total width Γ_Z, the cross section into hadrons at the pole position, as obtained from (3.251), viz.

[9] The remaining ambiguity between the solution ($c_\mathrm{V} \approx 0$, $c_\mathrm{A} \approx 1$) and ($c_\mathrm{V} \approx 1$, $c_\mathrm{A} \approx 0$) is resolved in favour of the former if one adds the information obtained from the forward–backward asymmetry in the process $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$, cf. Sect. 3.6.3.

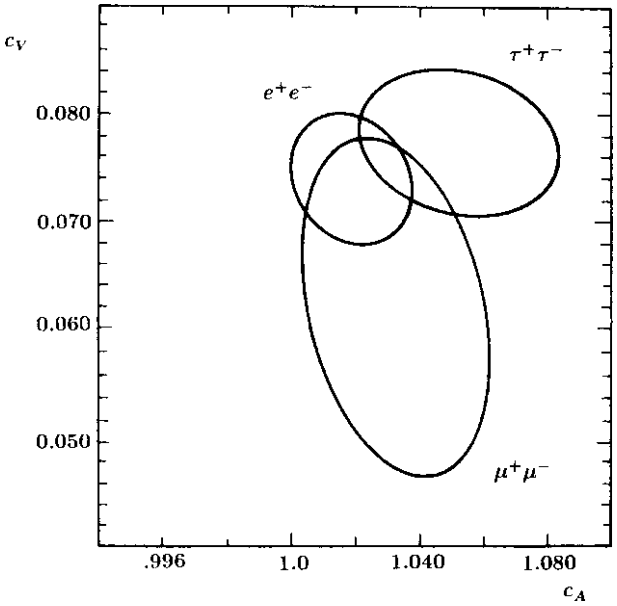

Fig. 3.11 Determination of c_A and c_v from $e^+e^- \to F^+F^-$ at the Z pole. Note the dramatic improvement in precision over neutrino reactions (Fig. 3.10)

$$\sigma^0_{\text{hadron}}(s = m_Z^2) = \frac{12\pi \Gamma_e \Gamma_{\text{hadrons}}}{m_Z^2 \Gamma_Z^2},$$

and the branching ratio $R_l = \Gamma_{\text{hadrons}}/\Gamma_l$ of the decay width into hadrons and the decay width into one species of charged lepton. These parameters may be replaced by the equivalent set ($\Gamma_{\text{invisible}}$, Γ_{hadrons}, Γ_l) where $\Gamma_{\text{invisible}}$ is the decay width into unseen, invisible, final states (i.e. into neutrinos),

$$\Gamma_{\text{invisible}} := \Gamma_Z - \Gamma_{\text{hadrons}} - \Gamma_l. \tag{3.266}$$

The invisible part of the Z^0 width is due to neutrinos which remained undetected. Let n_ν denote the number of (massless or light) neutrinos, Γ_ν the decay width (3.250) into a neutrino–antineutrino pair. Then $\Gamma_{\text{invisible}} = n_\nu \Gamma_\nu$. From (3.249) we know the ratio of the partial widths into a charged lepton pair $F^+ F^-$, and into a neutrino-antineutrino pair as predicted by the standard model,

$$\left(\frac{\Gamma_l}{\Gamma_\nu}\right)_{\text{St.M.}} = \frac{1}{2}[1 + (1 - 4\sin^2\theta_w)^2] \approx 0.5026,$$

where we have inserted the value $\sin^2\theta_w = 0.2319$. Inserting the measured values

$$\Gamma_{\text{invisible}} = 499.8\,\text{MeV},\ \Gamma_l = 83.96\,\text{MeV},$$

as well as the ratio above, one obtains

$$n_\nu = \frac{\Gamma_{\text{invisible}}}{\Gamma_l}\left(\frac{\Gamma_l}{\Gamma_\nu}\right)_{\text{St.M.}} = 2.99. \tag{3.267}$$

Thus, the number of lepton generations is found to be 3 (the error bar on the result (3.267) is less than 0.02).[10]

The simplified analysis given here is based on the assumption that the leptonic couplings are independent of the lepton family (lepton universality). However, the data have also been analysed without this assumption. The result was that universality is well confirmed.

In practice, the following set of primary parameters is used for a combined fit of the data around the Z pole

$$m_{\text{Z}}, \quad \Gamma_{\text{Z}}, \quad \sigma^0_{\text{hadron}}, \; R_l, \quad A_{\text{fb}}.$$

It turns out that this choice of parameters is optimal because they form the least correlated set. This has the advantage that data from different experiments can be combined into a general fit to the standard model including radiative corrections.

Finally, we note that experiments with Z^0 bosons have many more direct and indirect results. For instance, one determines partial widths for Z^0 decaying into $\text{b}\bar{\text{b}}$ and $\text{c}\bar{\text{c}}$ pairs. From these one obtains indirect information on the mass of the t-quark which is surprisingly accurate and, luckily, is found to be in agreement with the direct determination. Indeed, comparing all data for the Z^0 with the full standard model allows one to conclude that the mass of the t-quark is about 176 GeV, with an error bar of about 25 GeV. This is in excellent agreement with the direct determination of this mass which gave $m_{\text{t}} = 172.0 \pm 1.2\,\text{GeV}$.

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[10] We have calculated the width (3.250) on the assumption that the neutrino mass is zero or negligible. Taking this mass into account one finds that the conclusion from (3.267) remains unaltered even if one or several of the neutrino masses were rather large.

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Exercises

3.1. A quark and an antiquark of the same flavour form a bound state $(q\overline{q})^L_{l,s}$ with relative orbital angular momentum l and total spin S, coupled to angular momentum J. Show that this is an eigenstate of P and of C and give the corresponding eigenvalues. Apply the results to π^0, η, ρ^0, ω, ϕ, and a_2^0 mesons. What is the wave function of ω if ϕ contains only strange quarks? *Hint*: $(j_1 m_1 j_2 m_2 | JM) = (-)^{j_1+j_2-J}(j_2 m_2 j_1 m_1 | JM)$.

3.2. A meson is said to have natural parity if $P = (-)^J$. Show that in this case the bound states of exercise 3.1 necessarily have $P = C$.

3.3. Show that the matrices $\exp\{iH\}$ where $H = \Sigma_{i=1}^3 \alpha_i \sigma^{(i)}$ and α_i real are unitary and have determinant 1. *Hint*: First diagonalize the matrix H.

3.4. Show that in a local gauge theory built on the Lie algebra of $G = \mathrm{SU(P)} \times \mathrm{SU(Q)}$ there are two constants e_P and e_Q which can be chosen independently.

3.5. From (3.313) it is clear that A_α can be "gauged to zero", i.e. be transformed to $A'_\alpha \equiv 0$, if and only if there exists a $g(x)$ for which

$$-(\partial_\alpha g^{-1}(x))g(x) = A_\alpha(x).$$

This is a differential equation for $g(x)$ with a given inhomogeneity $A_\alpha(x)$. A condition of integrability for this equation is $(\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) g^{-1}(x) = 0$. Work this out and show that $A'_\alpha \equiv 0$ can be obtained if and only if the field tensor $F^{\alpha\beta}$ vanishes identically. What is the analogy to electrodynamics?

3.6. Work out the globally symmetric Lagrangian (3.125) as well as the locally invariant version (3.128) for the case $G = \mathrm{SO}(3)$ and with real boson and fermion multiplets.

3.7. Derive the matrix (3.152) that describes state mixing of the quarks d, s, b. *Hints*: The matrix V can be written as a product of three unitary matrices, $V_1(2)V_2(1)V_1(3)$, where $V_i(k) \equiv V_i(\psi_k, \theta_k, \phi_k)$ leaves invariant the component i. Some of the resulting phases can be absorbed in the fields.

3.8. Write out explicitly the generalized kinetic Lagrangian for the gauge boson fields in the GSW theory. In particular, isolate the couplings of $W^{\pm}$ to the photon field and compare to what one would have obtained from a Klein–Gordon equation for the W-field supplemented by minimal substitution.

3.9. Starting from (3.172, 3.173) and the assignment $y = -2t_3$ for the neutral Higgs field, construct the mass matrix of the vector bosons in the basis of the fields $A_{\alpha}^{(\mu)}$. Diagonalize this matrix.

3.10. Consider neutrino-electron scattering as in Sect. 3.6.2. In the case dealt with in (3.230) calculate the longitudinal polarization of the outgoing neutrino.

3.11. Suppose you had data on $(\nu_e e)$ scattering (integrated elastic cross section). Analyze the cross section in the context (3.227, 3.228).

3.12. Consider the cross section

$$\frac{d\sigma}{d\Omega^*}(e^+e^- \to \tau^+\tau^-)$$

for $m_\tau \neq 0$.B_+ and B_- can now interfere. Calculate $\Sigma(A_+B_-)(A_+B_-)^*$ and the cross section noting that in (3.232) t and u are replaced by $(t - m_\tau^2)$ and $(u - m_\tau^2)$, respectively.

3.13. Suppose the Z^0 had only vector couplings to electrons and only axial vector couplings to muons. What asymmetry would one predict?

Chapter 4
Beyond the Minimal Standard Model

The GSW theory is a great step forward in our understanding of electroweak interactions because it allows the well-known extremely successful theory of quantized electrodynamics and the theory of the weak CC and NC interactions to be cast into one unified, renormalizable local gauge theory. Renormalizability, in particular, is a very desirable property of the theory because it makes covariant perturbation theory a reasonable and well-defined approximation method for calculating physical quantities beyond the lowest order diagrams. Nevertheless, this model, very likely, is not the corner stone of a final theory of weak and electromagnetic interactions. It contains very many parameters which are not predicted and whose origin remains unclear. The most prominent and specific properties of the weak interactions are built into the model (see e.g. the discussion in Sects. 3.4.1 a,b) and are not predicted. One of these is parity violation: The fact that QED conserves parity whereas (bare) CC interactions as well as neutrino induced NC interactions break parity maximally, is introduced into the theory by hand. There is not even a hint at an answer to the question of why right-handed neutrino states decouple from the physical world. Furthermore, the accuracy to which some of the empirical information on weak interactions is known, is limited, and there is indeed room for deviations from this minimal picture.

4.1 Leptonic Charged and Neutral Current Interactions

Encouraged by the success of the basic concepts of gauge theories, many authors have proposed enlarged unified models of the elementary interactions which contain the GSW model as a limiting case but exhibit specific deviations from its predictions. While some of their features, such as the existence of further massive weak gauge bosons, can be tested through direct search at very high energy accelerators, others can be established (or disproved) by experiments of high precision at low and intermediate energies. Without going into these generalized models, we provide here the general frame of testing for specific deviations from the simple V–A picture

F. Scheck, *Electroweak and Strong Interactions*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-20241-4_4,

of CC interactions and the minimal neutral interactions of the GSW model. We give some typical examples referring to muon decay and NC interactions in atomic systems.

4.1.1 Effective SPVAT Interactions

As we saw earlier, weak interactions have very short ranges, of the order of 10^{-16}cm. For the application to processes at low and intermediate energies this means that the interaction effectively acts like a contact interaction that connects four fermion fields at the same point x in Minkowski space. In Sect. 3.2.2 we discussed the example of effective V and A interactions, cf. (3.64) and studied their behaviour under the discrete symmetries P, C, and T, in some detail. In this section we extend these considerations to the case of scalar, pseudoscalar and tensor interactions which are the only other Lorentz structures that can be formed on the basis of spinor fields.

a) Scalars and pseudoscalars

In analogy to the definitions (3.59, 3.60) let us introduce the following covariants

$$s(x; i \to k) = \overline{\psi^{(k)}(x)}\, \psi^{(i)}(x), \tag{4.1a}$$

$$p(x; i \to k) = \overline{\psi^{(k)}(x)}\, \gamma_5 \psi^{(i)}(x), \tag{4.1b}$$

When written in terms of spinors of first and second kind, the operators (4.1) have the structure

$$\left.\begin{matrix} s \\ p \end{matrix}\right\} = \chi^{(k)*a}(x)\phi^{(i)}{}_a(x) \pm \phi^{(k)*}{}_A(x)\chi^{(i)A}(x). \tag{4.2}$$

With the aid of (3.50) it is easy to show that s is a scalar under Lorentz transformations,

$$s(x; i \to k) \underset{P}{\longrightarrow} s(Px; i \to k), \tag{4.3a}$$

whilst p is a pseudoscalar,

$$p(x;\ i \to k) \underset{P}{\longrightarrow} -p(Px; i \to k). \tag{4.3b}$$

The transformation of s and p under charge conjugation is derived in the same way as for (3.55). One finds

$$s(x; i \to k) \underset{C}{\longrightarrow} s(x; k \to i), \tag{4.4a}$$

$$p(x; i \to k) \underset{C}{\longrightarrow} p(x; k \to i). \tag{4.4b}$$

The behaviour of s and p under time reversal is derived along the same lines as for the case of vector and axial vector covariants, cf. Sect. 3.2.1. For example,

$$\chi^{(k)*a}(x)\phi_a^{(i)}(x) \underset{T}{\longrightarrow} \varepsilon_{BD}(\hat{\sigma}^0)^{Da}(\sigma^0)_{aF}\phi^{(i)*F}(Tx)\chi^{(k)B}(Tx)$$
$$= \phi^{(i)*}{}_B(Tx)\chi^{(k)B}(Tx),$$

so that

$$s(x; i \to k) \underset{T}{\longrightarrow} s(Tx; k \to i), \tag{4.5a}$$
$$p(x; i \to k) \underset{T}{\longrightarrow} -p(Tx; k \to i). \tag{4.5b}$$

The transformation properties (4.3–4.5) hold to within the phase factors discussed in Sect. 3.2.1. We can now formulate the most general Lagrangian built from scalar and pseudoscalar covariants, which is invariant under proper orthochronous Lorentz transformations. Dropping the common argument x in the covariants it reads

$$\begin{aligned} -\mathscr{L}_{\mathrm{SP}} = \frac{G}{\sqrt{2}}\{&s(i \to k)[C_{\mathrm{S}}s(n \to m) + C'_{\mathrm{S}}p(n \to m)] \\ &+ [C^*_{\mathrm{S}}s(m \to n) - C^*_{\mathrm{S}}p(m \to n)]s(k \to i) \\ &- p(i \to k)[C_{\mathrm{P}}p(n \to m) + C'_{\mathrm{p}}s(n \to m)] \\ &- [C^*_{\mathrm{P}}p(m \to n) - C'^*_{\mathrm{P}}s(m \to n)]p(k \to i)\} \end{aligned} \tag{4.6}$$

For the sake of convenience, we have taken out a factor $G/\sqrt{2}$ such as to have the $C_{\mathrm{S}}, \ldots, C'_{\mathrm{P}}$ dimensionless and to make (4.6) directly comparable to L_{VA}, (3.64). The minus sign in front of C'^*_S and of C'^*_{P} is due to hermitean conjugation,

$$(\overline{\Psi^{(k)}}\gamma_5\,\Psi^{(i)})^\dagger = \overline{\Psi^{(i)}}\gamma^0\gamma_5\gamma^0\,\Psi^{(k)} = -\overline{\Psi^{(i)}}\gamma_5\,\Psi^{(k)}$$

whereas the minus sign of the pseudoscalar terms (third and fourth term on the r.h.s) is a matter of convention: It is useful (but not generally adopted) to define pseudoscalar invariants with an extra factor i,

$$\overline{\Psi^{(k)}(x)}\Gamma_P\Psi^{(i)}(x) \quad \text{with } \Gamma_{\mathrm{P}} := \mathrm{i}\gamma_5. \tag{4.7}$$

The advantage of this choice is evident in calculating traces because

$$\gamma^0\Gamma^\dagger_{\mathrm{P}}\gamma^0 = \Gamma_{\mathrm{p}}$$

and one need not worry about extra signs. Since we shall use this convention in the sequel but do not wish to redefine C_{P} and C'_{P} then, we introduce an i^2 already at this point.

It is easy to verify that the behaviour of $\mathscr{L}_{\mathrm{SP}}$ with respect to the discrete symmetries P, C, and T, is precisely as indicated in (3.66–3.68). In particular, $\mathscr{L}_{\mathrm{SP}}$ obeys the *PCT*-theorem in the same form as $\mathscr{L}_{\mathrm{VA}}$, equation (3.70).

b) Tensors and pseudotensors

A third type of covariants that can be defined by forming bilinears of two fermion fields are the following

$$t^{\alpha\beta}(x;\, i \to k) := \overline{\Psi^{(k)}(x)} \frac{1}{\sqrt{2}} \sigma^{\alpha\beta} \Psi^{(i)}(x), \tag{4.8a}$$

$$t'^{\alpha\beta}(x; i \to k) := \overline{\Psi^{(k)}(x)} \frac{1}{\sqrt{2}} \sigma^{\alpha\beta} \gamma_5 \Psi^{(i)}(x), \tag{4.8b}$$

where $\sigma^{\alpha\beta}$ stands for an antisymmetric product of γ-matrices which is generally defined by

$$\sigma^{\alpha\beta} := \frac{\mathrm{i}}{2}(\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha). \tag{4.9}$$

[The extra factor $1/\sqrt{2}$ in (4.8) is a matter of convention, see below].

The operator $t^{\alpha\beta}$ transforms like a Lorentz tensor, $t'^{\alpha\beta}$ transforms like a Lorentz pseudotensor, i.e. like a tensor times the determinant of the homogeneous Lorentz transformation. This can be seen most easily if we write these operators in terms of spinor fields ϕ and χ. For this purpose we decompose them as follows

$$t^{\alpha\beta} = \frac{1}{\sqrt{2}}(f^{\alpha\beta} + g^{\alpha\beta}), \quad t'^{\alpha\beta} = \frac{1}{\sqrt{2}}(f^{\alpha\beta} - g^{\alpha\beta}),$$

with

$$f^{\alpha\beta} := \chi^{(k)*a} \frac{\mathrm{i}}{2} \{(\sigma^\alpha)_{aB}(\hat{\sigma}^\alpha)^{Bd} - (\sigma^\beta)_{aB}(\hat{\sigma}^\alpha)^{Bd}\} \phi_d^{(i)}, \tag{4.10a}$$

$$g^{\alpha\beta} := \phi^{(k)*}{}_A \frac{\mathrm{i}}{2} \{(\hat{\sigma}^\alpha)^{Ab}(\sigma^\beta)_{bD} - (\hat{\sigma}^\beta)^{Ab}(\sigma^\alpha)_{bD}\} \chi^{(i)D}. \tag{4.10b}$$

Let us calculate the divergence ∂_β of these latter quantities. Making use of the relation (1.64) and noting that $f^{\alpha\beta}$ vanishes if $\alpha = \beta$, one obtains

$$\partial_\beta f^{\alpha\beta} = \frac{\mathrm{i}}{2} \chi^{(k)*} \{2\sigma^\alpha \hat{\sigma}^\beta - 2g^{\alpha\beta}\} \partial_\beta \phi^{(i)} - \frac{\mathrm{i}}{2} (\partial_\beta \chi^*) \{2\sigma^\beta \hat{\sigma}^\alpha - 2g^{\alpha\beta}\} \phi^{(i)}.$$

This expression can be simplified by means of the Dirac equations (1.69), giving

$$\partial_\beta f^{\alpha\beta} = m_i \chi^{(k)*} \sigma^\alpha \chi^{(i)} + m_k \phi^{(k)*} \hat{\sigma}^\alpha \phi^{(i)} - \mathrm{i} \chi^{(k)*} \overset{\leftrightarrow}{\partial}{}^\alpha \phi^{(i)}.$$

In a similar fashion one shows

$$\partial_\beta g^{\alpha\beta} = m_i \phi^{(k)*} \hat{\sigma}^\alpha \phi^{(i)} + m_k \chi^{(k)*} \sigma^\alpha \chi^{(i)} - \mathrm{i} \phi^{(k)*} \overset{\leftrightarrow}{\partial}{}^\alpha \chi^{(i)}.$$

From these equations we obtain

$$\begin{aligned} \partial_\beta t^{\alpha\beta} &= \frac{1}{\sqrt{2}}(m_k + m_i)\{\chi^{(k)*}\sigma^\alpha \chi^{(i)} + \phi^{(k)*}\hat{\sigma}^\alpha \phi^{(i)}\} \\ &\quad - \frac{\mathrm{i}}{\sqrt{2}}\{\chi^{(k)*} \overset{\leftrightarrow}{\partial}{}^\alpha \phi^{(i)} + \phi^{(k)*} \overset{\leftrightarrow}{\partial}{}^\alpha \chi^{(i)}\}, \qquad (4.11a) \\ \partial_\beta t'^{\alpha\beta} &= \frac{1}{\sqrt{2}}(m_k - m_i)\{-\chi^{(k)*}\sigma^{(i)} \chi^{(i)} + \phi^{(k)*}\hat{\sigma}^\alpha \phi^{(i)}\} \\ &\quad - \frac{\mathrm{i}}{\sqrt{2}}\{\chi^{(k)*} \overset{\leftrightarrow}{\partial}{}^\alpha \phi^{(i)} - \phi^{(k)*} \overset{\leftrightarrow}{\partial}{}^\alpha \chi^{(i)}\}. \qquad (4.11b) \end{aligned}$$

Looking back to at our discussion of vector and axial vector operators in Sect. 3.2.1 the behaviour of $t^{\alpha\beta}$ and $t'^{\alpha\beta}$ under (proper and improper) Lorentz transformations is now obvious

$$t^{\alpha\beta}(\Lambda x) = \Lambda^\alpha{}_\sigma \Lambda^\beta{}_\tau t^{\sigma\tau}(x), \qquad (4.12a)$$

$$t'^{\alpha\beta}(\Lambda x) = (\det \Lambda)\Lambda^\alpha{}_\sigma \Lambda^\beta{}_\tau t'^{\sigma\tau}(x). \qquad (4.12b)$$

The divergences (4.11) contain the vector and the axial-vector operators (3.59) and (3.60), respectively. This fact may be utilized to derive the behaviour of t and t' under charge conjugation. With respect to C, $\upsilon^\alpha(x; i \to k)$ is odd. Therefore, from (4.11a), $t^{\alpha\beta}$ is also odd under C. The axial vector $a^\alpha(x; i \to k)$, which appears on the r.h.s. of (4.11b), is even. However, C also interchanges i and k and yields an extra minus sign from the antisymmetric factor in (4.11b). Thus $t'^{\alpha\beta}$ is odd, too, and we have

$$t^{\alpha\beta}(x;\, i \to k) \underset{C}{\to} -t^{\alpha\beta}(x; k \to i), \qquad (4.13a)$$

$$t'^{\alpha\beta}(x; i \to k) \underset{C}{\to} -t'^{\alpha\beta}(x; k \to i). \qquad (4.13b)$$

The behaviour with respect to T, finally, is contained in (4.12).

A rather useful relation in calculating matrix elements of tensor and pseudotensor covariants is the following

$$\sigma^{\alpha\beta}\gamma_5 = -\frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta\mu\nu}\sigma_{\mu\nu}, \qquad (4.14)$$

where $\varepsilon_{\alpha\beta\mu\nu}$ is the totally antisymmetric tensor in four dimensions, with $\varepsilon_{0123} = +1$.

As for the case of V, A and S, P covariants, the general Lagrangian which is invariant under proper orthochronous Lorentz transformations has the form

$$-\mathscr{L}_{\mathrm{T}} = \frac{G}{\sqrt{2}}\{t^{\alpha\beta}(i \to k)[C_T t_{\alpha\beta}(n \to m) + C'_{\mathrm{T}} t'_{\alpha\beta}(n \to m)] + [C^*_{\mathrm{T}} t_{\alpha\beta}(m \to n) - C'^*_{\mathrm{T}} t'_{\alpha\beta}(m \to n)] t^{\alpha\beta}(k \to i)\}. \tag{4.15}$$

In writing the hermitean conjugate of the first term on the r.h.s. of (4.15) we have made use of the equations

$$\gamma^0(\sigma^{\alpha\beta})^\dagger\gamma_0 = \sigma^{\alpha\beta}, \quad \gamma^0(\sigma^{\alpha\beta}\gamma_5)^\dagger\gamma^0 = -\sigma^{\alpha\beta}\gamma_5.$$

Here too, one verifies that the behaviour of $\mathscr{L}_T$ under the discrete symmetries is that one of (3.66–3.68), and that $\mathscr{L}_{\mathrm{T}}$ obeys the *PCT*-theorem (3.70).

Remark: The relation (4.14) shows that the pseudotensor can be expressed in terms of another tensor term. Indeed, γ_5 is i times the product of four distinct γ-matrices two of which must be identical with the ones appearing in $\sigma^{\alpha\beta}$. Therefore, one may wonder why we distinguish the pseudotensor from the tensor and, in particular, why there are two different coupling constants C_{T} and C'_{T} in the effective Lagrangian (4.15), or (4.20) below, not one. An easy way to understand this point is to note that if one talks about tensor terms only, barring the pseudotensors altogether, then there are two independent, Lorentz-invariant ways of coupling two tensor terms, i.e.

$$t^{\alpha\beta} g_{\alpha\sigma} g_{\beta\tau} t^{\alpha\tau} \text{ and } t^{\alpha\beta}\varepsilon_{\alpha\sigma\beta\tau}t^{\sigma\tau}$$

One verifies, making use of relation (4.14), that the second term can be written alternatively in terms of the contraction of a tensor and a pseudotensor $t^{\alpha\beta}t'_{\alpha\beta}$. In practice, either representation is useful and both of them are being used in actual calculations.

There is a close analogy to electrodynamics where $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\widetilde{F}^{\mu\nu} = F_{\mu\nu}\varepsilon^{\mu\nu\sigma\tau}F_{\sigma\tau}$ are independent invariants constructed from the field strength tensor F and its dual $\tilde{F}$. Only the first of them appears in the Lagrangian (1.189) because electromagnetic interactions conserve parity.

c) General four-fermion contact interaction

With the tools of Sects. 3.2.2, 4.1.1a,b at hand we can now formulate the general, effective interaction Lagrangian connecting four fermion field operators $\Psi^{(\mathrm{f}_1)}(x)$ to $\Psi^{(\mathrm{f}_4)}(x)$ at the same point of space-time. For this purpose let us introduce the following, somewhat symbolic, notation.

$$\Gamma_{\mathrm{S}} \equiv \mathbb{1}, \quad \Gamma_{\mathrm{P}} \equiv \mathrm{i}\gamma_5, \tag{4.16}$$

$$\Gamma_{\mathrm{V}} \equiv \gamma^\alpha, \quad \Gamma_{\mathrm{A}} \equiv \gamma^\alpha\gamma_5, \tag{4.17}$$

$$\Gamma_{\mathrm{T}} \equiv \frac{1}{\sqrt{2}}\sigma^{\alpha\beta}, \tag{4.18}$$

with $\sigma^{\alpha\beta}$ as defined by (4.9) above. Note, in particular, the factor i in our definition of Γ_{P}. This operator, like all others in (4.16–4.18), has the property

$$\gamma^0(\Gamma_i)^\dagger\gamma^0 = \Gamma_i, \quad i = \mathrm{S,P,V,A,T}, \tag{4.19a}$$

which is particularly convenient in practical calculations. Besides these operators, also the products of $\Gamma_i\gamma_5$ will appear in the Lagrangian. The analogous relation of conjugation for these products is easily derived from (4.19a), viz.

$$\gamma^0(\Gamma_i\gamma_5)^\dagger\gamma^0 = (\gamma^0\gamma_5\gamma^0)(\gamma^0\Gamma_i^\dagger\gamma^0) = -\gamma_5\Gamma_i.$$

Depending on whether γ_5 commutes or anticommutes with Γ_i, this is equal to minus or plus $\Gamma_i\gamma_5$. Thus

$$\gamma^0(\Gamma_i\gamma_5)^\dagger\gamma^0 = \Gamma_i\gamma_5, \quad \text{for}\, i = \mathrm{V,A}, \tag{4.19b}$$

$$\gamma^0(\Gamma_k\gamma_5)^\dagger\gamma^0 = -\Gamma_k\gamma_5, \quad \text{for}\, k = \mathrm{S,P,T}. \tag{4.19c}$$

Denoting the field operators by the symbol of the particles that they describe, the most general effective Lagrangian (which does not contain derivative couplings) reads

$$-\mathscr{L} = \frac{G}{\sqrt{2}}\sum_i\{(\overline{f_1(x)}\Gamma_i f_2(x))[C_i(\overline{f_3(x)}\Gamma^i f_4(x)) + C_i'(\overline{f_3(x)}\Gamma^i\gamma_5 f_4(x))] + \mathrm{h.c.}\}. \tag{4.20}$$

It is understood that the fields $f_1, \ldots, f_4$ are selected such that $\mathscr{L}$ is electrically neutral and is a scalar with respect to all internal symmetries for which one wishes to impose a conservation law.

The Lagrangian (4.20) contains ten complex quantities, two for each Lorentz structure, i.e. twenty real constants. Only nineteen of these are physically relevant because an unobservable common phase can always be factored out. If all of these constants are different from zero then $\mathscr{L}$ neither conserves parity, nor charge conjugation, nor time reversal symmetry. If one of these discrete symmetries is conserved the number of coupling constants is reduced by a factor of two as can be seen from (3.66–3.68). For instance, if we impose T-invariance, (3.68) shows that all C_i and C_i' must be real. From the PCT-theorem (3.70) we then see that either P and C are both violated, in which case at least some unprimed and some primed coupling constants are different from zero, or both are conserved in which case there can be either unprimed or primed couplings, but not both.

Finally, it is instructive to convince oneself that the tensor and the pseudotensor (4.15) are indeed independent couplings, even though the pseudotensor does not appear in the set (4.16–4.18).

d) Fierz transformations

Of the Lorentz structures S,...,T that we considered above, the tensor is somewhat exotic but the scalar and pseudo-scalar are not. Indeed, if all fundamental interactions (except gravitation) are mediated by particles with spin 1 there cannot be an effective interaction of the type (4.15). The vector particles couple to V and A currents. So where do S and P covariants come in?

In order to see this, let us return to the general contact interaction (4.20). Once we have ascertained that the theory fulfills the proper conservation laws by suitably combining the quantum numbers of particles f_1 to f_4, we can write the field operators in the ordering

$$(\overline{f}_1 \cdots f_4)(\overline{f}_3 \cdots f_2)$$

as well, without changing any of the invariances of the Lagrangian. However, the specific linear combination of covariants in (4.20) is not the same combination when written in the reordered form.

Let us write the *same* Lagrangian (4.20) in a form where the field operators $f_2(x)$ and $f_4(x)$ are interchanged, viz.

$$-\mathscr{L} = \frac{G}{\sqrt{2}} \sum_i \{(\overline{f_1}\, \Gamma_i\, f_4)[D_i\, (\overline{f_3}\, \Gamma^i\, f_2) + D'_i (\overline{f_3}\, \Gamma^i\, \gamma_5 f_2)] + \text{h.c.}\}. \tag{4.21}$$

This reordering of operators (first studied by M. Fierz, 1936) maps the constants $\{C_i, C'_i\}$ onto the constants $\{D_i, D'_i\}$ according to the linear substitutions

$$\begin{pmatrix} D_\mathrm{S} \\ D_\mathrm{P} \\ D_\mathrm{V} \\ D_\mathrm{A} \\ D_\mathrm{T} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 1 & -4 & 4 & -6 \\ 1 & -1 & -4 & 4 & 6 \\ -1 & -1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ -1 & 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} C_\mathrm{S} \\ C_\mathrm{P} \\ C_\mathrm{V} \\ C_\mathrm{A} \\ C_\mathrm{T} \end{pmatrix}, \tag{4.22}$$

$$\begin{pmatrix} D'_\mathrm{S} \\ D'_\mathrm{P} \\ D'_\mathrm{V} \\ D'_\mathrm{A} \\ D'_\mathrm{T} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 1 & 4 & -4 & -6 \\ 1 & -1 & 4 & -4 & 6 \\ 1 & 1 & 2 & 2 & 0 \\ -1 & -1 & 2 & 2 & 0 \\ -1 & 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} C'_\mathrm{S} \\ C'_\mathrm{P} \\ C'_\mathrm{V} \\ C'_\mathrm{A} \\ C'_\mathrm{T} \end{pmatrix}. \tag{4.23}$$

[These transformations contain an extra minus sign due to the interchange of the two anticommuting fermion fields f_2 and f_4]. In a short-hand notation we shall also write these as

$$D_i = \sum_k M_{ik} C_k, \quad D'_i = \sum_k F_{ik} C'_k. \tag{4.24}$$

The matrices M and F have a number of important properties:

(i) The Fierz transformation from (4.20) to (4.21), of course, is the same as from (4.21) to (4.20). Therefore, M and F are each equal to their own inverse:

$$M^2 = \mathbb{1}, \qquad F^2 = \mathbb{1}.$$

(ii) It is remarkable that there is no matrix element connecting V, A and T. Thus, if there are only V and A couplings in the Lagrangian (4.20), i.e. $C_i = 0, C_i' = 0$ for $i = \mathrm{S}, \mathrm{P}, \mathrm{T}$, then (4.21) contains VA as well as SP couplings but no tensor couplings, viz.

$$\begin{aligned} D_\mathrm{S} &= D_\mathrm{P} = -C_\mathrm{V} + C_\mathrm{A}, \\ D_\mathrm{V} &= D_\mathrm{A} = \frac{1}{2}(C_\mathrm{V} + C_\mathrm{A}), \\ D_\mathrm{S}' &= D_\mathrm{P}' = C_\mathrm{V}' - C_\mathrm{A}', \\ D_\mathrm{V}' &= D_\mathrm{A}' = \frac{1}{2}(C_\mathrm{V}' + C_\mathrm{A}'), \\ D_\mathrm{T} &= D_\mathrm{T}' = 0 \end{aligned} \tag{4.25}$$

(iii) In particular, if in the representation (4.20) the interaction is of the form "V $\pm$ A", by which we mean

$$C_0(\overline{f}_1\gamma^\alpha(1 \pm \gamma_5) f_2)(\overline{f}_3\gamma_\alpha(1 \pm \gamma_5) f_4), \tag{4.26a}$$

then

$$C_\mathrm{V} = C_\mathrm{A} = C_0, \quad C_\mathrm{V}' = C_\mathrm{A}' = \pm C_0,$$

and, from (4.25),

$$\begin{aligned} &D_\mathrm{S} = D_\mathrm{P} = 0, \quad D_\mathrm{S}' = D_\mathrm{P}' = 0, \quad D_\mathrm{T} = D_\mathrm{T}' = 0, \\ &D_\mathrm{V} = D_\mathrm{A} = C_0, \quad D_\mathrm{V}' = D_\mathrm{A}' = \pm C_0. \end{aligned} \tag{4.26b}$$

Thus, the reordered Lagrangian (4.21) has the same form "V $\pm$ A" as the original (4.20), "V + A" is mapped onto "V + A", "V − A" onto "V − A".

If, on the other hand, the original Lagrangian is

$$C_0(\overline{f}_1(1 \pm \gamma_5) f_2)(\overline{f}_3(1 \pm \gamma_5) f_4), \tag{4.27a}$$

that is $C_\mathrm{s} = -C_\mathrm{p} = C_0, C_\mathrm{S}' = -C_\mathrm{P}' = \pm C_0$ (all others vanishing), then there are no V and A couplings in the reordered form (4.21), $D_\mathrm{V} = D_\mathrm{A} = D_\mathrm{V}' = D_\mathrm{A}' = 0$ and

$$D_\mathrm{S} = -D_\mathrm{P} = D_\mathrm{T} = -\frac{1}{2}C_0, \quad D_\mathrm{S}' = -D_\mathrm{P}' = D_\mathrm{T}' = \mp\frac{1}{2}C_0. \tag{4.27b}$$

[Clearly, these results have to do with the specific helicity selection rules of the covariants, cf. Sect. 3.1.2c.]

We see from these examples that if the interaction is of V and A type in one ordering, it contains S, P, V and A in the other, but no T, cf. (4.25). Only if it is precisely "V − A", or "V + A", are there no S and P terms in the reordered form, cf. (4.26). Thus, in a theory where all interactions are mediated by spin-1 bosons, the effective contact Lagrangian will not contain tensor couplings, no matter in which order the interaction is written. However, if the theory also contains genuine scalar and/or pseudo-scalar interactions then tensor couplings do occur in a Fierz reordering of the interaction Lagrangian, cf. (4.27).

*e) *Proof of Fierz reordering relations*

This section contains the proof of equations (4.22, 4.23). As this is somewhat technical the reader may wish to skip it in a first reading and go directly to the next section. However, even in passing, it may be worth noting relations (4.30, 4.32, 4.36) which are *invariant* linear combinations with respect to reordering.

There is a useful relation for the direct product of two σ-matrices (1.26) contracted over their Lorentz indices, which one derives by verification, (cf. exercise 1.8):

$$(\sigma^\alpha)_{aB}(\sigma_\alpha)_{dE} = -2\varepsilon_{ad}\varepsilon_{BE}. \tag{4.28}$$

As this equation is antisymmetric in B and E, one deduces immediately the relation

$$(\sigma^\alpha)_{aB}(\sigma_\alpha)_{dE} + (\sigma^\alpha)_{aE}(\sigma_\alpha)_{dB} = 0. \tag{4.28a}$$

It is easy to derive analogous relations for the matrices $\hat{\sigma}^\alpha$ by means of eq. (1.60). In particular, the analogue of (4.28a) reads

$$(\hat{\sigma}^\alpha)^{Ab}(\hat{\sigma}_\alpha)^{De} + (\hat{\sigma}^\alpha)^{Ae}(\hat{\sigma}_\alpha)^{Db} = 0. \tag{4.28b}$$

Using the hermiticity of σ_α, by which $(\sigma_\alpha)_{dE} = (\sigma_\alpha^*)_{Ed}$, and multiplying (4.28) by $\varepsilon^{DE}\varepsilon^{df}$ we obtain, finally,

$$(\sigma^\alpha)_{aB}(\hat{\sigma}_\alpha)^{Df} = 2\delta_a^f\delta_B^D. \tag{4.29}$$

It is convenient to write the covariants S,...,T in terms of two-component spinors and to combine them such as to project onto spinor fields of first and second kind. Thus, from (4.2)

$$(s+p)_{ki} = 2\chi^{(k)*a}\phi_a^{(i)},$$
$$(s-p)_{ki} = 2\phi^{(k)*}{}_A\chi^{(i)A},$$

where s and p are defined by (4.1) (p still without the factor i). Similarly, from (3.51, 3.53) we have

$$(\upsilon^\alpha + a^\alpha)_{ki} = 2\phi^{(k)^*}{}_A(\hat{\sigma}^\alpha)^{Ab}\phi_b^{(i)},$$
$$(\upsilon^\alpha - a^\alpha)_{ki} = 2\chi^{(k)^*a}(\sigma^\alpha)_{aB}\chi^{(i)B}.$$

Finally, from (4.8, 4.10), one obtains

$$(t^{\alpha\beta} + t'^{\alpha\beta})_{ki} = \frac{\mathrm{i}}{\sqrt{2}}\chi^{(k)^*a}\{(\sigma^\alpha)_{aB}(\hat{\sigma}^\beta)^{Bd} - (\sigma^\beta)_{aB}(\hat{\sigma}^\alpha)^{Bd}\}\phi_d^{(i)},$$
$$(t^{\alpha\beta} - t'^{\alpha\beta})_{ki} = \frac{\mathrm{i}}{\sqrt{2}}\phi^{(k)^*}{}_A\{(\hat{\sigma}^\alpha)^{Ab}(\sigma^\beta)_{bD} - (\hat{\sigma}^\beta)^{Ab}(\sigma^\alpha)_{bD}\}\chi^{(i)D}.$$

It is not difficult to show that interference terms between $f^{\alpha\beta}$, (4.10a) and $\mathrm{g}_{\alpha\beta}$, (4.10b), vanish, i.e. $(f^{\alpha\beta})_{ki}(\mathrm{g}_{\alpha\beta})_{nm} = 0$. Therefore, one has the relation

$$(t'^{\alpha\beta})_{ki}(t'_{\alpha\beta})_{nm} = (t^{\alpha\beta})_{ki}(t_{\alpha\beta})_{nm},$$

and, for instance

$$(t^{\alpha\beta} \pm t'^{\alpha\beta})(t_{a\beta} \pm t'_{\alpha\beta}) = 2t^{\alpha\beta}(t_{\alpha\beta} \pm t'_{\alpha\beta}).$$

Consider first the symmetric combination

$$\frac{1}{4}(\upsilon^\alpha + a^\alpha)_{12}(\upsilon_\alpha + a_\alpha)_{34} = (\hat{\sigma}^\alpha)^{Ab}(\hat{\sigma}_\alpha)^{Df}\phi_A^{(1)*}\phi_b^{(2)}\phi_D^{(3)*}\phi_f^{(4)}.$$

Commuting the operators 2 and 4, and applying (4.28b), this is equal to

$$-(\hat{\sigma}^\alpha)^{Af}(\hat{\sigma}_\alpha)^{Db}\{-\phi^{(1)*}{}_A\phi_f^{(4)}\phi^{(3)*}{}_D\phi_b^{(2)}\} = \frac{1}{4}(\upsilon^\alpha + a^\alpha)_{14}(\upsilon_\alpha + a_\alpha)_{32}.$$

A similar relation holds for the combination "V–A", so that we have

$$(\upsilon^\alpha \pm a^\alpha)_{12}(\upsilon_\alpha \pm a_\alpha)_{34} = (\upsilon^\alpha \pm a^\alpha)_{14}(\upsilon_\alpha \pm a_\alpha)_{32}. \tag{4.30}$$

The combinations $(\upsilon \pm a)(\upsilon \mp \mathrm{a})$ are reorderd by making use of relation (4.29)

$$\frac{1}{4}(\upsilon^\alpha + a^\alpha)_{12}(\upsilon_\alpha - a_\alpha)_{34} = (\hat{\sigma}^\alpha)^{Ab}(\sigma_\alpha)_{dF}\phi^{(1)*}{}_A\phi_b^{(2)}\chi^{(3)*d}\chi^{(4)F}$$
$$= -2\delta_a^b\delta_F^A\phi^{(1)*}{}_A\chi^{(4)F}\chi^{(3)*d}\phi_b^{(2)} = -\frac{1}{2}(s-p)_{14}(s+p)_{32}.$$

The same relation holds with the plus and minus signs exchanged on either side, so that we obtain

$$(\upsilon^\alpha \pm a^\alpha)_{12}(\upsilon_\alpha \mp a_\alpha)_{34} = -2(\mathrm{s} \mp \mathrm{p})_{14}(s \pm p)_{32}. \tag{4.31}$$

From this relation follows, in particular,

$$\begin{aligned} &2(s \pm p)_{12}(\mathrm{s} \mp \mathrm{p})_{34} - (\upsilon^\alpha \mp \mathrm{a}^\alpha)_{12}(\upsilon_\alpha \pm a_\alpha)_{34} \\ &\quad = 2(s \pm p)_{14}(s \mp p)_{32} - (\upsilon^\alpha \mp a^\alpha)_{14}(\upsilon_\alpha \pm a_\alpha)_{32}. \end{aligned} \tag{4.32}$$

Before we go on, let us apply (4.30) and (4.31) to the transformation of the most general VA Lagrangian from the ordering (12) (34) to the ordering (14)(32):

$$\begin{aligned} &C_\mathrm{V}(\upsilon^\alpha)_{12}(\upsilon_\alpha)_{34} + C'_\mathrm{V}(\upsilon^\alpha)_{12}(a_\alpha)_{34} + C'_\mathrm{A}(a^\alpha)_{12}(\upsilon_\alpha)_{34} + C_\mathrm{A}(a^\alpha)_{12}(a_\alpha)_{34} \\ &= \frac{1}{4}\{(C_\mathrm{V} + C'_\mathrm{V} + C'_\mathrm{A} + C_\mathrm{A})(\upsilon^\alpha + a^\alpha)_{12}(\upsilon_\alpha + a_\alpha)_{34} \\ &\quad + (C_\mathrm{V} - C'_\mathrm{V} - C'_\mathrm{A} + C_\mathrm{A})(\upsilon^\alpha - a^\alpha)_{12}(\upsilon_\alpha - a_\alpha)_{34} \\ &\quad + (C_\mathrm{V} - C'_\mathrm{V} + C'_\mathrm{A} - C_\mathrm{A})(\upsilon^\alpha + a^\alpha)_{12}(\upsilon_\alpha - a_\alpha)_{34} \\ &\quad + (C_\mathrm{V} + C'_\mathrm{V} - C'_\mathrm{A} - C_\mathrm{A})(\upsilon^\alpha - a^\alpha)_{12}(\upsilon_\alpha + a_\alpha)_{34}\}, \end{aligned}$$

The first two terms on the r.h.s., by (4.30) go over into the same forms in the ordering (14)(32). The remaining two terms are transformed according to (4.31) and give scalar and pseudoscalar couplings. Inserting (4.30, 4.31) and remembering that $p_{ki} = -\mathrm{i}(\bar{f}_k \Gamma_P f_i)$, the comparison with the general form in the ordering (14)(32) yields precisely (4.25), thus establishing the third and fourth columns of (4.22, 4.23).

In order to obtain the remainder of these equations, we need the transformation behaviour of S, P and T terms. Clearly (4.31) can also be read from right to left, with 4 and 2 interchanged. Therefore, in order to cope with the most general SP term, we also need relations for $(s \pm p)_{12}(s \pm p)_{34}$. These are obtained as follows. Using (1.64), one has

$$(\sigma^\alpha \hat{\sigma}^\beta)^d_a \{(\sigma_\alpha \hat{\sigma}_\beta)^b_c - (\sigma_\beta \hat{\sigma}_\alpha)^b_c\} = (\sigma^\alpha \hat{\sigma}^\beta)^d_a \{2\mathrm{g}_{\alpha\beta}\delta^b_c - 2(\sigma_\beta \hat{\sigma}_\alpha)^b_c\}.$$

The sums over the Lorentz indices α and β are performed by means of (4.29), giving eventually

$$= 8\delta^d_a \delta^b_c - 16\delta^b_a \delta^d_c. \tag{4.33}$$

Multiplying this equation by

$$\chi^{(1)*a}\phi^{(2)}_b \chi^{(3)*c}\phi^{(4)}_d$$

or by $-\chi^{(1)*}\phi^{(4)}\chi^{(3)*}\phi^{(2)}$, depending on which pair of indices must be contracted, this yields a relation for $(s + p)_{12}(s + p)_{34}$ in terms of $(s + p)_{14}(s + p)_{32}$ and $t_{14}(t + t')_{32}$. Clearly, an analogous identity holds if σ and $\hat{\sigma}$ are interchanged in (4.33). Thus we obtain the formulae

$$(s \pm p)_{12}(s \pm p)_{34} = -\frac{1}{2}(s \pm p)_{14}(s \pm p)_{32} - \frac{1}{2}(t^{\alpha\beta})_{14}(t_{\alpha\beta} \pm t'_{\alpha\beta})_{32}. \tag{4.34}$$

As one easily verifies, (4.34) and (4.31) with $(4 \longleftrightarrow 2)$, establish the first and second columns of (4.22) and (4.23). [Note also that relation (4.34) is identical with (4.27b).]

Regarding the tensor covariants, we now derive two more relations which, when combined with (4.34), yield the full transformation formula for T couplings. Let us return to (4.33) to which we add the term $8\,\delta_a^b\delta_c^d$,

$$(\sigma^\alpha\hat{\sigma}^\beta)_a^d\{(\sigma_\alpha\hat{\sigma}_\beta)_c^b - (\sigma_\beta\hat{\sigma}_\alpha)_c^b\} + 8\delta_a^b\delta_c^d\,. \tag{4.35a}$$

The first term in this expression can be reordered by means of (4.28a) to $-(\sigma^\alpha\hat{\sigma}^\beta)_a^b(\sigma_\alpha\hat{\sigma}_\beta)_c^d$. The second and third term cancel against each other, as one verifies by means of (4.29). Therefore, the combination (4.35a) is equal to

$$-(\sigma^\alpha\hat{\sigma}^\beta)_a^d\{(\sigma_\alpha\hat{\sigma}_\beta)_c^b - (\sigma_\beta\hat{\sigma}_\alpha)_c^d\} - 8\delta_a^d\delta_c^b. \tag{4.35b}$$

Multiplying this identity with $\chi^{(1)*a}\phi_b^{(2)}\chi^{(3)*c}\phi_d^{(4)}$ (or the combination with $\phi^{(2)}$ and $\phi^{(4)}$ interchanged), one obtains the following relation with the upper sign:

$$\begin{aligned}&(t^{\alpha\beta})_{12}(t_{\alpha\beta} \pm t'_{\alpha\beta})_{34} - (s \pm p)_{12}(s \pm p)_{34}\\ &= (t^{\alpha\beta})_{14}(t_{\alpha\beta} \pm t'_{\alpha\beta})_{32} - (s \pm p)_{14}(s \pm p)_{32}.\end{aligned} \tag{4.36}$$

The corresponding relation with the lower sign is derived in a similar way. Equations (4.36) and (4.34) are combined to yield the relations

$$(t^{\alpha\beta})_{12}(t_{\alpha\beta} \pm t'_{\alpha\beta})_{34} = \frac{1}{2}(t^{\alpha\beta})_{14}(t_{\alpha\beta} \pm t'_{\alpha\beta})_{32} - \frac{3}{2}(s \pm p)_{14}(s \pm p)_{32} \tag{4.37}$$

from which one reads off the fifth columns of (4.22) and (4.23). This completes the proof. Note the particular linear combinations (4.30, 4.32, 4.36) which are (the only) invariants under Fierz reordering.

4.1.2 Precision Tests in Muon Decay

Besides the nuclear β-transitions the ordinary, most frequent decay mode of μ^- (or μ^+)

$$\mu^- \to \mathrm{e}^-\overline{\nu}_\mathrm{e}\nu_\mu \quad (\mu^+ \to \mathrm{e}^+\nu_\mathrm{e}\overline{\nu}_\mu) \tag{4.38}$$

provides a very precise source of information on CC weak interactions. It is a purely leptonic process and, therefore, is particularly well suited for testing the standard theory and for identifying specific deviations from the simple "V– A" picture. Restricting the analysis to the electron (or positron) in the final state there are at least nine observables which are measurable realistically and all of which carry

some characteristic information on the basic interaction: The rate Γ_μ, or its inverse the lifetime τ_μ, three parameters (ρ, η, δ) determining shapes of spectra, two strength parameters (ξ, ξ') characterizing spin-momentum correlations, and four parameters $(\alpha, \beta, \alpha', \beta')$ determining the transverse polarization of the electron in the final state. [The spectrum parameter η is linearly dependent on α and β]. The precise definitions are given below.

This analysis disregards the neutrinos as these are not readily observable. There is, however, some information on the character of the neutrinos in the decay (4.38). If muonic lepton number, instead of being conserved additively, were some kind of parity quantum number, i.e. if instead of $\Sigma_i L_\mu(i)$ the "parity" $\Pi_i(-)^{L\mu^{(i)}}$ were conserved, then a μ^+ could decay following either of the two branches

$$\mu^+ \to \mathrm{e}^+ \nu_\mathrm{e} \overline{\nu}_\mu, \tag{4.39a}$$

$$\mu^+ \to \mathrm{e}^+ \overline{\nu}_e \nu_\mu, \tag{4.39b}$$

In the additive scheme, on the other hand, the mode (4.39b) is forbidden. An experiment which was designed to identify the $\overline{\nu}_\mathrm{e}$ through inverse β-decay, gave the following result for the decay width (Willis et al., 1980):

$$\frac{\Gamma(\mu^+ \to \mathrm{e}^+ \overline{\nu}_\mathrm{e} \nu_\mu)}{\Gamma(\mu^+ \to \text{all})} = -0.001 \pm 0.061 \,(90\%\,\text{C.L.}). \tag{4.40}$$

This important result shows that the decay mode (4.39b) is absent, at the level of a few percent, and that the additive mode of conserving L_μ is strongly supported (see also exercise 4.2).

In this section we analyze the differential decay probability of polarized muons in a rather general manner. By way of example and in order to demonstrate the sensitivity of the observables to specific types of couplings we discuss the case of S, P, V, and A interactions. This is not quite the most general one as it leaves out possible T couplings. We choose this example because it illustrates well the information content of muon decay while still being manageable as far as practical calculations are concerned. [The general case is found in the literature, see e.g. Scheck (1978).] We also discuss briefly the reaction $\nu_\mu \mathrm{e}^- \to \mu^- \nu_\mathrm{e}$, so-called inverse muon decay, and show in which respect it yields information which is complementary to the information from the decay.

a) Muon decay: kinematics

Let q be the momentum of the muon, p the momentum of the electron, k_1 and k_2 the momenta of the neutrinos, respectively, and let Q be the sum of k_1 and k_2:

$$q = p + k_1 + k_2 = p + Q,$$

where

$$Q = k_1 + k_2 = q - p. \tag{4.41}$$

From the general formulae of App. B the differential decay probability, when expressed in the muon's rest frame, is given by

$$\frac{\mathrm{d}^3\Gamma}{\mathrm{d}^3 p} = \frac{(2\pi)^7}{4m_\mu E}\int \frac{\mathrm{d}^3 k_1}{2E_1}\int \frac{\mathrm{d}^3 k_2}{2E_2}\delta(Q - k_1 - k_2)|T|^2, \tag{4.42}$$

where E is the energy of the electron in the final state. The physical domain of E is

$$m_\mathrm{e} \leq E \leq W,$$

with

$$W = \frac{1}{2m_\mu}(m_\mu^2 + m_\mathrm{e}^2). \tag{4.43}$$

The direction of $\boldsymbol{p}$, the three-momentum of the electron is understood to be taken with respect to the direction of the muon spin (see Fig. 4.1).

Obviously, the kinematics is symmetric in the momenta of the two neutrinos. As we integrate over them, only those covariants in $|T|^2$ will contribute which are symmetric in k_1 and k_2. Antisymmetric combinations do not contribute. Furthermore, the result of the integration in eq. (4.42) can only depend on Q. For S, P, V, and A couplings the pertinent integrals are

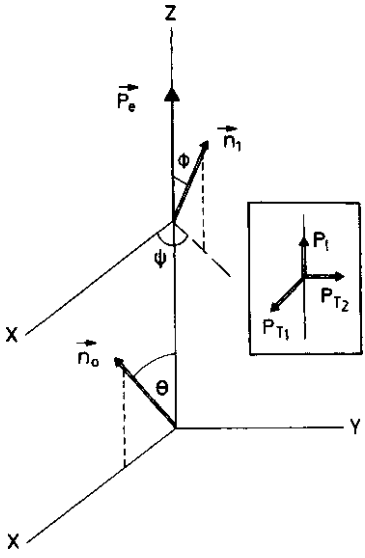

Fig. 4.1 Definitions of kinematic variables in muon decay. $\boldsymbol{n}_0$ is the direction of the muon's spin expectation value, $\boldsymbol{n}_1$ the direction of the electron's spin expectation value. The insert shows the decomposition of the electron polarization along its momentum (P_l), transverse to the momentum and in the plane spanned by $\boldsymbol{n}_0$ and $\boldsymbol{p}$(P_{T1}), and normal to that plane (P_{T2})

$$\int \frac{\mathrm{d}^3k_1}{2k_1^0}\int \frac{\mathrm{d}^3k_2}{2k_2^0}(k_1\cdot k_2)\delta(Q-k_1-k_2)=\frac{\pi}{4}Q^2, \tag{4.44a}$$

$$\int \frac{\mathrm{d}^3k_1}{2k_1^0}\int \frac{\mathrm{d}^3k_2}{2k_2^0}(k_1^\alpha k_2^\beta-(k_1\cdot k_2)g^{\alpha\beta}+k_2^\alpha k_1^\beta)\delta(Q-k_1-k_2)$$
$$=\frac{\pi}{6}(Q^\alpha Q^\beta-Q^2g^{\alpha\beta}). \tag{4.44b}$$

(See exercise 4.3)

Let $\boldsymbol{\zeta}_\mu = P_\mu \boldsymbol{n}_0$ be the expectation value of the spin of the decaying muon, where $\boldsymbol{n}_0$ is a unit vector and P_μ is the muon polarization. In the muon's rest frame where $q = (m_\mu;0)$, $p = (E;\boldsymbol{p})$ and where the covariant spin vector is $s_0 = (0;\zeta_\mu)$, we choose polar coordinates such that $\boldsymbol{p}$ defines the z-axis and that $\boldsymbol{n}_0$ lies in the xz-plane, see Fig. 4.1. Let $\boldsymbol{n}_1$ be another unit vector with polar coordinates (ϕ,ψ), as shown in the figure. Below we derive the probability of finding the electron spin pointing in the direction $\boldsymbol{n}_1$; this means that we must calculate the decay amplitude for a polarized electron state whose covariant spin vector (1.146) is

$$s_1=\left(\frac{1}{m_\mathrm{e}}\boldsymbol{p}\cdot\boldsymbol{n}_1;\boldsymbol{n}_1+\frac{\boldsymbol{p}\cdot\boldsymbol{n}_1}{m_\mathrm{e}(E+m_\mathrm{e})}\boldsymbol{p}\right).$$

In the muon's rest frame, according to Fig. 4.1, we have (setting $P_\mu = 1$)

$$\begin{aligned}
&\boldsymbol{n}_0=(\sin\theta,0,\cos\theta),\quad \boldsymbol{n}_1=(\sin\phi\cos\psi,\sin\phi\sin\psi,\cos\phi),\\
&(s_0\cdot p)=-|\boldsymbol{p}|\cos\theta\simeq -E\cos\theta,\\
&m_\mathrm{e}(s_1\cdot q)=m_\mu|\boldsymbol{p}|\cos\phi\simeq m_\mu E\cos\phi,\\
&m_\mathrm{e}(s_0\cdot s_1)=-E\cos\theta\cos\phi-m_\mathrm{e}\sin\theta\sin\phi\cos\psi\simeq -E\cos\theta\cos\phi.
\end{aligned} \tag{4.45}$$

The $\simeq$ sign refers to the case where E is large compared to the rest mass of the electron.

b) Muon decay: interaction, decay probability and observables

The general, effective interaction responsible for muon decay has the form given in (4.20), namely,

$$\mathscr{H}=-\mathscr{L}=\frac{G}{\sqrt{2}}\sum_i\{\overline{e(x)}\,\Gamma_i\nu_e(x)[G_i\,\overline{\nu_\mu(x)}\,\Gamma^i\mu(x)$$
$$+G_i'\overline{\nu_\mu(x)}\,\Gamma^i\gamma_5\mu(x)]+\text{h.c.}\}. \tag{4.46}$$

For the sake of simplicity, we consider only S, P, V and A interactions, but no tensors, and refer to Scheck (1978) for the complete case. Furthermore, on the basis of our experience gained in Sect. 4.1.1 d, e we rewrite the interaction in a *helicity* or *chirality projection* form (Mursula et al. 1985)

$$\begin{aligned}\mathscr{H} = &\frac{G}{\sqrt{2}}\{h_{\mathrm{LR}}(s+p)_{\bar{\mathrm{e}}\nu_e}(s+p)_{\bar{\nu}_\mu\mu} + h_{\mathrm{LL}}(s+p)(s-p)\\ &+ h_{\mathrm{RR}}(s-p)(s+p) + h_{\mathrm{RL}}(s-p)(s-p)\\ &+ g_{\mathrm{RR}}(\upsilon^\alpha + \mathrm{a}^\alpha)_{\bar{\mathrm{e}}\nu_e}(\upsilon_\alpha + \mathrm{a}_\alpha)_{\bar{\nu}_\mu\mu} + g_{\mathrm{RL}}(\upsilon^\alpha + a^\alpha)(\upsilon_\alpha - \mathrm{a}_\alpha)\\ &+ g_{\mathrm{LR}}(\upsilon^\alpha - \mathrm{a}^\alpha)(\upsilon_\alpha + \mathrm{a}_\alpha) + g_{\mathrm{LL}}(\upsilon^\alpha - \mathrm{a}^\alpha)(\upsilon_\alpha - \mathrm{a}_\alpha) + \mathrm{h.c.}\} \qquad (4.46')\end{aligned}$$

(we have written the field symbols only once for each class of covariants). The notation for the coupling constants refers to the handedness, R (right) or L (left), of the charged leptons, in accord with the selection rules discussed in Sect. 3.1.2c. Keeping in mind the definition of Γ_{P} in (4.7) and of p in eq. (4.1b) it is easy to express the h_{ik} in terms of the G_i and G_i' and vice versa:

$$\left.\begin{matrix} h_{\mathrm{LR}} \\ h_{\mathrm{RL}} \end{matrix}\right\} = \frac{1}{4}[G_{\mathrm{S}} - G_{\mathrm{P}}) \pm (G_{\mathrm{S}}' - G_{\mathrm{P}}')], \qquad (4.47a)$$

$$\left.\begin{matrix} h_{\mathrm{LL}} \\ h_{\mathrm{RR}} \end{matrix}\right\} = \frac{1}{4}[G_{\mathrm{S}} + G_{\mathrm{P}}) \mp (G_{\mathrm{S}}' + G_{\mathrm{P}}')]. \qquad (4.47b)$$

In a similar way, the g_{ik} are linear functions of $G_{\mathrm{V}}, \ldots, G_{\mathrm{A}}'$:

$$\left.\begin{matrix} g_{\mathrm{RR}} \\ g_{\mathrm{LL}} \end{matrix}\right\} = \frac{1}{4}[G_{\mathrm{V}} + G_{\mathrm{A}}) \pm (G_{\mathrm{V}}' + G_{\mathrm{A}}')], \qquad (4.47c)$$

$$\left.\begin{matrix} g_{\mathrm{RL}} \\ g_{\mathrm{LR}} \end{matrix}\right\} = \frac{1}{4}[G_{\mathrm{V}} - G_{\mathrm{A}}) \mp (G_{\mathrm{V}}' - G_{\mathrm{A}}')]. \qquad (4.47d)$$

The "V − A" interaction corresponds to

$$\begin{aligned} &g_{\mathrm{LL}} = 1, \quad g_{\mathrm{RR}} = g_{\mathrm{RL}} = g_{\mathrm{LR}} = 0.\\ &h_{\mathrm{LR}} = h_{\mathrm{LL}} = h_{\mathrm{RR}} = h_{\mathrm{RL}} = 0. \qquad (4.48)\end{aligned}$$

In view of the fact that the neutrinos are not detected it is appropriate to apply a Fierz transformation to (4.46′) to the effect of grouping the (observable) charged leptons in one covariant, and the two (unobserved) neutrinos in the other. In Sect. 4.1.1e we have learnt how to do this. Using (4.30, 4.31, 4.34), we find at once

$$\begin{aligned}\mathscr{H} = &\frac{G}{\sqrt{2}}\{-\frac{1}{2}h_{\mathrm{LR}}(s+p)_{\bar{\mathrm{e}}\mu}(s+p)_{\bar{\nu}_\mu\nu_{\mathrm{e}}} - 2g_{\mathrm{LR}}(s+p)(s-p)\\ &-2g_{\mathrm{RL}}(s-p)(s+p) - \frac{1}{2}h_{\mathrm{RL}}(s-p)(s-p)\\ &+g_{\mathrm{RR}}(\upsilon^\alpha + a^\alpha)_{\bar{\mathrm{e}}\mu}(\upsilon_\alpha + a_\alpha)_{\bar{\nu}_\mu\nu_{\mathrm{e}}} - \frac{1}{2}h_{\mathrm{RR}}(\upsilon^\alpha + a^\alpha)(\upsilon_\alpha - a_\alpha)\\ &-\frac{1}{2}h_{\mathrm{LL}}(\upsilon^\alpha - a^\alpha)(\upsilon_\alpha + a_\alpha) + g_{\mathrm{LL}}(\upsilon^\alpha - a^\alpha)(\upsilon_\alpha - a_\alpha)\\ &-\frac{1}{2}h_{\mathrm{LR}}(t^{\alpha\beta})_{\bar{\mathrm{e}}\mu}(t_{\alpha\beta} + t'_{\alpha\beta})_{\bar{\nu}_\mu\nu_{\mathrm{e}}} - \frac{1}{2}h_{\mathrm{RL}}t^{\alpha\beta}(t_{\alpha\beta} - t'_{\alpha\beta}) + \mathrm{h.c.}\}. \qquad (4.49)\end{aligned}$$

The form (4.46) or (4.46′) of the Lagrangian is usually referred to as the *charge changing* form. The same Hamiltonian is said to be given in *charge retention* form if it is written as done in (4.49).

The calculation of the amplitude and of $|T|^2$ is straightforward but rather lengthy. It is simplified somewhat by the observation that due to the symmetric integration over the neutrino momenta, all terms which are antisymmetric in k_1 and k_2 can be skipped. One finds the following result for the decay of a negative muon:

$$\begin{aligned}
\frac{d^3\Gamma}{d^3p}\Bigg/\left\{\frac{\pi G^2}{8m_\mu E(2\pi)^5}\right\} &= aQ^2[(pq)-m_\mathrm{e}m_\mu(s_0s_1)]\\
&+\alpha Q^2[m_\mathrm{e}m_\mu+(s_1q)(s_0p)-(pq)(s_0s_1)]-a'Q^2[m_\mu(s_0p)-m_\mathrm{e}(s_1q)]\\
&+\frac{2}{3}b[m_\mathrm{e}m_\mu(s_0s_1)Q^2-2m_\mu m_\mathrm{e}(s_1q)(s_0p)+(pq)Q^2+2(Qq)(Qp)]\\
&+\frac{2}{3}\beta[(s_0s_1)(2m_\mathrm{e}^2m_\mu^2-(m_\mathrm{e}^2+m_\mu^2)(pq))\\
&+(m_\mathrm{e}^2+m_\mu^2)(s_1q)(s_0p)-3m_\mathrm{e}m_\mu Q^2]\\
&+\frac{2}{3}b'[m_\mathrm{e}(2(Qq)(s_1q)+Q^2(s_1q))+m_\mu(-2(Qp)(s_0p)+Q^2(s_0p))]\\
&+\frac{2}{3}c[4(Qq)(Qp)-Q^2(qp)+m_\mathrm{e}m_\mu(4(s_0p)(s_1q)+Q^2(s_0s_1))]\\
&+\frac{2}{3}c'[4m_\mu(Qp)(s_0p)+4m_\mathrm{e}(Qq)(s_1q)+Q^2(m_\mu(s_0p)\\
&-m_\mathrm{e}(s_1q))]-(\alpha'Q^2+\frac{2}{3}\beta'(m_\mu^2-m_\mathrm{e}^2))\varepsilon_{\alpha\beta\sigma\tau}q^\alpha s_0^\beta p^\sigma s_1^\tau. \qquad (4.50)
\end{aligned}$$

In this expression the real constants $a,\ldots,\beta'$ are given by

$$\left.\begin{matrix}a\\a'\end{matrix}\right\} = 16(|g_\mathrm{RL}|^2 \pm |g_\mathrm{LR}|^2)+|h_\mathrm{RL}|^2 \pm |h_\mathrm{LR}|^2, \qquad (4.51a)$$

$$\left.\begin{matrix}b\\b'\end{matrix}\right\} = 4(|g_\mathrm{RR}|^2 \pm |g_\mathrm{LL}|^2)+|h_\mathrm{RR}|^2 \pm |h_\mathrm{LL}|^2, \qquad (4.51b)$$

$$\left.\begin{matrix}c\\c'\end{matrix}\right\} = \frac{1}{2}(|h_\mathrm{RL}|^2 \pm |h_\mathrm{LR}|^2), \qquad (4.51c)$$

$$\left.\begin{matrix}\alpha\\\alpha'\end{matrix}\right\} = 8\left\{\begin{matrix}\mathrm{Re}\\\mathrm{Im}\end{matrix}\right\}(g_\mathrm{LR}h_\mathrm{RL}^* \pm g_\mathrm{RL}h_\mathrm{LR}^*), \qquad (4.51d)$$

$$\left.\begin{matrix}\beta\\\beta'\end{matrix}\right\} = -4\left\{\begin{matrix}\mathrm{Re}\\\mathrm{Im}\end{matrix}\right\}(g_\mathrm{LL}h_\mathrm{RR}^* \pm g_\mathrm{RR}h_\mathrm{LL}^*) \qquad (4.51e)$$

We do not evaluate (4.50) in its full complexity but rather assume the energy E of the electron to be large as compared to the rest mass. Then m_e may be set equal to zero except where it appears multiplied with s_1, in which case the equations (4.45) apply. In this approximation

$$W \simeq \frac{1}{2}m_\mu, \qquad Q^2 = (q-p)^2 \simeq m_\mu(m_\mu - 2E),$$
$$\mathrm{d}^3p = |\boldsymbol{p}|^2\mathrm{d}|\boldsymbol{p}|\mathrm{d}\Omega \simeq E^2\mathrm{d}E\mathrm{d}\Omega.$$

It is convenient to replace E with the dimensionless variable

$$x := \frac{E}{W} \simeq \frac{2E}{m_\mu}, \tag{4.52}$$

whose range of variation is

$$x_0 \le x \le 1 \qquad \text{with} \qquad x_0 = m_\mathrm{e}/W,$$

or, in the approximation $m_\mathrm{e} = 0 : 0 \le x \le 1$.

The differential decay probability for emission of an electron with energy between x and $x + \mathrm{d}x$ at an angle between θ and $\theta + \mathrm{d}\theta$ with respect to the muon spin, is then found to be[1]

$$\begin{aligned}
&\frac{\mathrm{d}^2\Gamma(x,\theta,\phi,\psi)}{\mathrm{d}x\mathrm{d}(\cos\theta)} \\
&\simeq \frac{Am_\mu^5G^2}{2^{10}\pi^36}x^2\{[6(1-x)+\frac{4}{3}\rho(4x-3)] + \xi\cos\theta[2(x-1)+\frac{4}{3}\delta(3-4x)] \\
&+\xi'\cos\phi[6(x-1)+4\delta'(3-4x)] + \xi''\cos\theta\cos\phi[2(1-x)+\frac{4}{3}\rho'(4x-3)] \\
&+2\sin\theta\sin\phi\left[\left((3(1-x)\frac{\alpha}{A}+2\frac{\beta}{A}\right)\cos\psi + \left(3(1-x)\frac{\alpha'}{A}+2\frac{\beta'}{A}\right)\sin\psi\right]\right\}.
\end{aligned} \tag{4.53}$$

In this expression $\alpha,\ \beta,\ \alpha',\ \beta'$ are as given above in (4.51 d,e). The other parameters are appropriately chosen combinations of $a, \ldots, c'$, viz.

$$A = a + 4b + 6c, \tag{4.54a}$$
$$\rho = \frac{1}{A}(3b + 6c), \tag{4.54b}$$
$$\xi = -\frac{1}{A}(3a' + 4b' - 14c'), \tag{4.54c}$$
$$\delta = \frac{1}{A\xi}(-3b' + 6c'), \tag{4.54d}$$

[1]This holds for a fully polarized muon. If the muon carries partial polarization P_μ, the terms in $\cos\theta$ and $\sin\theta$ are multiplied by P_μ.

$$\xi' = -\frac{1}{A}(a' + 4b' + 6c'), \tag{4.54e}$$

$$\delta' = -\frac{1}{A\xi'}(b' + 2c'), \tag{4.54f}$$

$$\xi'' = \frac{1}{A}(3a + 4b - 14c), \tag{4.54g}$$

$$\rho' = \frac{1}{A\xi''}(3b - 6c). \tag{4.54h}$$

ρ is called the Michel parameter. It was introduced by L. Michel who gave the first general analysis of muon decay. [As may be seen from (4.50) there are also spin-independent terms in the spectrum which are proportional to m_e and which depend on α and β. They are proportional to $\eta x_0(1-x)$ with $\eta = (1/A)(\alpha - 2\beta)$. As we neglect m_e they do not appear in (4.53).] Equation (4.53) holds for μ^- decay. For μ^+ decay the signs of the terms $\xi \cos\theta$ and $\xi' \cos\phi$ must be reversed.

What are the observables that can be determined in an experiment on muon decay? The parameters that were measured in the past are the following: The spectrum parameters ρ and δ are known to rather high precision (see below). The Michel parameter ρ, for instance, determines the shape of the spectrum for the decay of unpolarized muons, see Fig. 4.2. ξ is determined by a measurement of the asymmetry of electron emission with respect to the muon spin direction. ξ' determines the longitudinal polarization of the outgoing electron, $\{\alpha, \beta\}$ the transverse component P_{T1}, and $\{\alpha', \beta'\}$ the normal component P_{T2} of the electron polarization, all of which have been measured (for definitions see Fig. 4.1 and next

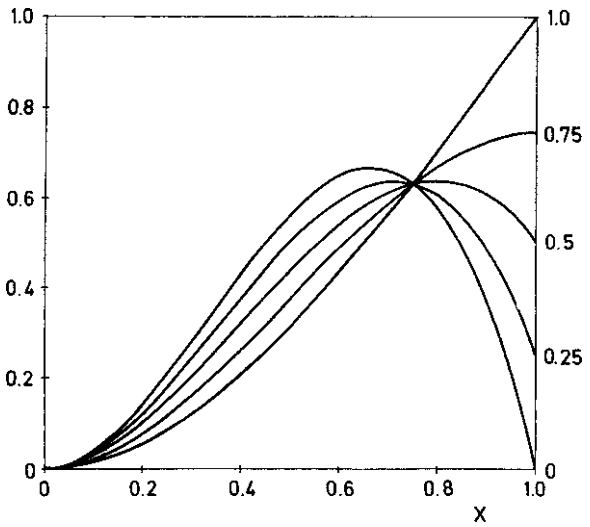

Fig. 4.2 Michel spectrum, i.e. isotropic part of differential decay probability (4.53), for various values of ρ. The spectrum is normalized such that it equals ρ at the maximal energy $x = 1$

section (c)). The quantity A, finally, appears in the expression for the total rate. The remaining parameters δ' and ρ' have not been measured.

c) Analysis and examples

As an example, let us analyze muon decay in terms of the interaction (4.46′) and compare the predictions for the lifetime and the decay parameters to their measured values.

The total *rate* is obtained from (4.53) by integrating over x and θ, and by summing over the spin directions of the outgoing electron. One finds

$$\begin{aligned}\Gamma &= \frac{Am_\mu^5 G^2}{2^8\pi^3 6}\int_0^1 x^2\left[6(1-x)+\frac{4}{3}\rho(4x-3)\right]\mathrm{d}x \\ &= \frac{m_\mu^5 G^2}{192\pi^3}\frac{A}{16}\end{aligned} \tag{4.55}$$

Without loss of generality one can choose the normalization $A = 16$ because this amounts to no more than fixing G in terms of the decay rate Γ.

Note that this result is independent of the value of ρ. In the limit of the exact "V − A" interaction, $g_{LL} = 1$ whilst all other constants g_{ik} and h_{ik} vanish. From (4.51) we then have $a = a' = 0, b = -b' = 4, c = c' = 0, \alpha = \beta = 0, \alpha' = \beta' = 0$, or, from (4.54),

$$\begin{aligned}&A = 16, \quad \rho = \frac{3}{4} = \delta = \rho', \quad \delta' = \frac{1}{4}, \\ &\xi = 1 = \xi' = \xi''.\end{aligned} \tag{4.56}$$

ρ and δ are indeed found to be $\frac{3}{4}$ within 0.3% and 0.5%, respectively [RPP94],

$$\rho = 0.7503(4), \qquad \delta = 0.7504(6) \tag{4.57a}$$

From (4.54 b, d) this means that $a \simeq 2c$ and $a' \simeq 2c'$ and $\delta' = \frac{1}{4}$, $\rho' = \frac{3}{4}$, or, from (4.51 a,c), $g_{LR} \simeq g_{RL} \simeq 0$. Let us assume that, indeed, $g_{LR} = 0 = g_{RL}$. Then, from (4.51)

$$\begin{aligned}A &= 4(a+b) \simeq 16(|g_{RR}|^2 + |g_{LL}|^2) + 4(|h_{LR}|^2 + |h_{RL}|^2 \\ &\quad + |h_{LL}|^2 + |h_{RR}|^2) = 16, \\ 1-\xi &= \frac{a+b-a'+b'}{a+b} = \frac{8}{A}(4|g_{RR}|^2 + |h_{RR}|^2 + |h_{LR}|^2), \\ 1-\xi' &= \frac{a+b+a'+b'}{a+b} = \frac{8}{A}(4|g_{RR}|^2 + |h_{RR}|^2 + |h_{RL}|^2), \\ \alpha &\simeq \alpha' \simeq 0,\end{aligned}$$

β, β' as given by (4.51e).

With these restrictions on the parameters, (4.53) reduces to

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\mathrm{d}(\cos\theta)} \simeq \frac{4m_\mu^5 G^2}{2^8 6\pi^3} x^2 \Big\{(3-2x)(1-\xi'\cos\phi) - (2x-1)\cos\theta(\xi-\xi''\cos\phi) + \sin\theta\sin\phi\frac{1}{4}(\beta\cos\psi+\beta'\sin\psi)\Big\}. \tag{4.58}$$

[If the initial polarization P_μ is not unity, the second and third term are to be multiplied by P_μ].

The experimental values for ξ and ξ' are (assuming fully polarized muons from pion decay)[2]

$$\xi = 1.0007(35), \qquad \xi' = 1.00(4). \tag{4.57b}$$

The value of ξ', in particular, is obtained by measuring the longitudinal polarization of the electron, cf. (3.25), which is given by

$$P_\ell = \frac{\mathrm{d}\Gamma(x,\theta,\phi=0,\psi=0) - \mathrm{d}\Gamma(x,\theta,\phi=\pi,\psi=0)}{\mathrm{d}\Gamma(x,\theta,0,0) + \mathrm{d}\Gamma(x,\theta,\pi,0)}.$$

With $\rho = \delta = \rho' = \frac{3}{4}$ and $\delta' = \frac{1}{4}$ this is approximately

$$P_\ell \simeq -\xi' + \frac{(2x-1)\cos\theta}{3-2x+\xi(2x-1)\cos\theta}(\xi''-\xi\xi'). \tag{4.59}$$

β and β', as well as α and α', can be determined, for instance, by measuring the two transverse components of the electron polarization (cf. Fig. 4.1), viz.

$$P_{\mathrm{T1}} = \frac{\mathrm{d}\Gamma(x,\,\theta,\,\pi/2,0) - \mathrm{d}\Gamma(x,\,\theta,\,-\pi/2,0)}{\mathrm{d}\Gamma(x,\,\theta,\,\pi/2,0) + \mathrm{d}\Gamma(x,\,\theta,-\pi/2,0)},$$

which, in the situation considered above, is

$$P_{\mathrm{T1}} \simeq \frac{4\beta/A}{3-2x+\xi(2x-1)\cos\theta}\sin\theta. \tag{4.60}$$

Similarly,

$$P_{\mathrm{T2}} = \frac{\mathrm{d}\Gamma(x,\theta,\pi/2,\pi/2) - \mathrm{d}\Gamma(x,\theta,-\pi/2,\pi/2)}{\mathrm{d}\Gamma(x,\theta,\pi/2,\pi/2) + \mathrm{d}\Gamma(x,\theta,-\pi/2,\pi/2)},$$

[2]The general case is analyzed along these lines in the review by W. Fetscher and H.J. Gerber, [RPP10, p. 521].

which is approximately

$$P_{\mathrm{T2}} \simeq \frac{4\beta'/A}{3 - 2x + \xi(2x-1)\cos\theta} \sin\theta. \tag{4.61}$$

P_{T2}, the component perpendicular to the plane spanned by the muon spin and by the electron momentum, is particularly interesting because it can only be different from zero if the interaction (4.46′) is not invariant under time reversal. Indeed, we see from (4.51d,e) that α' and β' are different from zero only if some of the coupling constants are relatively complex.

The measurement of P_{T1} and P_{T2} gave values compatible with zero (Corriveau 1983, Burkard et al. 1985) (with $\alpha = \alpha' = 0$),

$$\beta/A = 0.0039(62), \quad \beta'/A = 0.002(7) \tag{4.62}$$

[The measurement of P_{T1} also fixes the parameter η mentioned above].

In conclusion, the comparison of the experimental results with the theoretical predictions shows that there is good evidence for the "V – A" interaction but that the data do not exclude contributions of the order of a few percent, from other types of interaction in the Lagrangian (4.49).

A case of special interest is the class of left-right symmetric unified gauge theories which are extensions of the GSW theory and which aim at explaining parity violation as a phenomenon typical for low energies. In such theories there is a second charged gauge boson $\mathrm{W_R}$ which couples to V + A currents. In order to obtain the observed "V − A" interaction at low energies the mass m_{R} of $\mathrm{W_R}$ must be significantly larger than the mass m_{L} of its sister boson $\mathrm{W_L}$. Since the two bosons are gauge bosons of the same local gauge theory, they couple to the matter fields with the same coupling constant g. Thus, the effective weak CC Lagrangian is

$$\left\{ \frac{g^2}{8m_{\mathrm{L}}^2}(\upsilon^\alpha - a^\alpha)(\upsilon_\alpha - a_\alpha) + \frac{g^2}{8m_{\mathrm{R}}^2}(\upsilon^\alpha + a^\alpha)(\upsilon_\alpha + a_\alpha) \right\}.$$

Setting

$$\frac{G}{\sqrt{2}} = \frac{g^2\sqrt{m_{\mathrm{R}}^4 + m_{\mathrm{L}}^4}}{8m_{\mathrm{L}}^2 m_{\mathrm{R}}^2},$$

we have

$$g_{\mathrm{RR}} = \frac{m_{\mathrm{L}}^2}{\sqrt{m_{\mathrm{R}}^4 + m_{\mathrm{L}}^4}}, \qquad g_{\mathrm{LL}} = \frac{m_{\mathrm{L}}^2}{\sqrt{m_{\mathrm{R}}^4 + m_{\mathrm{L}}^4}},$$

$g_{\mathrm{RL}} = 0 = g_{\mathrm{LR}}$. This is a special case of the analysis given above. All parameters except ξ and ξ' have their standard "V – A" values. ξ and ξ' carry the same information on the mass ratio $m_{\mathrm{R}}/m_{\mathrm{L}}$, i.e.

$$\xi = 1 - \frac{2m_{\mathrm{L}}^4}{m_{\mathrm{R}}^4 + m_{\mathrm{L}}^4}.$$

Present data, (4.57), to which one must add the available information on $\pi \to \mu\nu_\mu$ and nuclear β-decay (cf. (3.28)) yield a lower limit on the mass ratio $m_{\mathrm{R}}/m_{\mathrm{L}}$ of about 2.8, i.e. $m_R \gtrsim 230\,\mathrm{Ge\,V}$. To see the sensitivity of muon decay to the right-handed interactions induced by the existence of $\mathrm{W_R}$, let us assume that ξ is known to be 1 with an error bar of 0.1%. From the result above one would then conclude that $m_{\mathrm{R}}/m_{\mathrm{L}} \gtrsim 6.7$ or $m_{\mathrm{R}} \gtrsim 550\,\mathrm{Ge\,V}$.

In addition, the two physical W-boson states $\mathrm{W_1}$ and $\mathrm{W_2}$ may be orthogonal mixtures of the states $\mathrm{W_R}$ and $\mathrm{W_L}$. This happens if the mass Lagrangian is not diagonal in the basis of the states $\mathrm{W_R}$ and $\mathrm{W_L}$. In this case the state mixture leads to additional interaction terms of the type $(v^\alpha \pm a^\alpha)(v_\alpha \mp a_\alpha)$ (Bég et al. 1977). It is not difficult to show that, in our notation, g_{RL} and g_{LR} are proportional to

$$(m_{\mathrm{R}}^2 - m_{\mathrm{L}}^2)\mathrm{tg}\phi$$

where ϕ is the mixing angle. The limits on ϕ come primarily from the Michel parameter ρ and are of the order of $|\phi| \lesssim 0.05$.

d) Additional remarks

(i) Clearly, the expression (4.54) which is exact and quite general holds also for the decays

$$\tau^- \to \mu^- \overline{\nu}_\mu \nu_\tau,$$
$$\tau^- \to \mathrm{e}^- \overline{\nu}_\mathrm{e} \nu_\tau, \tag{4.63}$$

or for semileptonic decays of bare quarks. In deriving (4.53) one must check, of course, whether or not the mass of the daughter lepton can be neglected. We have not written out the additional mass dependent terms in (4.53) for the sake of economy, but it is easy to identify them in (4.50) and to insert them into (4.53). In muon decay (4.38) they are negligible unless one measures decay electrons with very low energies. The only exception is the rate for which a very accurate experimental value is available and where the mass terms must be taken into account, cf. (4.65, 4.66) below.

(ii) Radiative corrections to muon decay are important.[3] For instance, in the spectrum they amount to about 6% in the determination of the Michel parameter. [The experimental value quoted above was already corrected for this effect.] In the rate, the radiative correction is of the order of 0.4%. One finds, to order αG,

[3]For a summary see Scheck (1978), T. van Ritbergen and R. E. Stewart (1999), Phys. Rev. Lett. 82, 488.

$$\Gamma = \Gamma^{(0)} \left[1 + \frac{\alpha}{2\pi} \left(\frac{25}{4} - \pi^2 \right) + \frac{3}{5} \frac{m_\mu^2}{m_{\rm w}^2} \right], \tag{4.64}$$

where $\Gamma^{(0)}$ is the uncorrected expression.[4] In the general case, and including the electron mass terms, it is given by

$$\Gamma^{(0)} = \frac{m_\mu^5 G^2}{192\pi^3} \left\{ 1 + \frac{m_e}{m_\mu} \frac{\alpha - 2\beta}{4} - 8 \frac{m_{\rm e}^2}{m_\mu^2} + {\rm O}\left(\frac{m_{\rm e}^3}{m_\mu^3} \right) \right\}. \tag{4.65}$$

The lifetime is indeed known to very high accuracy

$$\tau_\mu = 2.197034(21) \times 10^{-6}\,{\rm sec}\,. \tag{4.66}$$

In the case of the "V – A" interaction we have $\alpha = \beta = 0$, so that we can extract G from this datum as

$$G = 1.16639(2) \times 10^{-5}\,{\rm Ge\,V}^{-2}. \tag{4.67}$$

Radiative corrections in the electron polarization $P = \{P_{\rm T1}, P_{\rm T2}, P_\ell\}$ are large at low energies (of the order of 10%) but become very small for, say, $x \gtrsim 0.25$ (Mehr and Scheck 1979, Fischer et al. 1974, Kuznetsov, 1981).

e) Inverse muon decay

The scattering reaction (known as inverse muon decay)

$$\nu_\mu e^- \to \mu^- \nu_e \tag{4.68}$$

provides additional information about the weak CC Lagrangian responsible for muon decay. This reaction was measured at high energies (Jonker et al. 1980, Bergsma et al. 1983, Mishra et al. 1990, Geiregat 1990) and the cross section was compared to its value as predicted by the pure "V – A" Lagrangian (3.74) and (3.186). In the experiment certain integrals of the cross section over the kinematics of the neutrino beam are determined and are compared to the same integrals over the theoretical expressions.

In this section we give a somewhat simplified analysis of reaction (4.68) by calculating the cross sections at fixed energy and by ignoring these integrations (see however exercise 4.4). Let k and k' be the initial and final neutrino momenta, respectively, p the electron momentum and q the muon momentum. Neglecting the electron and muon masses, the cross section in the c.m. system reads

$$\frac{{\rm d}\sigma}{d\,\Omega^*} \simeq \frac{1}{64\pi^2 s} \frac{1}{2} (2\pi)^{12} \sum |T|^2.$$

[4] The last term on the r.h.s. of (4.64) is a propagator effect taking into account the finiteness of the W-mass. It is, however, negligibly small.

ν_μ is the incident particle, the muon is the outgoing particle which is detected, so the invariant squared momentum transfer is $t = (k-q)^2 \simeq -(s/2)(1-\cos\theta^*)$. The invariant differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} 2\pi \frac{\mathrm{d}(\cos\theta^*)}{\mathrm{d}t} = \frac{1}{32\pi s^2}(2\pi)^{12} \sum |T|^2 .$$

The T-matrix element for the interaction (4.46′) and the spin summations over $|T|^2$ are calculated along standard lines. If h is the helicity of the incoming ν_μ, the cross section is found to be

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{G^2}{8\pi s^2} \big\{ & 4s^2(|g_{\mathrm{RR}}|^2(1+h) + |g_{\mathrm{LL}}|^2(1-h)) \\ & + 4(s+t)^2(|g_{\mathrm{RL}}|^2(1-h) + |g_{\mathrm{LR}}|^2(1+h)) \\ & + t^2(|h_{\mathrm{LR}}|^2 + |h_{\mathrm{RR}}|^2)(1-h) + t^2(|h_{\mathrm{RL}}|^2 + |h_{\mathrm{LL}}|^2)(1+h) \big\}. \end{aligned} \tag{4.69}$$

The muon neutrino in the laboratory stems from pion and kaon decays and, therefore, h is -1, to a very good approximation. If one compares the result (4.69) to the prediction of the standard interaction (3.186) one finds

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} \Big/ \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)_{V-A} = \frac{16}{A} \left\{ |g_{\mathrm{LL}}|^2 + \frac{(s+t)^2}{s^2}|g_{\mathrm{RL}}|^2 + \frac{t^2}{4s^2}(|h_{\mathrm{LR}}|^2 + |h_{\mathrm{RR}}|^2) \right\}, \tag{4.70}$$

with $A = 16$ as in Sect. 4.2.3. Remember that we had concluded $g_{\mathrm{LR}} \simeq g_{\mathrm{RL}} \simeq 0$. Reaction (4.68) therefore gives additional limits on the scalar couplings h_{LR} and h_{RR}. Experimentally, the ratio (4.70), after integration over the experimental spectrum, is indeed found to be 1 within about 5%. For the ratio of the total cross sections one finds

$$S = 1.006 \pm 0.048 \tag{4.71}$$

(Mishra et al. 1990, Geiregat 1990).

The result (4.71) is particularly interesting because it can be used to place a *lower* limit on the coupling constant g_{LL}. This follows from the expression (4.70) for the ratio of cross sections: the terms in the couplings g_{RL}, h_{LR} and h_{RR} are known to be small from the measurements of muon decay parameters discussed above. In turn, the lower limit on g_{LL}, through the normalization condition on the rate parameter $A = 16$, (4.54a), yields an *upper* limit on h_{LL} which is

$$|h_{\mathrm{LL}}| < 4(1-S).$$

As first pointed out by Fetscher et al. (1986) the combined analysis of muon decay and inverse muon decay, in principle, allows a complete determination of all coupling constants even though the number of observables is smaller than the number of parameters (including the T-type couplings there are 19 real constants!).

We do not work out the details here but give the essential argument. The specific representation (4.46′) [or, for that matter, the Fierz reordered form (4.49)] of the interaction makes use of the projection operators onto states of definite chirality (helicity for massless fermions). As a consequence, most of the observables depend on absolute squares of coupling constants–as illustrated by the examples we gave above. Now, if quantities such as $(1-\xi)$ or $(1-\xi')$ etc. are found to be zero, within a certain error bar, one concludes at once that several coupling constants must vanish within these error bars. For example, one can calculate the total probability for a muon with chirality μ to turn into an electron with chirality ε, (irrespective of the helicities of the neutrinos). The result is (again omitting T-type couplings),

$$Q(\mu \to \varepsilon) = \frac{1}{4}|h_{\varepsilon\mu}|^2 + |g_{\varepsilon\mu}|^2,\ \mu, \varepsilon = \text{R or L},$$

their sum being normalized to 1 by the rate parameter, viz.

$$Q(\text{R} \to \text{R}) + Q(\text{R} \to \text{L}) + Q(\text{L} \to \text{R}) + Q(\text{L} \to \text{L}) = 1.$$

From muon decay alone, the first three of these are found to be small or zero, while $Q(\text{L} \to \text{L})$ is 1 or close to that value. Obviously, this result, however accurate, does not allow one to discriminate between the couplings g_{LL} and h_{LL}, both of which give the correct chirality/helicity transfer. If one adds the information from inverse muon decay, then, as shown above, g_{LL} is found to be the dominant coupling. Actual limits on couplings other than g_{LL} can be found in Fetscher et al. (1986) and in [RPP10]

f) neutrino helicity from muon decay

An interesting special case where a precision measurement was performed is the following: In most experimental situations muons originate from pion decay $\pi^- \to \mu^- + \overline{\nu}_\mu$(or $\pi^+ \to \mu^+ + \nu_\mu$) and carry longitudinal polarization P_μ. Taking into account this polarization in the decay $\mu^- \to e^- \nu_\mu \overline{\nu}_\text{e}$, (but summing over the polarization of the electron in the final state), the decay asymmetry of the electron with respect to the muon spin, close to the upper end of the spectrum is calculated from (4.53) for $x = 1$, viz.

$$\left(\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\,\mathrm{d}\cos\theta}\right)_{x\to 1} = \frac{m_\mu^5 G^2}{144\pi^3}\rho\left\{1 - P_\mu\frac{\xi\delta}{\rho}\cos\theta\right\}. \tag{4.72}$$

The expression in curly brackets must be positive-semidefinite. Therefore, without returning to the explicit form of the decay parameters we know at once that the specific combination $|P_\mu \xi\delta/\rho|$ cannot exceed 1. The experimental result (Jodidio et al. 1986)

$$P_\mu\frac{\xi\delta}{\rho} > 0.99682 \tag{4.73}$$

being very close to the maximal value 1, and noting that the longitudinal polarization, by definition, cannot be larger than 1, one concludes that both P_μ and $\xi\delta/\rho$

separately are very close to 1. The latter result is relevant for the determination of coupling constants, the former has an immediate consequence for the muon neutrino emitted in pion decay. Indeed, by conservation of angular momentum alone, the helicity $h(\bar{\nu}_\mu)$ of the muon neutrino is equal and opposite to P_μ. Therefore, the experimental result (4.73) implies (Fetscher 1984)

$$1 - |h(\bar{\nu}_\mu)| < 0.003 \text{ at } 90\% \text{ C.L.} \tag{4.74}$$

This result may be converted to information on the helicity of ν_μ from positive pion decay. The sign of $h(\nu_\mu)$ is known to be negative. Equation (4.74) then tells us that the helicity of the ν_μ is -1, within a very small error bar. This is by far the most accurate information on a neutrino helicity.

4.1.3 Neutral Currents in Muonic Atoms

Muonic atoms provide a promising, but still largely unexplored, system for testing parity-violating effects due to weak neutral currents. There are three main reasons why precision tests of NC interactions in muonic atoms are superior, except for problems of purely experimental nature, to tests in electronic atoms or even in electron scattering:

(i) On atomic scales the range of the weak NC interaction is practically zero, so that the effective lepton-nucleon interaction acts like a contact interaction. Due to its larger mass the bound muon moves closer to the nucleus than a bound electron, and, therefore, weak NC effects in muonic atoms are enhanced by some power of the mass ratio m_μ/m_e over such effects in electronic atoms.

(ii) The muon, from pion decay, is fully polarized when it is captured in the Coulomb field of the nucleus. The amount of depolarization (through hyperfine interactions) during the cascade can be measured as well as calculated; the residual polarization in the lowest states of the cascade is finite and is well under control. This is important if one wishes to identify the NC effects by means of some spin-momentum correlation such as the asymmetry of a cascade X-ray with respect to the muon spin.

(iii) The muonic energies and wave functions are calculable to any desired accuracy and the analysis of measured effects in terms of the elementary muon-nucleon NC interaction is well defined. This is in contrast to the case of electronic atoms where the analysis of measured effects (dichroism and optical rotation) is beset with uncertainties about the atomic configuration.

The elementary effective Lagrangian is given by the neutral current part of (3.186) (with $\rho = 1$),

$$-\mathscr{L}_{\mathrm{NC}}^{\mathrm{eff}} = \frac{G}{2\sqrt{2}} K_\alpha^\dagger(x) K^\alpha(x), \tag{4.75}$$

where the cross product of the muonic neutral current and of the nucleonic current, cf. (3.184b), is relevant here. The Lagrangian contains Lorentz *scalar* pieces which stem from the product of leptonic vector V_ℓ and the nucleonic vector V_n, as well as from the product of leptonic axial vector A_ℓ and nucleonic axial vector A_n. These terms will hardly be observable, as they have to compete with the electromagnetic interaction which is of the type $V_\ell V_\mathrm{n}$. It also contains Lorentz *pseudo*scalar pieces which stem from the couplings $V_\ell A_\mathrm{n}$ and $A_\ell V_\mathrm{n}$. These latter terms are parity-odd relative to the electromagnetic interaction and give rise to observable, parity-violating effects in the cascade of a muonic atom.

As an example, let us consider the $n = 2$ system in a light muonic atom. Without the weak NC interaction, the 2p-state decays very quickly into the ls-state through emission of an X-ray photon with the characteristics of an E1-field. The 2s-state is metastable and it will disintegrate to the ground state primarily by emitting two photons. As another possibility it can emit a single photon with the quantum numbers of an M1-field. This latter possibility is due to the relativistic components in the s-states, (remember that a state with $\kappa = -1$, $j = \frac{1}{2}$ contains a "large" component with $l = 0$ and a small component with $l = 1$). Clearly, as a light muonic atom is only moderately relativistic, this M1-transition will be very weak.

If we add the static, Lorentz pseudoscalar NC interaction to the electromagnetic one, the atomic states are no more eigenstates of parity. In particular, the 2s-state will receive a small admixture from the 2p-state, and vice versa. As a consequence, what was a weak (2s–1s) M1-transition before will now have an admixture of the strong (2p–1s) E1-transition, see Fig. 4.3a. The admixture coefficient η is proportional to the matrix element of the parity-odd interaction between the 2s and 2p states divided by an energy denominator which contains the difference of the real energies and the sum of the radiative widths of the 2s and 2p states, respectively, i.e.

$$\eta \simeq \frac{\langle 2\mathrm{p}|H^{\mathrm{NC}}|2\mathrm{s}\rangle}{E_{2s} - E_{2p} + (\mathrm{i}/2)(\Gamma_{2\mathrm{s}} + \Gamma_{2\mathrm{p}})}. \tag{4.76}$$

Here H^{NC} is an effective Hamiltonian which stems from the $V_\ell A_\mathrm{n}$ and $A_\ell V_\mathrm{n}$ couplings in (4.75). An admixture of this kind is detectable by measuring the angular distribution of the photon (with the characteristics of the M1-, E1- mixture), with respect to the orientation of the muon's spin.

Another example is provided by the $n = 3$ system in muonic atoms, cf. Fig. 4.3b. The 3d-state decays preferentially via E1-emission to the 2p-state (and then on to the 1s-state via another E1-transition). However, it also has a small but nonvanishing probability of decaying directly to the 1s-state through emission of an E2 X-ray. The neutral current interaction will cause a small admixture of the 3p- to the 3d-state (and vice versa), so that the weak E2 transition receives a small contribution from the strong (3p-1s) E1 transition. Here again this admixture will show up in the angular distribution of the photon relative to some spin orientation. [In fact, this need not necessarily be the muon spin. It could as well be the spin of the nucleus provided the atom is formed in a polarized target.]

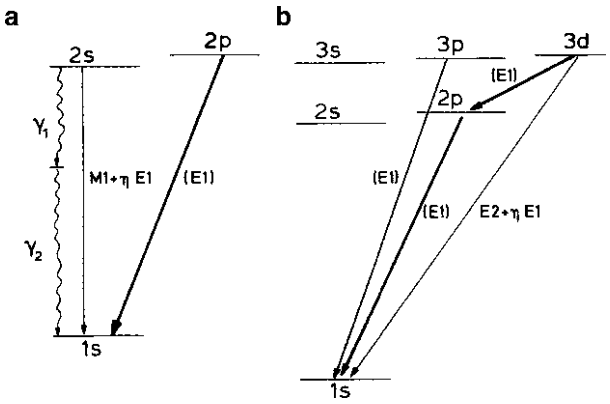

Fig. 4.3 Observable, parity-violating effects in the muonic atom cascade as caused by NC weak interactions. (**a**) The 2p-2s state mixing causes a small admixture of E1 to the direct M1-transition from 2s- to 1s-states. (**b**) The 3d-3p state mixing gives rise to a small admixture of E1 to the dominant (3d-1s) E2-transition

These two examples, as well as many others which have been examined (Missimer et al. 1984), have in common that one studies a weak transition T_0 to which H^{NC} admixes some other, strong transition T_1, such that the parity violating effect in the resulting transition amplitude

$$T \simeq T_0(1 + \eta T_1/T_0) \tag{4.77}$$

is enhanced by the factor T_1/T_0. [Clearly, T_0 must not be suppressed too much because otherwise the transition will have no measureable yield.] Let us briefly analyze H^{NC} as it follows from the basic interaction (3.186). The motion of the nucleons in the nucleus is essentially nonrelativistic. Furthermore, in first approximation the bound nucleons have the same weak coupling constants as when they are free. In the nonrelativistic reduction of the nucleonic vector current only the time component $\langle p'|\upsilon_0(0)|p\rangle$ survives, the space components $\langle p'|\boldsymbol{\upsilon}(0)|p\rangle$ being of higher order in (υ/c). This is easy to see if one examines these matrix elements in the standard representation and in the limit $\upsilon/c \to 0$. Therefore, the interaction term due to the $A_\ell V_{\mathrm{n}}$-couplings becomes

$$H_1 \simeq \frac{G}{\sqrt{2}} \left\{ C_{1\mathrm{p}} \sum_{i-1}^{Z} \delta(\boldsymbol{r}_i - \boldsymbol{r}_\mu) + C_{1\mathrm{n}} \sum_{k=1}^{N} \delta(\boldsymbol{r}_k - \boldsymbol{r}_\mu) \right\} \gamma_5, \tag{4.78}$$

where the first sum runs over the protons, the second over the neutrons in the nucleus, and where γ_5 acts on the relativistic bound state wave functions of the

muon. The parameters in (4.78) are easy to identify from (3.186, 3.184, 3.208):[5]

$$C_{1\mathrm{p}} = \frac{1}{2}(1 - 4\sin^2\theta_{\mathrm{w}}),$$
$$C_{1\mathrm{n}} = -\frac{1}{2}.$$

Similarly, it is easy to see that the nonrelativistic limit of the nucleonic axial current $\langle p'|a_\alpha(0)|p\rangle$ vanishes if $\alpha = 0$, and gives a matrix element of the spin operator $\sigma^{(i)}$ for $\alpha = i$. Thus the interaction due to the $V_\ell\ A_{\mathrm{n}}$ couplings is approximately

$$H_2 \simeq \frac{G}{\sqrt{2}}\left\{C_{2p}\sum_{i=1}^{Z}\boldsymbol{\sigma}_i\delta(\boldsymbol{r}_i - \boldsymbol{r}_\mu) + C_{2\mathrm{n}}\sum_{k=1}^{N}\boldsymbol{\sigma}_k\delta(\boldsymbol{r}_k - \boldsymbol{r}_\mu)\right\}\cdot\boldsymbol{\alpha} \tag{4.79}$$

where the matrices $\boldsymbol{\alpha}$ are defined in eqs. (1.81) and act on the muonic wave functions. The constants in (4.79) are found from (3.208, 3.209) to be

$$C_{2\mathrm{p}} = -\frac{1}{2}F_A(0)(1 - 4\sin^2\theta_{\mathrm{w}}),$$
$$C_{2\mathrm{n}} = \frac{1}{2}F_A(0)(1 - 4\sin^2\theta_{\mathrm{w}}).$$

Without going into the detailed analysis of nuclear matrix elements of H_1, (4.78) and H_2, (4.79), we can make the following general remarks. The expectation value of H_1 in the nuclear ground state is proportional to a linear combination of proton and neutron densities,

$$ZC_{1\mathrm{p}}\rho_{\mathrm{p}}(\boldsymbol{r}_\mu) + NC_{1\mathrm{n}}\rho_{\mathrm{n}}(\boldsymbol{r}_\mu)$$

with $C_{1\mathrm{P}} \simeq 0.04$ (for $\sin^2\theta_{\mathrm{w}} \simeq 0.23$ and $C_{1\mathrm{n}} = -0.50$. In a static situation where the nucleus remains in the ground state, the nucleons contribute coherently, each with its own weak neutral coupling $(1 - 4\sin^2\theta_{\mathrm{w}})$ for protons, 1 for neutrons.

In the same situation, the nuclear matrix element of H_2 either vanishes, when the nucleus has spin $J = 0$, or else, if $J \geq \frac{1}{2}$, is proportional to $< \boldsymbol{\sigma} >$, the matrix element of the spin operator of the unpaired valence nucleon. Except for the possibility of collective state admixtures, this matrix element is a typical one-particle matrix element. Therefore, in the static situation, one expects the interaction H_1 to dominate over the term H_2. This is indeed the result of detailed investigations of this problem. Therefore, experiments with muonic atoms test primarily the $A_\ell\ V_{\mathrm{n}}$ couplings. The couplings of the type $V_\ell A_{\mathrm{n}}$ which appear in H_2 (4.79), should be accessible, too, but will be more difficult to isolate.

[5]Here we are using the fact that the "charge" form factor of the vector current at zero momentum transfer is not renormalized by strong interactions if CVC holds. See the discussion of CVC in Sect. 4.2.2 below.

In order to give a feeling for the order of magnitude, we quote the example of muonic thulium $^{169}_{69}\mathrm{Tm}_{\mu}$: The forward-backward asymmetry of X-rays from the 3d-1s transition in this atom (with respect to the muonic or nuclear spin direction). due to parity violating NC interactions is calculated to be -1.2×10^{-5}.

Finally, we note and emphasize that the muonic atom is perhaps the best possibility of testing muon-electron universality of the weak NC interactions.

4.2 Hadronic Charged Current Interactions

In this section we return to another facet of weak interactions that we mentioned in the introduction to Chap 3: Assuming the Lorentz structure and selection rules of hadronic weak interactions to be known, the weak currents can be regarded as still another class of probes for the structure of hadronic targets. In close analogy to the role of the electromagnetic current in electron scattering on hadrons, the weak vector and axial vector currents give access to internal properties of mesons and baryons, mainly through the characteristics of their weak decay modes. Nucleons and nuclei, in particular, can also be probed in muon capture and in exclusive and inclusive neutrino scattering.

While muon capture is touched upon briefly in Sect. 4.2.4b, we concentrate primarily on *semileptonic* decays of mesons and nucleons for which we choose some illustrative and instructive examples.

4.2.1 Semileptonic Interactions, Structure of Hadronic Charged Current

Our starting point is the effective current × current interaction (3.186) with $J_\alpha(x)$, the weak charged current, as given by (3.184a). In the product of this current with its hermitean conjugate there appear terms of the form

$$-\mathscr{L}^{\mathrm{CC}}_{\mathrm{semileptonic}} = \frac{G}{\sqrt{2}} \sum_{\mathrm{f,f'}} \{(\overline{f(x)}\gamma^\alpha(1-\gamma_5)\nu_{\mathrm{f}}(x)) \times \overline{(u_{\mathrm{f'}}(x)}\gamma_\alpha(1-\gamma_5)d_{\mathrm{f'}}x) - \mathrm{h.c.}\}, \tag{4.80}$$

which couple two lepton fields to two quark fields at the same point of space-time. Equation (4.80), as well as its analogue with the neutral current K_α, (3.184b), defines and describes the so-called *semileptonic processes*. After the purely leptonic weak interactions this is a second, simple class of interactions: A semileptonic process involves one hadronic matrix element of the charged current multiplied by a leptonic vertex which is assumed to be known and which, therefore, is perfectly

calculable. In other words, semileptonic processes give information on matrix elements of the hadronic pieces of J_α, viz.

$$\left\langle \mathrm{B} \left| \sum_{\mathrm{f}=1}^{3} \overline{u_{\mathrm{f}}(0)} \gamma_\alpha (1-\gamma_5) d_{\mathrm{f}}(0) \right| \mathrm{A} \right\rangle . \tag{4.81}$$

Here A is a one-boson or one-baryon state, while B can be the vacuum, another one-boson or one-fermion state, etc., depending on the selection rules which apply to the matrix elements (4.81). Examples we shall encounter are

$$\begin{aligned} &\langle 0|J_\alpha(0)|\pi^+\rangle,\ \langle 0|J_\alpha(0)|\mathrm{K}^+\rangle,\ \langle \pi^0|J_\alpha(0)|\pi^+\rangle, \\ &\langle \gamma|J_\alpha(0)|\pi^+\rangle,\ \langle \mathrm{p}|J_\alpha(0)|\mathrm{n}\rangle,\ \text{etc.} \end{aligned} \tag{4.81'}$$

All effective coupling constants in (4.80) are known combinations of G and the elements of the mixing matrix V (3.152). What we do not know are the effects of the *strong* interactions on the matrix elements (4.81) which originate in the nature of hadrons being bound states of quarks and in hadronic corrections of higher order. These effects are parametrized in the form of Lorentz scalar form factors which then are purely hadronic objects, i.e. in some general sense are "structure functions" of the hadronic system under investigation.

The charged current (3.184a) contains nine different terms

$$\sum_{f=1}^{3} \sum_{f'=1}^{3} \overline{u_{\mathrm{f}}(x)} \Gamma^\alpha V_{\mathrm{ff}'} b_{\mathrm{f}'}(x), \qquad \Gamma^\alpha \equiv \gamma^\alpha (1-\gamma_5),$$

which have different selection rules as far as the internal quantum numbers of hadronic states are concerned. In Table 4.1 we list the pieces $\bar{u}\mathrm{d}$ and $\bar{u}\mathrm{s}$ which are relevant for the discussion in this section, together with their strong isospin, spin/parity, and strangeness properties. Unlike the case of the electromagnetic current, cf. Table 2.1, the $\bar{u}\mathrm{d}$-current is not diagonal under charge conjugation: Indeed, from the discussion in Sect. 3.2.1 we know that

$$\begin{aligned} \overline{u(x)}\,\gamma^\alpha\, d(x) &\underset{C}{\longrightarrow} -\overline{d(x)}\,\gamma^\alpha\, u(x), \\ \overline{u(x)}\,\gamma^\alpha\,\gamma_5\, d(x) &\underset{C}{\longrightarrow} +\overline{d(x)}\,\gamma^\alpha\,\gamma_5\, u(x). \end{aligned}$$

However, if we apply a rotation $\exp\{\mathrm{i}\pi I_2\}$ in isospin space to the fields $u(x)$ and $d(x)$, the currents on the r.h.s. are mapped back onto the originals on the l.h.s. From (1.45) we have

$$\mathrm{e}^{\mathrm{i}\pi I_2}\, u(x) = d(x),\ \mathrm{e}^{\mathrm{i}\pi I_2}\, d(x) = -u(x).$$

Thus, if one applies the combined operation of G-parity to the $\bar{u}$ d-currents $G = \exp\{\mathrm{i}\pi I_2\}\, C$, the vector current is seen to be even, the axial current to be odd.

Table 4.1 Quantum numbers of hadronic weak currents in the u, d, s-sector of the Lagrangian

Current	Strong Isospin I	Strong Isospin I_3	Spin/Parity J^π	ΔS	G-parity G	Analogue meson states
$\bar{u}\gamma^\alpha d \cos\theta_1$	1	1	$\left\{ \begin{matrix} 0^+ \\ 1^- \end{matrix} \right\}$	0	+	$\rho(770)$
$\bar{u}\gamma^\alpha\gamma_5 d \cos\theta_1$	1	1	$\left\{ \begin{matrix} 0^- \\ 1^+ \end{matrix} \right\}$	0	−	$\pi(139)$, $a_1(1260)$
$\bar{u}\gamma^\alpha s \sin\theta_1 \cos\theta_3$	$\frac{1}{2}$	$\frac{1}{2}$	$\left\{ \begin{matrix} 0^+ \\ 1^- \end{matrix} \right\}$	1		$K^*(892)$
$\bar{u}\gamma^\alpha\gamma_5 s \sin\theta_1 \cos\theta_3$	$\frac{1}{2}$	$\frac{1}{2}$	$\left\{ \begin{matrix} 0^- \\ 1^+ \end{matrix} \right\}$	1		$K(494)$, $K_1(1270)$

In the last column of Table 4.1, finally, we list a few meson states that have the same quantum numbers as the currents listed in the first column. It is worth nothing that the $\bar{u}$ d vector current has the same quantum numbers as the isovector part of the electromagnetic current as shown in the second line of Table 2.1, p. 102.

4.2.2 *Pion Beta Decay and Conserved Vector Current (CVC)*

We noted above the close similarity of the $\bar{u}$d vector current, first line of Table 4.1, and of $j_\alpha^{(1)}$, the isovector part of the electromagnetic current, second line of Table 2.1. In a world of free u- and d-quarks these currents are, respectively,

$$j_\alpha^{(1)}(x) = \frac{1}{2}\left\{\overline{u(x)}\,\gamma_\alpha\, u(x) - \overline{d(x)}\,\gamma_\alpha\, d(x)\right\} = \frac{1}{2}\,\overline{N(x)}\,\gamma_\alpha\,\tau_3\,N(x), \tag{4.82a}$$

where $N \equiv \begin{pmatrix} u \\ d \end{pmatrix}$, and

$$\upsilon_\alpha^{(1+i2)}(x) = \overline{u(x)}\,\gamma_\alpha\, d(x) = \frac{1}{2}\,\overline{N(x)}\,\gamma_\alpha\,(\tau_1 + i\tau_2)\,N(x), \tag{4.82b}$$

where (4.82a) follows from (3.155). It is obvious that these currents are components of one and the same triplet of isovector operators. If the masses of u and d quarks are exactly equal then υ_α is conserved,

$$\partial^\alpha\,\upsilon_\alpha^{(1+i2)}(x) = 0. \tag{4.83}$$

The hypothesis of the conserved vector current (CVC) states that these properties hold for the strangeness-conserving vector current, even when it is "dressed" by the strong interactions, viz.

(i) $\upsilon_\alpha^{(1\pm i2)}$ is conserved,
(ii) $\upsilon_\alpha^{(1+i2)}$ and $\upsilon_\alpha^{(1-i2)}$ are isospin components of a triplet of operators whose third member is $j_\alpha^{(1)}$, the isovector part of the electromagnetic current.

As an example let us consider pion β-decay

$$\pi^-(q) \rightarrow \pi^0(q') + \mathrm{e}(p) + \bar{\nu}_e(k), \tag{4.84}$$

where we have written the momenta in parentheses. The T-matrix element reads

$$T(\pi^- \rightarrow \pi^0 \mathrm{e}^- \bar{\nu}_e) = \frac{G}{\sqrt{2}} \cos\theta_1 \frac{1}{(2\pi)^3} \overline{u_e(p)}\, \gamma^\alpha (1-\gamma_5) \upsilon_\nu(k) \times \langle \pi^0(q') \mid \upsilon_\alpha^{(1+i2)}(0) \mid \pi^-(q) \rangle. \tag{4.85}$$

A general form factor decomposition of the pionic matrix element is

$$\langle \pi^0(q') \mid \upsilon_\alpha^{(1+i2)}(0) \mid \pi^{-(q)} \rangle = \frac{1}{(2\pi)^3} \{ (q_\alpha + q'_\alpha) f_+(Q^2) + (q_\alpha - q'_\alpha) f_-(Q^2) \}, \tag{4.86}$$

with $Q^2 := (q-q\prime)^2 = m^2 + m_0^2 - 2m\sqrt{m_0^2+\kappa^2}$ and where κ is the magnitude of the 3-momentum of π^0 in the rest system of the decaying π^- · m denotes the mass of the charged pion, m_0 the mass of π^0. If υ_α is conserved, we have the condition

$$(m^2 - m_0^2) f_+(Q^2) + Q^2 f_-(Q^2) = 0.$$

The contribution of f_- to T, (4.85), is multiplied by $(q - q') = (p + k)$ and, therefore, by a factor m_e which is small compared to the term $(q + q')\, f_+$. In the limit $m = m_0$ (exact isospin symmetry), one has $f_-(Q^2) \equiv 0$. Nelecting the second form factor one obtains the following expression for the decay width (see below)

$$\Gamma(\pi^- \rightarrow \pi^0 \mathrm{e}^- \bar{\nu}_\mathrm{e}) \simeq \frac{G^2 \cos^2\theta_1\, \Delta^5}{60\pi^3} \left(1 - \frac{3}{2}\frac{\Delta}{m}\right) K(\varepsilon)\, f_+^2(0), \tag{4.87}$$

Where $\Delta = m - m_0 \simeq 4.6$ MeV, $\varepsilon \equiv m_\mathrm{e}^2 / \Delta^2$; $K(\varepsilon)$ is a correction factor given by

$$K(\varepsilon) = \sqrt{1-\varepsilon}(1 - \tfrac{9}{2}\varepsilon + 4\varepsilon^2) + \tfrac{15}{2}\varepsilon^2 \ln\left(\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\right) \simeq 1 - 5\varepsilon.$$

We have replaced $f_+(Q^2)$ by its value at $Q^2 = 0$ because the domain of variation is very small in this decay, $m_\mathrm{e}^2 \leq Q^2 \leq \Delta^2$.

The value of $f_+(0)$ is fixed by the second part of the CVC hypothesis. In order to see this let us note that $I_+ := \int \mathrm{d}^3 x\, \upsilon_0^{(1+i2)}(x)$ is a generator of infinitesimal transformations in isospin space, provided isospin is an exact global symmetry. Thus

$$\langle \pi^0 \Big| \int \mathrm{d}^3 x\, \upsilon_0^{(1+i2)}(x) \Big| \pi^- \rangle = \langle \pi^0 \mid I_+ \mid \pi^- \rangle = \sqrt{2}\, 2E_q\, \delta(\boldsymbol{q} - \boldsymbol{q}').$$

On the other hand, making use of the translation formula (2.42), the left-hand side is equal to

$$\int \mathrm{d}^3\, x\, \mathrm{e}^{\mathrm{i}x(q'-q)}\, \langle \pi^0|\, \upsilon_0^{(1+\mathrm{i}2)}\,(0)\,|\pi^-\rangle = (2\pi)^3\,\delta\,(\boldsymbol{q}-\boldsymbol{q}')\,\frac{1}{(2\pi)^3}\,2E_q\,f_+(0),$$

which gives the result $f_+(0) = \sqrt{2}$. Note that in this derivation we have used exact isospin symmetry in the pion states and, consequently (except for the kinematics), we have set $m = m_0$.

The absolute square of the matrix element, summed over the spin is worked out to be

$$\begin{aligned}(2\pi)^{12}\sum|T|^2 = 4G^2\,\cos^2\,\theta_1\{&f_+^2\,(2(ap)(ak)-a^2(kp))\\ &+f_-^2\,m_\mathrm{e}^2\,(kp)+2f_+f_-\,m_\mathrm{e}^2\,(ak)\},\end{aligned}\tag{4.88}$$

where we have set $a = q+q' = 2q-Q$. The terms which contain the factor m_e^2 are negligibly small, of order m_e^2/m_π^2, as compared to the others. For the calculation of the total width we use the analogue of (4.42) above, viz.

$$\frac{\mathrm{d}^3\Gamma}{\mathrm{d}^3q'} = \frac{1}{4m\,E_{q'}(2\pi)^5}\int\frac{\mathrm{d}^3p}{2E_p}\int\frac{\mathrm{d}^3k}{2E_k}\,\delta(Q-p-k)\,(2\pi)^{12}\sum|T|^2.\tag{4.89}$$

The integrations over $\boldsymbol{p}$ and $\boldsymbol{k}$ are performed in the same way as the integration over the neutrino momenta in the case of muon decay, except that here we must keep the mass of the electron. Explicit calculation (exercise 4.5) shows that (4.44b) is

$$\begin{aligned}&\int\frac{\mathrm{d}^3\,p}{2E_p}\int\frac{\mathrm{d}^3\,k}{2E_k}\,(p^\alpha k^\beta-(pk)\,g^{\alpha\beta}+k^\alpha p^\beta)\,\delta\,(Q-p-k)\\ &=\frac{\pi}{6}\frac{(Q^2-m_\mathrm{e}^2)^2}{(Q^2)^2}\left\{\left(1+2\frac{m_\mathrm{e}^2}{Q^2}\right)Q^\alpha\,Q^\beta-\left(1+\frac{m_\mathrm{e}^2}{2Q^2}\right)Q^2\,g^{\alpha\beta}\right\}.\end{aligned}\tag{4.90}$$

It is not possible to expand this expression in terms of m_e^2 because m_e^2 is not small compared to Q^2. In calculating

$$\Gamma = \int \boldsymbol{q}'^2\,\mathrm{d}|\boldsymbol{q}'|\,\frac{1}{E_{q'}}\ldots$$

it is useful to transform both $|\boldsymbol{q}'|$ and $E_{q'}$ to the variable Q^2. From

$$E_{q'} = \frac{1}{2m}\,(m^2+m_0^2-Q^2)$$

we have

$$\frac{1}{E_{q'}}\,\boldsymbol{q}'^2\,\mathrm{d}|\boldsymbol{q}'| = |\boldsymbol{q}'|\;\mathrm{d}\,E_{q'} = -\frac{1}{2m}\,|\boldsymbol{q}'|\mathrm{d}(Q^2)$$

with

$$\begin{aligned}|\boldsymbol{q}'| &= \frac{1}{2m}\,\sqrt{(m^2+\,m_0^2-Q^2)^2-4m^2m_0^2}\\ &= \frac{1}{2m}\,\sqrt{(\Delta^2-Q^2)\;(\Delta^2-Q^2+4mm_0)}\\ &= \frac{1}{m}\,\sqrt{mm_0}\;\sqrt{(\Delta^2-Q^2)\;\;\left(1+\frac{\Delta^2-Q^2}{4mm_0}\right)}.\end{aligned}$$

The following relations are needed when (4.90) is contracted with $a_\alpha a_\beta$, as prescribed by (4.88),

$$(aQ) = \Delta(m+m_0),\; a^2 = 2(m^2+m_0^2) - Q^2.$$

Finally, it is convenient to introduce the dimensionless integration variable

$$z := Q^2/\,\Delta^2.$$

Neglecting small terms of the order of Δ^2/m^2 in the integrand one then has the following expression for the total width:

$$\begin{aligned}\Gamma \;\simeq\; &\frac{G^2\,\cos^2\,\theta_1\,f_+^2\,(0)}{24\pi^3 m}\,\frac{(m+m_0)^2}{4m^2}\;\sqrt{mm_0}\,\Delta^5\\ &\int_\varepsilon^1 \mathrm{d}z\;\sqrt{1-z}\,\frac{(z-\varepsilon)^2}{z^2}\left\{1-\frac{\varepsilon}{2}-z+2\frac{\varepsilon}{z}\right\}.\end{aligned}$$

The mass factors can be approximated as follows: With $m_0 = m - \Delta$ one has

$$\frac{1}{4m^3}(m+m_0)^2\,\sqrt{mm_0} \simeq 1-\frac{3}{2}\,\frac{\Delta}{m}.$$

The integral over z can be performed by elementary means [KAL64], giving $\int_\varepsilon^1 \mathrm{d}z\,\cdots = \frac{2}{5}\,K(\varepsilon)$ with $K(\varepsilon)$ as given above. Thus, (4.87) is proven.

The experimental branching ratio of pion beta decay

$$\frac{\Gamma(\pi^-\to\pi^0\,\mathrm{e}^-\bar{\nu}_\mathrm{e})}{\Gamma(\pi\;\to\;\mu\nu_\mu)}$$

is found to be $1.036(6)\times10^{-8}$ which is in a very good agreement with the theoretical prediction of $1.035(5)\times10^{-8}$ (which includes radiative corrections), and provides

the most precise test of the CVC hypothesis at this time. The applications and predictions of CVC in nucleonic currents are dealt with in Sect. 4.2.4 below.

4.2.3 The Strangeness-Conserving Axial Current

a) Pion decays $\pi\ell 2$ and $\tau \to \pi\nu$ decay

The decay modes

$$\pi^- \to \mu^- \bar{\nu}_\mu, \quad \pi^- \to \mathrm{e}^- \bar{\nu}_\mathrm{e}, \tag{4.91a}$$

$$\tau^+ \to \pi^+ \bar{\nu}_\tau, \tag{4.91b}$$

all involve a leptonic factor times a matrix element of the hadronic weak current between a one-pion state and the vacuum. In reactions (4.91a) it is $\langle 0|\, J_\alpha^{(1+\mathrm{i}2)}\,(0)\,|\,\pi^-(q)\rangle$, in (4.91b) it is $\langle \pi^+(q)|\, J_\alpha^{(1+\mathrm{i}2)}\,(0)\,|\,0\rangle$. The discussion in Sect. 3.1 showed that only the axial part of the current can contribute to this matrix element, cf. (3.6). Furthermore, knowing the behaviour of the axial current under time reversal and charge conjugation one shows that the factor F_π in (3.6) is pure imaginary (exercise 4.6) thus

$$\langle 0|\, a_\alpha^{(1+\mathrm{i}2)}\,(0)\,|\,\pi^-(q)\rangle = \frac{\mathrm{i}}{(2\pi)^{3/2}}\, f_\pi\, q_\alpha. \tag{4.92}$$

The amplitude for the decays (4.91a) then reads

$$\begin{aligned} T_{\pi^- \to f^- \bar{\nu}_f} &= \frac{\mathrm{i}}{(2\pi)^{9/2}}\, \frac{f_\pi\; G\; \cos\,\theta_1}{\sqrt{2}}\; \overline{u_\mathrm{f}(p)} \not{q} (1-\gamma_5) \upsilon_\nu\,(k) \\ &= \frac{\mathrm{i}}{(2\pi)^{9/2}}\, \frac{f_\pi\, G\, \cos\theta_1}{\sqrt{2}}\, m_\mathrm{f}\, \overline{u_\mathrm{f}(p)} (1-\gamma_5) \upsilon_\nu\,(k), \end{aligned}$$

with f = e or μ.

It is not difficult to calculate the decay width from this amplitude. One finds

$$\Gamma(\pi \to \mathrm{f}\nu_\mathrm{f}) = \frac{G^2\, \cos^2\,\theta_1\, f_\pi^2\, m_\pi}{8\pi}\; m_\mathrm{f}^2\, (1 - m_\mathrm{f}^2/m_\pi^2)^2, \tag{4.93}$$

from which one obtains the ratio of $\pi \to \mathrm{e}\nu_e$ to $\pi \to \mu\nu_\mu$ that we anticipated in (3.38) (see also the discussion there). Knowing that the mode $\pi \to \mu\nu_\mu$ amounts to practically 100% of the pion decays, one deduces from the experimental lifetime $\tau_\pi = 2.6030(24) \times 10^{-8}$ s and from $|\cos\theta_1| = 0.974$ a value for f_π:

$$f_\pi \simeq 0.944 m_{\pi^+} \simeq 0.132\,\mathrm{GeV}. \tag{4.94}$$

In a similar way the decay (4.91b) is described by the amplitude

$$T_{\tau^+\to\pi^+\bar{\nu}_\tau} = \frac{\mathrm{i}}{(2\pi)^{9/2}}\frac{f_\pi \mathrm{G}\cos\theta_1}{\sqrt{2}}\overline{\upsilon_\nu(k)}\not{q}(1-\gamma_5)\upsilon_\tau(p)$$
$$= \frac{\mathrm{i}}{(2\pi)^{9/2}}\frac{f_\pi \mathrm{G}\cos\theta_1}{\sqrt{2}} m_\tau \overline{\upsilon_\nu(k)}(1+\gamma_5)\upsilon_\tau(p). \tag{4.95}$$

In this case it is useful to keep the correlation term between the spin of the τ and the pion momentum. If the pion is emitted at an angle θ with respect to the expectation value of s_τ, the differential decay probability is found to be, from (4.95)

$$\frac{\mathrm{d}\Gamma(\tau^+\to\pi^+\bar{\nu}_\tau)}{\mathrm{d}(\cos\theta)} = \Gamma(\tau\to\pi\nu_\tau)\frac{1}{2}(1-\cos\theta), \tag{4.96}$$

where

$$\Gamma(\tau\to\pi\nu_\tau) = \frac{G^2\cos^2\theta_1 f_\pi^2 m_\tau^3}{16\pi}(1-m_\pi^2/m_\tau^2)^2. \tag{4.97}$$

The decay mode has a branching ratio of about 10% and is found in good agreement with the prediction (4.97), (see exercise 4.7)

There are two different definitions of f_π (4.92), in the literature on particle physics. The first is the one we have adopted in (4.92), where a *physical* matrix element of the isospin raising operator $a_\alpha^{(1+\mathrm{i}2)}$ between a one-pion state and the vacuum is decomposed in terms of covariants. We can write this current in terms of Cartesian components, viz. $a_\alpha^{(1+\mathrm{i}2)} = a_\alpha^{(1)} + \mathrm{i}a_\alpha^{(2)}$. If these were given in terms of free quark fields they would read

$$a_\alpha^{(i)} = \overline{N(x)}\gamma_\alpha\gamma_5\frac{\tau^{(i)}}{2}N(x).$$

Obviously, the π^- state can be written in Cartesian coordinates, too,

$$|\pi^-\rangle = \frac{1}{\sqrt{2}}\{|\pi_1\rangle - \mathrm{i}|\pi_1\rangle\},$$

so that

$$\langle 0|a_\alpha^{(1+\mathrm{i}2)}|\pi^-\rangle = \frac{1}{\sqrt{2}}\left\{\langle 0|a_\alpha^{(1)}|\pi_1\rangle + \langle 0|a_\alpha^{(2)}|\pi_2.\rangle\right\}$$

This implies that the Cartesian matrix elements are

$$\langle 0|a_\alpha^{(j)}(0)|\pi_k(q)\rangle = \frac{\mathrm{i}}{(2\pi)^{3/2}}\delta_{jk}\frac{1}{\sqrt{2}}f_\pi q_\alpha. \tag{4.98}$$

On many occasions it is convenient to work with Cartesian, rather than spherical coordinates. For example, in studying commutators of current densities $\upsilon_\alpha^{(i)}(x)$ and

$a_\alpha^{(j)}(x)$ one usually makes use of the Cartesian notation which treats the three isospin components in a symmetric (in fact, cyclic) way. For this reason some authors perfer to use (4.98) as the defining equation for the pion decay constant, but without the factor $1/\sqrt{2}$ on the r.h.s. Thus, another definition (not used in this book) is

$$\bar{f_\pi} = \frac{1}{\sqrt{2}} f_\pi \simeq 0.667 m_{\pi^+} \simeq 0.0933\,\text{GeV}. \tag{4.99}$$

Additional remark. Unfortunately this is not the only point where confusion may arise. In accord with standard phase conventions for the rotation group it would seem appropriate to express a π^+ -state in the form

$$|\pi^+\rangle = -\frac{1}{\sqrt{2}}\{|\pi_1\rangle + \mathrm{i}|\pi_2\rangle\}, \tag{4.100}$$

with the characteristic minus sign of the spherical basis. With this convention the isospin rotation contained in the definition of G-parity is indeed $\mathrm{e}^{\mathrm{i}\pi I_2}$ whose matrix representation in the space of a unitary irreducible multiplet is $(-)^{I-\mu}\delta_{\mu-m}$. This rotation (which effects the transition from cogredience to contragredience) correctly transforms the generators of the strong isospin, see exercise 1.3. Unfortunately, since the early days of analyzing pion-nucleon scattering it has become customary to define the $|\pi^+\rangle$ state with a plus sign, in disaccord with the spherical basis. In this latter convention the rotation factor in G must be chosen to be $\mathrm{e}^{\mathrm{i}\pi I_1}$.

There is no problem in adopting the spherical basis (4.100) provided it is done consistently throughout the calculation. Attention to phases must be paid, however, in handling matrix elements of isospin raising and lowering currents $j_\alpha^{(1\pm\mathrm{i}2)} = j_\alpha^{(1)} \pm \mathrm{i} j_\alpha^{(2)}$ which neither are normalized nor are written in the spherical basis.

b) Axial current and pion field: PCAC

It is remarkable that the axial currents $a_\alpha^{(3)}$ and $a_\alpha^{(1\pm\mathrm{i}2)}(x)$ carry the internal quantum numbers of the pion states π^0, and $\pi^\pm$, respectively. Their divergences $\partial^\alpha a_\alpha^{(\lambda)}$, in particular, define pseudoscalar fields which have the same behaviour with respect to Lorentz transformations as the pion fields. For example, the pion-to-vacuum matrix element of the divergence of the isospin raising component can be calculated from (4.92). Using the translation formula (2.42) we have

$$\begin{aligned}\partial_x^\alpha \left\langle 0|a_\alpha^{(1+\mathrm{i}2)}(x)|\pi^-(q)\right\rangle &= \frac{\mathrm{i}}{(2\pi)^{3/2}} f_\pi q_\alpha \partial^\alpha \mathrm{e}^{-\mathrm{i}qx} = \frac{1}{(2\pi)^{3/2}} f_\pi q^2 \mathrm{e}^{-\mathrm{i}qx} \\ &= \frac{1}{(2\pi)^{3/2}} f_\pi m_\pi^2 \mathrm{e}^{-\mathrm{i}qx}. \end{aligned} \tag{4.101}$$

This equation shows clearly that the axial current cannot be conserved. If it were then either f_π would have to vanish, in which case pions would not decay, or m_π^2 would have to be zero. [The second alternative is very interesting because it shows that in models with a conserved axial current pions may appear as massless Goldstone bosons.]

The divergence $\partial^\alpha a_\alpha^{(\lambda)}(x)$ is a pseudoscalar field operator which has all properties of a pion field $\phi^{(\lambda)}(x)$, except for the normalization. Indeed, an interpolating pion field must be normalized such that a one-pion matrix element is

$$\langle 0|\phi_\pi(x)|\pi(q)\rangle = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-iqx}.$$

Thus (expressed in Cartesian coordinates) we can define

$$\phi_\pi^{(i)}(x) := \frac{\sqrt{2}}{f_\pi m_\pi^2} \partial^\alpha a_\alpha^{(i)}(x) \tag{4.102}$$

and use this operator as interpolating field for the pion. As it stands, this definition is just one possible choice and contains no more than the statements that the divergence carries the internal quantum numbers of the pion and that it is correctly normalized.

By the assumption of PCAC (partial conservation of the axial current) it is understood that typical one-particle matrix elements of the kind

$$\left\langle \mathrm{B}; q'|(\Box + m_\pi^2)\phi_\pi(x)|\mathrm{A}; q\right\rangle \tag{4.103}$$

are smooth functions of the invariant, squared momentum transfer $Q^2 = (q-q')^2$, in the interval $-m_\pi^2 < Q^2 \lesssim m_\pi^2$. In other words, it is assumed that a vertex function $\langle \mathrm{B}|\phi_\pi|\mathrm{A}\rangle$ is dominated by the pion pole $1/(Q^2 - m_\pi^2)$ whose residue is a slowly varying function of Q^2.

In the next section we shall discuss two examples of applying PCAC to pion-nucleon vertices. Generally speaking, PCAC is a useful approximation whenever one has to estimate simple matrix elements of weak axial currents, or combinations of such currents with other currents. It is also useful when one wishes to take external pions off their mass shell, replace their interpolating field by the divergence $\partial^\alpha a_\alpha(x)$ and extrapolate to $q^2 = 0$ (so-called *soft-pion* method). In practice, this approximation is usually found to be a rather good one, in those examples where it can actually be tested. This, presumably, is a reflection of the observation that the physical world of hadrons is not very far from an idealized world in which the axial current is strictly conserved, like the vector current, and in which pions are massless Goldstone bosons of a spontaneously broken symmetry. This limit is called the limit of *chiral symmetry*, (see also Sect. 3.5).

4.2.4 CVC and PCAC as Applied to Nucleonic Currents

a) Nucleonic vector currents and CVC

The weak charged vector current, taken between a neutron state of momentum q and a proton state of momentum q' has the general decomposition into covariants,

$$\langle \mathrm{p}; q' | \upsilon_\alpha^{(1+\mathrm{i}2)}(0) | \mathrm{n}; q \rangle = \frac{1}{(2\pi)^3} \overline{u_p(q')} \left\{ \gamma_\alpha F_\mathrm{V}(Q^2) + \mathrm{i}\sigma_{\alpha\beta} \frac{Q^\beta}{2m_\mathrm{N}} F_\mathrm{M}(Q^2) + \frac{Q_\alpha}{2m_\mathrm{N}} F_3(Q^2) \right\} u_n(q), \tag{4.104}$$

with $Q = q' - q$ [see (2.46)]. The CVC hypothesis, when applied to this specific case, has two consequences:

(i) Very much like in the case of the electromagnetic current, the conservation condition (4.83) implies that $F_3(Q^2)$ vanishes identically.

(ii) The isotriplet character of $\upsilon_\alpha^{(1\pm\mathrm{i}2)}$ and $j_\alpha^{(1)}$ implies that the form factors F_V and F_M are identical with the nucleonic isovector form factors (2.66b), to within a factor 2,

$$F_\mathrm{V}(Q^2) = F_1^{(\mathrm{p})}(Q^2) - F_1^{(\mathrm{n})}(Q^2), \tag{4.105a}$$

$$F_\mathrm{M}(Q^2) = F_2^{(\mathrm{p})}(Q^2) - F_2^{(\mathrm{n})}(Q^2), \tag{4.105b}$$

(see also exercise 4.8). In particular, CVC makes the important prediction

$$F_\mathrm{V}(0) = 1, \tag{4.106}$$

which implies that superallowed nuclear Fermi transitions have the strength $G \cos\theta_1$, without any renormalization by the strong interaction. Furthermore, (4.105b) implies

$$F_\mathrm{M}(0) = \mu_\mathrm{an}^\mathrm{p} - \mu_\mathrm{an}^\mathrm{n} \simeq 3.706, \tag{4.107}$$

so that the "magnetic" terms in (4.104) are fixed at $Q^2 = 0$. This latter prediction is the basis of a famous test of CVC in the isotriplet of 1^+ -states in ^{12}B(g.s), ^{12} N(g.s.), and ^{12}C*, which decay to the state ^{12}C(0^+ g.s.) through β^-, β^+ and M1-γ-decay, respectively.

Finally, CVC predicts that the dependence of the form factors (4.105) on Q^2 is the same as that of the isovector, electric and magnetic form factors.

b) Nucleonic axial vector currents and PCAC

Let us consider the vertex pnπ, Fig. 4.4, with the pion on or off its mass shell, and carrying the momentum $Q = q' - q$.

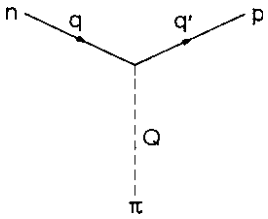

Fig. 4.4 Strong interaction vertex proton-neutron-pion

The pion's source term j_π is defined by the action of the Klein–Gordon operator on the field $\phi_\pi(x)$:

$$j_\pi(x) := (\Box + m_\pi^2)\phi_\pi(x). \tag{4.108a}$$

$j_\pi(x)$ is a complicated operator function which contains the couplings of the pion to the other particles in the theory. Owing to this definition, a matrix element of the type of (4.103) is

$$\left\langle \mathrm{B}; q' | (\Box + m_\pi^2)\phi_\pi(x) | \mathrm{A}; q \right\rangle = (-Q^2 + m_\pi^2) \left\langle \mathrm{B}; q' | \phi_\pi(x) | \mathrm{A}; q \right\rangle,$$

so that we have

$$\left\langle \mathrm{B}; q' | \phi_\pi(0) | \mathrm{A}; q \right\rangle = \frac{1}{m_\pi^2 - Q^2} \left\langle \mathrm{B}; q' | j_\pi(0) | \mathrm{A}; q \right\rangle. \tag{4.108b}$$

For the example of Fig 4.4, a decomposition in Lorentz covariants is

$$\left\langle \mathrm{p}; q' | j_\pi(0) | \mathrm{n}; q \right\rangle = \frac{\mathrm{i}\sqrt{2}}{(2\pi)^3} g(Q^2) \overline{u_\mathrm{p}(q')} \gamma_5 u_\mathrm{n}(q). \tag{4.109}$$

Here $\mathrm{g}(Q^2)$ is a Lorentz scalar form factor; at the point $Q^2 = m_\pi^2$ it is the conventional pion–nucleon coupling constant

$$\mathrm{g}(Q^2 = m_\pi^2) \equiv \mathrm{g}_{\pi\mathrm{NN}} \quad \text{with} \quad \mathrm{g}_{\pi\mathrm{NN}}^2/4\pi = 14.64. \tag{4.110}$$

Making use of (4.102) and (4.108b) we can write

$$\begin{aligned} \left\langle \mathrm{p}; q' | j_\pi(0) | n; \mathrm{q} \right\rangle &= (m_\pi^2 - Q^2) \left\langle \mathrm{p}; q' | \partial^\alpha a_\alpha^{(1+\mathrm{i}2)}(0) | \mathrm{n}; q \right\rangle \\ &= (m_\pi^2 - Q^2) \frac{\mathrm{i}Q^\alpha}{f_\pi m_\pi^2} \left\langle \mathrm{p}; q' | a_\alpha^{(1+\mathrm{i}2)}(0) | \mathrm{n}; q \right\rangle. \end{aligned}$$

On the r.h.s. of this equation we insert the form factor decomposition (3.209) (with $F_{\mathrm{T}} \equiv 0$), and obtain

$$= \frac{\mathrm{i}}{(2\pi)^3} \frac{m_\pi^2 - Q^2}{f_\pi m_\pi^2} \overline{u_{\mathrm{p}}(q')} \left\{ \not{Q} \gamma_5 F_A(Q^2) + \frac{Q^2}{2m_{\mathrm{N}}} \gamma_5 F_{\mathrm{P}}(Q^2) \right\} u_{\mathrm{n}}(q).$$

Upon comparison with (4.109), and making use of the Dirac equations by means of which

$$\overline{u_{\mathrm{p}}(q')} \not{Q} \gamma_5 u_{\mathrm{n}}(q) = (m_{\mathrm{p}} + m_{\mathrm{n}}) \overline{u_{\mathrm{p}}(q')} \gamma_5 u_{\mathrm{n}}(q) \simeq 2m_{\mathrm{N}} \bar{u}_{\mathrm{p}} \gamma_5 u_{\mathrm{n}},$$

we obtain the equation

$$\sqrt{2} g(Q^2) = \frac{m_\pi^2 - Q^2}{f_\pi m_\pi^2} \left\{ 2m_{\mathrm{N}} F_{\mathrm{A}}(Q^2) + \frac{Q^2}{2m_{\mathrm{N}}} F_{\mathrm{P}}(Q^2) \right\}. \tag{4.111}$$

As such and without further knowledge about the form factors g, F_{A}, and F_{P}, this equation is barely more than an identity because at least one of its two sides must be off-shell: $g(Q^2)$ is physical, i.e. measurable at $Q^2 = m_\pi^2$, whilst $F_{\mathrm{A}}(Q^2)$ and $F_{\mathrm{P}}(Q^2)$ are measurable in the domain $Q^2 \leq 0$ which does not contain the point m_π^2. It is precisely at this stage that the assumption of PCAC becomes effective: According to this assumption the extrapolation of the form factors away from their physical domain should be as smooth as possible.

Example 1. Goldberger–Treiman relation. As an application let us evaluate (4.111) at $Q^2 = 0$, and let us assume that $\mathrm{g}(Q^2)$ is essentially constant when extrapolated from the physical point m_π^2 to the point zero, $\mathrm{g}(0) \simeq \mathrm{g}(m_\pi^2)$. In this approximation we obtain the relation

$$f_\pi \simeq \frac{\sqrt{2} m_{\mathrm{N}} F_{\mathrm{A}}(0)}{\mathrm{g}_{\pi \mathrm{NN}}} \tag{4.112}$$

(so-called Goldberger–Treiman relation), which now relates measurable quantities, $F_{\mathrm{A}}(0)$ is determined in the decay $\mathrm{n} \to \mathrm{p} \mathrm{e}^- \bar{\nu}_{\mathrm{e}}$ with the result $F_{\mathrm{A}}(0) = 1.2573(28)$. The r.h.s. of (4.112) then gives $0.880\ m_{\pi+} = 0.123\,\mathrm{GeV}$ which is indeed rather close to the measured value of f_π, (4.94).

Example 2. Muon capture in nuclei. As a second application consider capture of a muon from the 1s-state in a muonic atom via the reaction $\mu^- \mathrm{p} \to \mathrm{n} \nu_\mu$. For the sake of simplicty let us assume that the muon (momentum p) and the proton (momentum q) on which it is captured, are in a relative s-state and move with small, nonrelativistic velocities, i.e. $p \simeq (m_\mu, \mathbf{0}), q \simeq (m_{\mathrm{N}}, \mathbf{0})$. The energy of the neutron in the final state is then approximately

$$E_{q'} \simeq m_{\mathrm{N}} \{1 + m_\mu^2 / 2m_{\mathrm{N}}^2\},$$

and the squared momentum transfer is $Q^2 = (q' - q)^2 = 2m_{\mathrm{N}}(m_{\mathrm{N}} - 2E'_q) \simeq -m_\mu^2$. In the theory of muon capture it is customary to redefine the form factor F_{P} as

follows

$$G_{\mathrm{P}} := \frac{m_\mu}{2m_{\mathrm{N}}} F_{\mathrm{P}}(-m_\mu^2).$$

At $Q^2 = -m_\mu^2$ and assuming the extrapolations

$$F_{\mathrm{A}}(-m_\mu^2) \simeq F_{\mathrm{A}}(0), \qquad g(-m_\mu^2) \simeq g(m_\pi^2),$$

equation (4.111) leads to the approximate equality

$$m_\mu G_{\mathrm{P}} \simeq 2m_{\mathrm{N}} F_{\mathrm{A}}(0) - \frac{m_\pi^2 \sqrt{2}}{m_\pi^2 + m_\mu^2} g_{\pi\mathrm{NN}} f_\pi,$$

or, by making use of the relation (4.112),

$$\frac{G_{\mathrm{P}}}{F_{\mathrm{A}}(0)} \simeq \frac{2m_\mu m_{\mathrm{N}}}{m_\pi^2 + m_\mu^2}. \tag{4.113}$$

Thus, PCAC predicts $G_{\mathrm{P}} \simeq 6.5\ F_{\mathrm{A}}(0) = 8.13$, in good agreement with the results of muon capture experiments.

Example 3. Adler–Weisberger relation. One can apply PCAC and the method of soft pions to pion–nucleon scattering at threshold. With an additional input from current algebra, stating, in essence, that the equal time commutator of an axial vector charge $Q_{\mathrm{A}}^{(i)}(x^0) := \int \mathrm{d}^3x a_0^{(i)}(x)$ with another axial vector current is proportional to a vector current, viz.

$$[Q_{\mathrm{A}}^{(i)}(0), a_\alpha^{(k)}(0)] = \mathrm{i}\varepsilon_{ikl} v_\alpha^{(l)}(0),$$

one derives the following relation between $F_{\mathrm{A}}(0)$, $g_{\pi\mathrm{NN}}$ and the isovector pion–nucleon scattering length:[6]

$$F_{\mathrm{A}}^2(0) \simeq 6\frac{g_{\pi\mathrm{NN}}^2}{4\pi}\left(\frac{m_\pi}{2m_{\mathrm{N}}}\right)^2 \frac{1}{m_\pi(1 + m_\pi/m_{\mathrm{N}})(a_1 - a_3)}. \tag{4.114}$$

This relation predicts that $(a_3 - a_1)$ is negative; from the values of $g_{\pi\mathrm{NN}}$ and F_{A} quoted above one finds $(a_3 - a_1)\ m_\pi \simeq -0.27$, in striking agreement with the experimental value for this quantity.

[6] We omit the derivation of (4.114) and refer to the orignal literature (Adler 1965, Weisberger 1966, Weinberg 1966).

4.2.5 Another Example: Pion Radiative Decay

A further example which illustrates the properties of hadronic weak currents in a transparent and instructive manner is the decay

$$\pi^+ \to \mathrm{e}^+ \nu_\mathrm{e} \gamma \quad (\pi^- \to \mathrm{e}^- \overline{\nu_\mathrm{e}} \gamma). \tag{4.115}$$

This decay mode has a branching ratio of $1.61(23) \times 10^{-7}$, and, therefore, is almost as rare as pion beta decay. Although the analogous muonic mode

$$\pi^+ \to \mu^+ \nu_\mu \gamma \tag{4.116}$$

is much more frequent (branching ratio $1.24(25) \times 10^{-4}$), it is less interesting than the electronic mode (4.115) because, as well shall see, it is completely dominated by internal bremsstrahlung (that is, by processes in which the photon is shaken off by the external charged particles). In the decay mode (4.115), to the contrary, there are sizeable contributions from genuine structure terms which are comparable to the contributions from bremsstrahlung diagrams.

It is convenient to analyze this process in several steps, as follows.

a) Internal bremsstrahlung

The diagrams describing internal bremsstrahlung are shown in Fig. 4.5. In the diagram 4.5a the photon is emitted by the outgoing positron, whilst in diagram 4.5b it is emitted by the incoming pion. In either case the process contains the decay amplitude $\pi^+ \to \mathrm{e}^+ \nu_\mathrm{e}$ (on or off-shell) and an ordinary bremsstrahlung process. Following standard rules, the diagram of Fig. 4.5a corresponds to the amplitude

$$T_\mathrm{IB}^{(\mathrm{e})} = -\frac{\mathrm{i}e f_\pi G}{(2\pi)^6 \sqrt{2}} \varepsilon_\alpha \overline{u_\nu(p_2)} \not{q} (1-\gamma_5) \frac{-(\not{p}_1 + \not{k}) + m_\mathrm{e}}{2(p_1 k)} \gamma^\alpha \upsilon_\mathrm{e}(p_1), \tag{4.117a}$$

where q is the momentum of the decaying pion; p_1, p_2 and k are the momenta of the positron, neutrino, and photon, respectively, whilst ε_α is the photon's polarization vector. The energy denominator in the propagator is

$$(p_1 + k)^2 - m_\mathrm{e}^2 = 2(p_1 k).$$

The amplitude 4.5b (the photon being emitted by the pion) reads

$$T_\mathrm{IB}^{\pi} = \frac{\mathrm{i}e f_\pi G}{(2\pi)^6 \sqrt{2}} \varepsilon_\alpha \frac{(q+Q)^\alpha Q^\beta}{(q-k)^2 - m_\pi^2} l_\beta, \tag{4.117b}$$

where

$$l_\beta := \overline{u_\nu(p_2)} \gamma_\beta (1-\gamma_5) \upsilon_e(p_1) \tag{4.118}$$

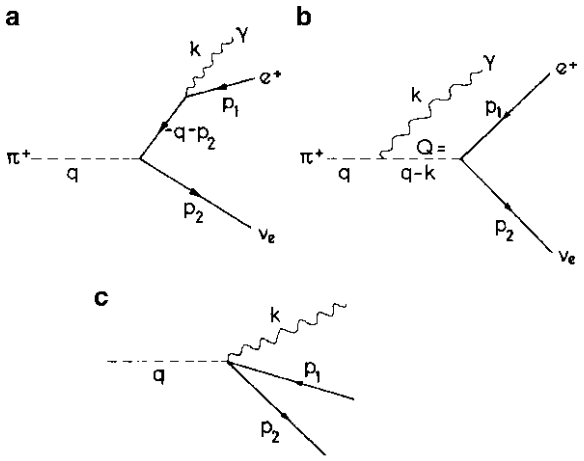

Fig. 4.5 Internal Bremsstrahlung in the decay $\pi^+ \to e^+ \nu_e \gamma$. In diagram (**a**) it is the positron, in diagram (**b**) it is the pion that radiates. Diagram (**c**) is the contact term. Only the sum of the three graphs is gauge invariant

is the leptonic vertex and where $Q := q - k$ is the momentum of the pion in the intermediate state.

Equation (4.117b) deserves some more explanation and a word of caution. The pion in the intermediate state is not on its mass shell. Therefore, at the weak vertex we have to insert $f_\pi(Q^2)$ at a squared momentum transfer off the pion's mass shell. Furthermore, the electromagnetic vertex describes the transition from an external pion to an offshell pion state and, therefore, must be described by a general form factor decomposition of the form of (4.86), viz.

$$\begin{aligned} &f_+(Q^2,k^2)(q+Q)^\alpha + f_-(Q^2,k^2)(q-Q)^\alpha \\ &\qquad = f_+(Q^2,k^2)(2q-k)^\alpha + f_-(Q^2,k^2)k^\alpha, \end{aligned} \tag{4.119a}$$

where $f_+(Q^2 = m_\pi^2, k^2)$ is the pion's electric form factor, (with $f_+(0) = 1$), whilst f_- is a form factor that would not contribute if Q^2 were equal to m_π^2. Indeed one can show (see remarks at the end of this section) that f_+ and f_- are linearly dependent and are related by

$$(m_\pi^2 - Q^2) f_+(Q^2,k^2) + k^2 f_-(Q^2,k^2) = m_\pi^2 - Q^2. \tag{4.119b}$$

This relation (taken at $k^2 \neq 0$) is instructive as it says that f_- is proportional to the squared r.m.s radius of the pion.

$$\lim_{Q^2 \to m_\pi^2} \left(\frac{1}{m_\pi^2 - Q^2} f_-(Q^2,k^2) \right) = \frac{1}{k^2}(1 - f_+(m_\pi^2,k^2)) = -\frac{1}{6}\langle r^2 \rangle + \mathrm{O}(k^2). \tag{4.119c}$$

In the amplitude (4.117b) we have replaced $f_\pi(Q^2)$ by its on-shell value $f_\pi(m_\pi^2)$ and have set $f_+(Q^2, k^2) \equiv 1$. In terms of physics this means that we have taken the pion to be a pointlike particle which possesses no internal structure. This is legitimate provided we lump the extra terms into what we shall call the structure terms proper (see below).

The sum of the amplitudes (4.117a) and (4.117b) is not gauge invariant because, if we replace ε_α by k_α, we obtain

$$\begin{aligned} k_\alpha \overline{u_\nu(p_2)} \Big\{ & \frac{1}{2(p_1k)} \not{q}(1-\gamma_5)(m_\mathrm{e} - \not{p}_1 - \not{k})\gamma^\alpha \\ & + \frac{1}{2(qk)}(2q-k)^\alpha (\not{q} - \not{k})(1-\gamma_5) \Big\} \upsilon_\mathrm{e}(p_1) \\ & = \overline{u_\nu}\{(-\not{q} + \not{q} - \not{k})(1-\gamma_5)\}\upsilon_\mathrm{e} = -\overline{u_\nu}\not{k}(1-\gamma_5)\upsilon_\mathrm{e}. \end{aligned}$$

In this equation we substituted $-\not{p}_1\not{k} = \not{k}\not{p}_1 - 2(p_1k)$, applied the Dirac equation and used the mass shell condition $k^2 = 0$. This calculation shows that gauge invariance can be restored by adding the following amplitude to (4.117a, b):

$$T_\mathrm{IB}^{(\pi\mathrm{e})} = -\frac{\mathrm{i}ef_\pi G}{(2\pi)^6\sqrt{2}}\varepsilon_\alpha g^{\alpha\beta} l_\beta. \tag{4.117c}$$

Indeed, the sum of $T_\mathrm{IB}^{(\mathrm{e})}$, T_IB^{π}, and $T_\mathrm{IB}^{(\pi\mathrm{e})}$ is now gauge invariant.

There is no arbitrariness in this procedure. This additional term, needed to make internal bremsstrahlung gauge invariant by itself, is fixed uniquely to the order $(k)^0$ in an expansion in terms of the photon momentum. Physically it represents a contact term (see Fig. 4.5c) where the photon and the lepton pair emerge at the same point of space–time. It arises naturally in models in which the pions remain structureless such as the nonrelativistic quark model (Scheck et al. 1973), we leave this as an exercise. Finally, having ascertained that the contributions from bremsstrahlung are gauge invariant, the structure terms proper must also be gauge invariant by themselves. This is a useful restriction in formulating form factor decompositions of those structure terms. With the abbreviations

$$s := (qk), \qquad \kappa := -\frac{eG}{(2\pi)^6\sqrt{2}}, \tag{4.120}$$

the sum of the three amplitudes (4.117) is

$$\begin{aligned} T_\mathrm{IB} = \kappa \mathrm{i} f_\pi \varepsilon_\alpha \Big\{ & \left[g^{\alpha\beta} + \frac{1}{s} q^\alpha (q^\beta - k^\beta) \right] l_\beta \\ & + \frac{1}{2(p_1k)} \overline{u_\nu(p_2)}\, \not{q}(1-\gamma_5)(m_\mathrm{e} - \not{p}_1 - \not{k})\gamma^\alpha \upsilon_\mathrm{e}(p_1) \Big\}, \end{aligned} \tag{4.121}$$

with l_β as defined by (4.118).

b) Structure terms

Let us now turn to the structure terms which, by definition, describe all contributions to the decay (4.115) which are not contained in the amplitude (4.121). In this amplitude the pion was taken to be structureless and no other intermediate states than those of Figs. 4.5a, b were admitted.

In the conversion of a pion into a photon via the weak hadronic currents both the vector and the axial vector current can contribute. This is so because the photon contains either G -parity, plus and minus (cf. Table 2.1). The pion has $G(\pi) = -1$. Therefore, in the matrix element $\langle\gamma|\upsilon_\alpha(0)|\pi\rangle$ it is the iso*scalar* part of the photon that contributes (cf. Table 4.1),

$$\langle\gamma(I=0,G=-)|\upsilon_\beta(I=1,G=+)|\pi(I=1,G=-)\rangle. \tag{4.122a}$$

Similarly, in the matrix element $\langle\gamma|a_\alpha(0)|\pi\rangle$ it is the iso*vector* piece that contributes,

$$\langle\gamma(I=1,G=+)|a_\beta(I=1,G=-)|\pi(I=1,G=-)\rangle. \tag{4.122b}$$

In decomposing the matrix elements (4.122) in terms of Lorentz covariants, we have to keep in mind the following points:

(i) Both expressions (4.122) must be proportional to ε_α, the polarization vector of the photon. ε_α is an axial vector.
(ii) The decomposition of either matrix element, (4.122a) or (4.122b), must be gauge invariant on its own.
(iii) If $\varepsilon_\alpha M^{\alpha\beta}$ is to be a vector (an axial vector), $M^{\alpha\beta}$ must be a pseudotensor (genuine tensor) constructed on the basis of $g^{\mu\nu}$, $\varepsilon^{\mu\nu\sigma\tau}$, and the momenta k and q.

The only pseudotensor we can form is $\varepsilon^{\alpha\beta\sigma\tau}k_\sigma q_\tau$. It is automatically gauge invariant since its contraction with k_α vanishes. Thus, the matrix element (4.122a) must be proportional to

$$\varepsilon_\alpha(k)\varepsilon^{\alpha\beta\sigma\tau}k_\sigma q_\tau.$$

Regarding proper tensors the only possible forms are $g^{\alpha\beta}, q^\alpha k^\beta, k^\alpha q^\beta, k^\alpha k^\beta$ and $q^\alpha q^\beta$, the first two of which can be combined to a gauge invariant form,

$$(qk)g^{\alpha\beta} - q^\alpha k^\beta.$$

The third and fourth give zero upon constraction with ε_α, whilst the fifth is not gauge invariant.

In conclusion, the structure dependent contribution to the process (4.115) can be written in the following effective form:

$$T_s = \kappa\frac{1}{m_\pi}\varepsilon_\alpha\{F(s)\varepsilon^{\alpha\beta\sigma\tau}k_\sigma q_\tau + \mathrm{i}a(s)(sg^{\alpha\beta} - q^\alpha k^\beta)\}l_\beta. \tag{4.123}$$

Here $F(s = (qk))$ is an invariant form factor that describes the vector structure (4.122a), $a(s)$ is the form factor that describes the axial vector structure (4.122b); s and κ are defined by eqs. (4.120). We have taken out a factor $1/m_\pi$ in order to make $F(s)$ and $a(s)$ dimensionless. [We note, but do not prove here, that from time reversal and charge conjugation invariance F and a are real.]

The hypothesis of CVC implies that the vector form factor $F(s)$ is related to the amplitude for the decay of the neutral pion into two photons,

$$\pi^0 \to \gamma\gamma. \tag{4.124}$$

This is seen as follows. In analyzing the properties of the matrix element $\langle\gamma_1\gamma_2|\pi^0\rangle$ with respect to internal quantum numbers we realize that the isopins and G -parities contained in the photon states must occur in either of the two combinations

$$(\text{photon}\,1: \quad I^G = 1^+, \text{photon}\,2: I^G = 0^-),$$

or

$$(\text{photon}\,1: I^G = 0^-, \text{photon}\,2: I^G = 1^+), \tag{4.125}$$

in order to match the quantum numbers $I^G = 1^-$ of the pion. However, due to their boson nature, the two photons are indistinguishable and therefore, these two possibilities are identical. In other words, the decay (4.124) is characterized by one single amplitude. Furthermore, the structure of the matrix element (4.122a) is exactly as indicated in (4.125). With the CVC relation between the electromagnetic current (the source of the photon field) and the weak vector current υ_α, it follows that the form factor $F(s)$ must be proportional to the amplitude characteristic for π^0-decay.

The exact relationship is this: Let k, k' be the momenta of the two photons, respectively, $\varepsilon^\alpha, \varepsilon'^\beta$ their polarization vectors, and let q be the momentum of the pion. For the same reasons as for (4.122a), the amplitude for the decay (4.124) must have the form

$$T_{\pi^0\to\gamma\gamma} = \frac{e^2}{(2\pi)^{9/2}} \frac{1}{m_\pi} 2a_0 \varepsilon_{\alpha\beta\sigma\tau} \varepsilon^\alpha \varepsilon'^\beta k'^\sigma k^\tau. \tag{4.126a}$$

[We have, arbitrarily, inserted a factor 2 because of the two identical possibilities (4.125). The factor $1/m_\pi$ is introduced in order to make a_0 dimensionless.]

The decay width is easily worked out to be

$$\Gamma(\pi^0 \to \gamma\gamma) = \alpha^2 \pi a_0^2 m_\pi. \tag{4.126b}$$

From the measured value $\Gamma(\pi^0 \to \gamma\gamma) = 7.85(54)\,\text{eV}$ one finds

$$|a_0| = \frac{\sqrt{\Gamma(\pi^0 \to \gamma\gamma)}}{\alpha\sqrt{\pi m_\pi}} = 1.865(64) \times 10^{-2}. \tag{4.127}$$

Using the Wigner–Eckart theorem we have

$$\langle \gamma(I^G = 0^-)|\, j_\alpha^{(1)}(0)\,|\,\pi^0\rangle = \frac{1}{\sqrt{3}}(0\,||\,j_\alpha\,||\,1), \tag{4.128a}$$

$$\langle \gamma(I^G = 0^-)\,|\,\frac{1}{\sqrt{2}}\upsilon_\alpha^{(1-i2)}\,(0)\,|\,\pi^0\rangle = -\frac{1}{\sqrt{3}}(0\,||\,j_\alpha\,||\,1), \tag{4.128b}$$

where the factor on the r.h.s. is the reduced (isospin) matrix element of the triplet current. The same matrix elements, when expressed in terms of the form factors $F(s)$ and a_0, read

$$\begin{aligned}\langle \gamma(0^-)\,|\,j_\alpha^{(1)}(0)\,|\,\pi^0\rangle &= \frac{e}{(2\pi)^6}\frac{1}{m_\pi}2a_0\varepsilon_{\alpha\beta\sigma\tau}\varepsilon^\beta k'^\sigma k^\tau \\ &= \frac{e}{(2\pi)^6}\frac{1}{m_\pi}2a_0\varepsilon_{\alpha\beta\sigma\tau}\varepsilon^\beta q^\sigma k^\tau, \\ \langle \gamma(0^-)\,|\,\upsilon_\alpha^{(1-i2)}(0)\,|\,\pi^+\rangle &= \frac{e}{(2\pi)^6}\frac{1}{m_\pi}F(s)\varepsilon_{\alpha\beta\sigma\tau}\varepsilon^\beta k^\sigma q^\tau.\end{aligned}$$

Upon comparison with the isospin decomposition (4.128), and noting that in the decay $\pi^0 \to \gamma\gamma$ the variable s has the value $s = (qk) = \frac{1}{2}q^2 = \frac{1}{2}m_\pi^2$, one obtains the CVC relation

$$F\left(s = \frac{1}{2}m_\pi^2\right) = a_0\sqrt{2}. \tag{4.129}$$

One important aspect of this result is that pion radiative decay now depends on only one unknown form factor, $a(s)$, defined in (4.123). Thus, assuming CVC, the data can be analyzed in terms of the ratio

$$R := a(s)/F(s). \tag{4.130}$$

The axial form factor is an interesting quantity for testing theoretical models of chiral symmetry. We do not go into these more theoretical speculations and we just note that, at the very least, it must contain the form factor effect that we discussed in connection with (4.117b).[7] Indeed, from (4.119c) we see that we must have

$$a = \bar{a} + 2m_\pi\, f_\pi \left.\frac{\partial f_+}{\partial k^2}\right|_{k^2=0} = \bar{a} + \frac{1}{3}m_\pi f_\pi \langle r_\pi^2\rangle, \tag{4.131}$$

[7]The difference between $f_\pi^2(Q^2)$ and $f_\pi(q^2 = m_\pi^2)$ is negligibly small since $Q^2 - q^2 = -2s$, with $s = (qk)$ varying between 0 and $m_\pi^2/2$. So when one expands,

$$f_\pi(Q^2) \simeq f_\pi(q^2) - 2s\partial f_\pi/\partial Q^2|_{Q^2=m_\pi^2},$$

the derivative is multiplied by a small number.

where $\bar{a}$ describes other structure terms contributing to the axial matrix element. We note that the second term in (4.131), with $\langle r_\pi^2 \rangle = 0.46\,\mathrm{fm}^2$, would give $\frac{1}{3} m_\pi f_\pi \langle r_\pi^2 \rangle \simeq 7.24 \times 10^{-2}$. The curious observation is that the data seem to favour $a \simeq 0$. this would imply a rather strong cancellation in (4.131). An analysis in the framework of quantum chromodynamics based on sum rules (Nasrallah et al. 1982) does indeed give a close to zero.

c) The differential decay spectrum

The calculation of the differential decay probability from the amplitudes (4.121, 4.123) is rather tedious and we skip its details. It is convenient to choose the energies of the positron (E_e) and of the photon (E_γ) as the independent variables, rather than one energy and the opening angle θ between the photon and the positron. The standard formulae of App. B give, in the pion's frame,

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}(\cos\theta)\mathrm{d}E_\gamma} = \frac{(2\pi)^9}{8m_\pi} \int \mathrm{d}E_\mathrm{e} \sum |T|^2 \frac{|\boldsymbol{p}_1| E_\gamma}{E_\nu} \delta(E_\nu + E_\gamma + E_\mathrm{e} - m_\pi).$$

Neglecting the electron mass, we have $E_e \simeq |\boldsymbol{p}_1|$ and, from energy and momentum conservation,

$$2E_\mathrm{e}E_\gamma(1 - \cos\theta) - 2m_\pi(E_\mathrm{e} + E_\gamma) + m_\pi^2 = 0. \tag{4.132}$$

With $m_e \simeq 0$ the maximal energy of the positron as well as that of the photon is $E_\mathrm{max} = m_\pi/2$. It is convenient to introduce the dimensionless variables

$$x := \frac{E_\gamma}{E_\mathrm{max}} \simeq \frac{2E_\gamma}{m_\pi}, \qquad y := \frac{E_\mathrm{e}}{E_\mathrm{max}} \simeq \frac{2E_\mathrm{e}}{m_\pi}, \tag{4.133}$$

and to transform the differential decay rate accordingly. The Jacobian is

$$\frac{\partial(E_\gamma, \cos\theta)}{\partial(x, y)} = \frac{x - 1}{xy^2} m_\pi.$$

Working out the traces and transforming to the variables x and y, one finds eventually:

$$\frac{\mathrm{d}^2\Gamma(\pi \to \mathrm{e}\nu\gamma)}{\mathrm{d}x\mathrm{d}y} = \frac{\alpha}{2\pi}\Gamma(\pi \to \mathrm{e}\nu)\{W_\mathrm{IB}(x, y) + W_\mathrm{int}(x, y) + W_\mathrm{S}(x, y)\}, \tag{4.134}$$

where $\Gamma(\pi \to \mathrm{e}\nu)$ is the decay width (4.93), with $\mathrm{f} \equiv \mathrm{e}$, W_IB is due to the bremsstrahlung diagrams, W_S to the structure terms, whilst W_int contains the interference terms:

$$W_{\mathrm{IB}}(x,y) = \frac{1-y}{x^2}\frac{(x-1)^2+1}{x+y-1}, \tag{4.135a}$$

$$W_{\mathrm{int}}(x,y) = \frac{m_\pi}{f_\pi}\frac{1-y}{x}\left\{(F+a)(1-x) - (F-a)\left(2-y+\frac{(1-y)^2}{x+y-1}\right)\right\}, \tag{4.135b}$$

$$W_{\mathrm{S}}(x,y) = \left(\frac{m_\pi}{f_\pi}\right)^2\left(\frac{m_\pi}{2m_{\mathrm{e}}}\right)^2(1-x)\{(F+a)^2(x+y-1)^2 + (F-a)^2(1-y)^2\}. \tag{4.135c}$$

Let us now discuss these formulae and add a few remarks on this decay. With $m_{\mathrm{e}} \simeq 0$ the kinematically allowed region for x and y is the triangle indicated in Fig. 4.6. According to (4.132), which now reads

$$\frac{1}{2}(1-\cos\theta)xy - (x+y) + 1 = 0. \tag{4.132'}$$

the outer boundaries correspond to $\theta = 180°$, the lower side of the triangle corresponds to $\theta = 0°$. The dashed line corresponds to an intermediate opening angle ($\theta = 120°$).

If in an actual experiment one integrates the rates over the upper part of the triangle, say from some x_0 to 1 and from y_0 to 1 with e.g. $x_0 \simeq y_0 \simeq 0.7$, then it is not difficult to see that W_{IB} and W_{S} give contributions of comparable magnitude, whilst W_{int} contributes much less. Actually, within W_{S} and under identical experimental conditions, the term with the factor $(F+a)^2$ would contribute about ten times more than the term term with $(F-a)^2$. Therefore, it is primarily the combination $(F+a)^2$ which is determined by experiment; (see, however, note at end of this section.)

As one sees from (4.134, 4.135a), the contribution from bremsstrahlung has the same suppression factor m_{e}^2 which was characteristic for the decay $\pi \to \mathrm{e}\nu$ and

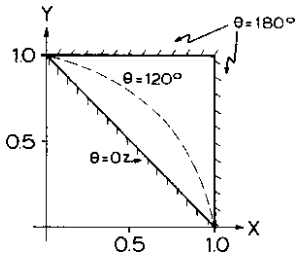

Fig. 4.6 The triangle indicates the domain of allowed photon and positron energies in the decay $\pi^+ \to \mathrm{e}^+\nu_{\mathrm{e}}\gamma$. x is the reduced photon energy $x = 2E_\gamma/m_\pi$, y is the reduced positron energy $y = 2E_{\mathrm{e}}/m_\pi$. θ is the opening angle between photon and positron momentum

whose origin we discussed in Sect. 3.1.2d in terms of the helicity transfer at V and A vertices. Clearly, the helicity selection rules do not depend on whether the pion (or positron) is on or off its mass shell. In the structure term proper (4.135c), there is no such suppression factor (the factor m_e^2 cancels out), because the helicity of the photon can always compensate for the helicity mismatch at the leptonic vertex, without any conflict with the conservation of total angular momentum.

The extra factor $m_\pi^2 / 2m_e^2$ in W_S compensates for the smallness of $F = a_0\sqrt{2}$, with a_0 as given in (4.127). For example, at $x = y = 0.7$ we find $W_{IB} \simeq 1.67$ while $W_S \simeq 1 \times 10^3 (F + a)^2 \simeq 0.70$ (assuming $a = 0$), so that the structure terms are of the same order of magnitude as the bremsstrahlung. On the basis of these results we can now comment on the muonic decay mode (4.116). The structure terms are proportional to $\Gamma(\pi \to \mu\nu_\mu) m_\pi^2 / 2m_\mu^2$, and are of the same order of magnitude as in the electric mode (4.115). However, the bremsstrahlung terms are larger by a factor $(m_\mu / m_e)^2 \simeq 4.3 \times 10^4$. The phase space available for the branch (4.116) is smaller than for the branch (4.115) (see exercise 4.10), and there are some additional mass terms in the decay probabilities. Nevertheless, the estimate is essentially correct. The branching ratio for the decay (4.116) is about four orders of magnitude larger than for the decay (4.115). In the muonic mode internal bremsstrahlung predominates by about 10^3 over the structure terms.

Additional remarks on theory. Both the vector form factor F and the axial vector form factor concern intrinsic properties of the pion and, therefore, are purely hadronic properties. One can show, using PCAC, that the amplitude a_0 for the decay $\pi^0 \to \gamma\gamma$ is given by (Adler 1968)

$$a_0 = \frac{g_{\pi NN}}{4\pi^2 F_A(0)} \frac{m_\pi}{m_N} S, \tag{4.136}$$

where S is a pure number that depends on the charges Q_f of the u- and d-quarks: $S = \Sigma_i a_i Q_i^2$. The a_i are the coefficients of the individual quark terms in $a_\alpha^{(3)}(x)$, the isospin partner of $a_\alpha^{(1\pm i2)}(x)$, viz.

$$a_\alpha^{(3)}(x) = \sum_{i=u,d} a_i \sum_{c=1}^{3} \overline{q_{ic}(x)} \gamma_\alpha \gamma_5 q_{ic}(x) = \sum_c (\bar{u}_c \bar{d}_c) \gamma_\alpha \gamma_5 \frac{\tau_3}{2} \begin{pmatrix} u_c \\ d_c \end{pmatrix}$$

(c being the colour index), so that $a_u = \frac{1}{2}, a_d = -\frac{1}{2}, a_s = 0$, and $S = 3\{\frac{1}{2}(\frac{2}{3})^2 - \frac{1}{2}(\frac{1}{3})^2\} = \frac{1}{2}$. The prediction (4.136) gives $a_0 = 1.96 \times 10^{-2}$, in good agreement with the experimental number (4.127). This quantitative agreement rests on the threefold degeneracy in the colour degree of freedom and is an important argument in favour of QCD.

Regarding the axial form factor $a(s)$ of (4.123), one can show that the natural scale which determines the orders of magnitude is the ratio of magnitude of $f_\pi m_\pi$ to the square of the mass of the ρ-meson (770)

$$f_\pi m_\pi / m_\rho^2 \simeq 3.1 \times 10^{-2}. \tag{4.137}$$

This is plausible from (4.131) if one assumes the pion's charge form factor to be dominated by the ρ-meson, because in this case $\langle r_\pi^2 \rangle$ is proportional to m_ρ^{-2}.

The remaining structure term $\bar{a}$ (4.131), has the same scale (4.137) in all models where it is expressed in terms of commutators of currents which are saturated with ρ- and a_1-vector meson states.

Finally we wish to comment on equations (4.119). If the pion in the intermediate state is not on its mass shell, the matrix element of the electromagnetic current $\langle Q | j_\alpha | q \rangle$ is to be replaced by the expression

$$\frac{\mathrm{i}}{(2\pi)^{3/2}} \int \mathrm{d}x \mathrm{e}^{\mathrm{i}Qx} (\Box + m_\pi^2) \langle 0 | T \phi_\pi(x) j_\alpha(0) | q \rangle$$
$$= \frac{1}{(2\pi)^3} \{ f_+(Q^2, k^2)(q+Q)_\alpha + f_-(Q^2, k^2)(q-Q)_\alpha \}. \quad (4.138)$$

Making use of translational invariance the integral on the l.h.s. can also be written as follows:

$$\int \mathrm{d}x \mathrm{e}^{-\mathrm{i}kx} (\Box + m_\pi^2) \langle 0 \,|\, T \phi_\pi(0) j_\alpha(-x) \,|\, q \rangle,$$

with $k = q$. Transforming x to $-x$, then multiplying with k^α gives the divergence of the integrand,

$$\partial^\alpha \langle 0 \,|\, T \phi_\pi(0) j_\alpha(-x) \,|\, q \rangle$$
$$= \langle 0 \,|\, (\partial^0 \theta(x^0)) j_0(x^0, x) \phi_\pi(0) + (\partial^0 \theta(-x^0)) \phi_\pi(0) j_0(x^0, \boldsymbol{x}) \,|\, q \rangle$$
$$= \langle 0 \,|\, [j_0(0, x), \phi_\pi(0)] \,|\, q \rangle \delta(x^0)$$

(noting that the divergence of j_α vanishes). The commutator of j_0 with ϕ_π is

$$[j_0(0, \boldsymbol{x}), \phi_\pi(0)] = -\phi_\pi(0) \delta(\boldsymbol{x}).$$

Therefore, contracting (4.138) with k^α we obtain for the l.h.s.

$$\frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x \mathrm{e}^{\mathrm{i}Qx} (\Box + m_\pi^2) \langle 0 \,|\, \phi_\pi(x) \,|\, q \rangle \delta(x^0) \delta(\boldsymbol{x}).$$

Integrating the $\Box$-operator by parts this gives, finally,

$$\frac{1}{(2\pi)^3} (m_\pi^2 - Q^2),$$

which is equal to the r.h.s. of (4.138) contracted with k^α,

$$\frac{1}{(2\pi)^3}\{f_+(m_\pi^2 - Q^2) + f_- k^2\}.$$

This completes the proof of (4.119b).

Remarks on experiments. Both the radiative decay $\pi^+ \to e^+ \nu_e \gamma$ and the analogous kaon decay $K^+ \to e^+ \nu_e \gamma$ have been measured to high accuracy (for a list of references see [RPP10]) and have been analysed in terms of the vector and axial vector structure terms. In addition, the decay $\pi^+ \to e^+ \nu_e e^+ e^-$, with the photon replaced by an electron–positron pair, has also been measured. A detailed analysis of this latter decay (Kersch et al. 1986) shows that it has a different sensitivity to the form factors F and a, and that it also allows a separate determination of the term

$$R := \frac{1}{3} m_\pi f_\pi \langle r_\pi^2 \rangle,$$

which depends on the mean square radius of the pion, cf. (4.131) above. Indeed, all three terms were determined from the decay with an electron–positron pair in the final state. The results are in fair agreement with the quantitative analysis sketched above.

4.3 New Perspectives and Open Problems

There are many open questions in the present theory of the weak interactions some of which were already mentioned in Sect. 3.4.4 and in the introduction. In this section we touch on some topics where present and future experimentation is likely to yield further insight and, perhaps, some clues for further progress in our understanding of weak interactions.

In Sect. 4.1 we discussed some possibilities for precision tests of the Lorentz structure of weak CC and NC interactions. In the present section we deal with two topics which have some, though model dependent, relation to this problem, but which are also of great interest in their own right: The question of neutrino masses and the nature of the leptonic family numbers.

The present upper limits on possible neutrino masses are not very stringent. The best mass limit for $\bar{\nu}_e$ comes from the β-decay of the triton,

$$^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \bar{\nu}_e,$$

where one measures the spectrum of the electron near its kinematical endpoint. The present upper limit is 2 eV. For the case of the muon neutrino ν_μ the best limit comes from a precision measurement of the muon momentum $|\boldsymbol{p}|$ in pion decay at rest,

$$\pi^+ \to \mu^+ \nu_\mu .$$

Using the muon and pion masses as input, information on $m(\nu_\mu)$ is deduced from the kinematic relation

$$m_{\nu_\mu}^2 = m_\pi^2 + m_\mu^2 - 2m_\pi \sqrt{\boldsymbol{p}^2 + m_\mu^2}. \tag{4.139}$$

This method yields an upper limit of about 270 keV.

Regarding the τ-neutrino direct mass limits are obtained from τ^- decays such as, e.g., $\tau^- \to 3\pi^- 2\pi^+ \nu_\tau$ which constrain, as much as possible, the phase space available for the emerging neutrino. The weighted average over the results obtained from τ decay studies gives an upper limit of about 30 MeV.

Indirect and, to some extent, model-dependent limits on neutrino masses are obtained from cosmological and astrophysical considerations, the latter concerning the supernova SN 1987 A that was observed in 1987. It would take too much space to discuss the details of these arguments and their implications for possible nonvanishing neutrino masses. The result is that standard cosmology is compatible with either very light neutrinos, the sum of the three neutrino masses being bound to be smaller than 100–200 eV, or, sufficiently heavy neutrinos, with bounds larger than the direct limits quoted above. One should note, however, that the experiments at the Z-pole quoted in Sect. 3.7, seem to exclude additional, heavy, neutrinos up to about 40 GeV.

In view of this state of affairs, one must be prepared to investigate two extreme, in fact complementary, possibilities: if neutrinos are massive then either

(i) some of the masses are "large", in the sense that $m(\nu_\mu)$ and $m(\nu_\tau)$ could be of the same order of magnitude as m_e and/or m_μ, respectively, or
(ii) Some or all masses are different from zero but small, say $\lesssim 1\,\mathrm{eV}$, and at least one mass *difference* is small as compared to a typical experimental energy resolution.

In the first case there is hope that it might be possible to find direct evidence for a nonvanishing mass in the kinematics of a leptonic or semileptonic process. The chances for this are even better, as well shall see, if there is neutrino state mixing. In the second case, and if the leptonic family numbers are not conserved, oscillations between different neutrino states become observable. If these occur, and if the state mixing involves no more than two or three neutrino states, then it is possible to extract information on masses and mixing matrix elements.

It is plausible that neutrinos which have finite, nondegenerate masses, will occur as mixed states in the weak interactions. In other terms, the weak eigenstates "ν_e", "ν_μ", "ν_τ", which couple to CC vertices (i.e. vertices of the type (fν_fW)) may not be identical with the mass eigenstates "n_1", "n_2", "n_3". For the case of the three lepton families, we would have

$$(\nu_\mathrm{f})_L = \sum_{i=1}^{3} U_{fi}(\alpha_1, \alpha_2, \alpha_3, \varepsilon)(n_i)_L, \qquad \mathrm{f} = \mathrm{e}, \mu, \tau \tag{4.140}$$

where $U_{\mathrm{f}i}$ is the leptonic analogue of the quark mixing matrix (3.152). More precisely, this is true if the physical neutrinos are purely left-handed. If, in addition, also right-handed neutrino states couple to other particles, then these can be mixed states, too, viz.

$$(\nu_{\mathrm{f}})_R = \sum_{i=1}^{3} V_{\mathrm{f}i}\,(n_i)_{\mathrm{R}}, \tag{4.141}$$

with a mixing matrix a priori independent of $U_{\mathrm{f}i}$, equation (4.140).

We emphasize that the ansatz (4.140) should not be analyzed in isolation and disregarding the mass sector of the theory since it is the structure of the mass Lagrangian that determines the mixing matrix elements. Particularly the limit of all masses going to zero must be studied with some caution.

4.3.1 "Heavy" Neutrinos

a) Neutrino masses from two-body decays

Relation (4.139) shows that the momentum of the charged lepton in the final state depends on the square of the neutrino mass. Therefore, the value of this momentum is not very sensitive to masses in a range of masses small as compared to the mass of the parent particle. On the other hand the two-body decay of a stopped particle has the advantage that the charged lepton in the final state is monochromatic and, therefore, provides a clear and unique signature for the number and masses of companion neutrino states. Indeed, suppose that in the decay of a pseudoscalar meson P

$$\mathrm{P}^+ \to \mathrm{f}^+ \nu_{\mathrm{f}}, \tag{4.142}$$

the weak interaction state "ν_{f}" is a superposition of different mass eigenstates n_i with masses m_i, as indicated in (4.140). For every mass state with m_i in the interval

$$0 \leqslant m_i < m_{\mathrm{P}} - m_{\mathrm{f}} \tag{4.143}$$

there is a monochromatic state of the charged lepton f^+ with momentum

$$|\boldsymbol{p}^{(i)}| = \frac{1}{2} m_{\mathrm{P}} \sqrt{(1 - r_{\mathrm{f}}^2)^2 + (1 - r_i^2)^2 - (1 + 2 r_i^2 r_{\mathrm{f}}^2)}, \tag{4.144}$$

where we have introduced the mass ratios

$$r_{\mathrm{f}} := m_{\mathrm{f}}/m_{\mathrm{P}}, \qquad r_i = m_i/m_{\mathrm{P}}. \tag{4.145}$$

In other words, the decay (4.142) in reality consists of several branches

$$\mathrm{P} \to \mathrm{f}(\boldsymbol{p}^{(i)}) + \mathrm{n}_i\,(m_i).$$

The corresponding partial decay width is calculated as follows. According to App. B we have

$$\Gamma = \frac{(2\pi)^7 \pi}{2m_{\mathrm{P}}^2} |\boldsymbol{p}^{(i)}| \sum |T|^2.$$

The T-matrix element being

$$T = \frac{\mathrm{i}}{(2\pi)^{9/2}} \frac{\kappa_{\mathrm{P}} G}{\sqrt{2}} U_{fi} \overline{u_i(k)} \not{q} (1 - \gamma_5) \upsilon_{\mathrm{f}}(p),$$

where $\kappa_{\mathrm{P}} \equiv f_\pi \cos\theta_1$ in the case of pions, $\kappa_{\mathrm{P}} = f_{\mathrm{K}} \sin\theta_1 \cos\theta_3$ in the case of kaons (f_{K} being defined in analogy to (4.92)), and where $q = p + k$. Making use of the Dirac equations (1.85, 1.84) this gives

$$T = -\frac{\mathrm{i}}{(2\pi)^{9/2}} \frac{\kappa_{\mathrm{P}} G}{\sqrt{2}} U_{fi} \overline{u_i(k)} \{m_{\mathrm{f}} - m_i + (m_{\mathrm{f}} + m_i)\gamma_5\} \upsilon_{\mathrm{f}}(p),$$

and therefore

$$\sum |T|^2 = \frac{4\kappa_{\mathrm{P}}^2 G^2}{(2\pi)^9} |U_{\mathrm{fi}}|^2 \{(m_{\mathrm{f}}^2 + m_i^2)(pk) + 2m_i^2 m_{\mathrm{f}}^2\},$$

with $(pk) = \frac{1}{2}(m_{\mathrm{P}}^2 - m_{\mathrm{f}}^2 - m_i^2)$. The partial decay width is then found to be

$$\Gamma(\mathrm{P} \to \mathrm{f}^{(i)} \nu_i) = \frac{\kappa_{\mathrm{P}}^2 |U_{\mathrm{fi}}|^2 G^2 |\boldsymbol{p}^{(i)}| m_{\mathrm{P}}^2}{4\pi} \{r_{\mathrm{f}}^2 + r_i^2 - (r_{\mathrm{f}}^2 - r_i^2)^2\}, \tag{4.146}$$

where $|\boldsymbol{p}^{(i)}|$ is given by eq.(4.144). [As one verifies easily, (4.146) with $r_i = 0$, $\kappa_{\mathrm{P}} = f_\pi \cos\theta_1$ and $U = 1$ reduces to (4.93) for the decay $\pi \to \mathrm{f}\nu_{\mathrm{f}}$. Note also the complete symmetry of (4.146) in i and f.]

Let us discuss the result (4.146) in a little more detail. Possible reactions that can be investigated in the laboratory are

$$\pi^+ \to \mu^+ \nu_\mu, \tag{4.147a}$$

$$\pi^+ \to \mathrm{e}^+ \nu_{\mathrm{e}}, \tag{4.147b}$$

$$\mathrm{K}^+ \to \mu^+ \nu_\mu, \tag{4.147c}$$

$$\mathrm{K}^+ \to \mathrm{e}^+ \nu_{\mathrm{e}}, \tag{4.147d}$$

A good starting point is the idea that the predominant states in ν_{e} and ν_μ, respectively, have rather small masses and that one (or several) heavy mass eigenstates are mixed into them. This would imply that both $|U_{\mathrm{e}1}|$ and $|U_{\mu 2}|$ are large as compared to $|U_{\mathrm{e}3}|$ or $|U_{\mu 3}|$ etc. The strengths of the heavy neutrino branches relative to the dominant light neutrino branch is determined by $|U_{\mathrm{e}3}/U_{\mathrm{e}1}|^2$ or $|U_{\mu 3}/U_{\mu 2}|^2$, respectively (which is a small number), and by the mass factor in curly brackets on

the r.h.s. of (4.146). This mass factor indicates to which extent the decay is inhibited by the helicity selection rule. For this reason, the electronic modes (4.147b) and (4.147d) are more sensitive to heavy neutrinos than the muonic modes: the dominant mode is suppressed because both e and n_1 are light, the heavy neutrino mode is not because n_3 is heavy. As an example consider the decay (4.147b) with $m_1, m_2 \ll m_\mathrm{e}$, but $m_3 = 80\,\mathrm{MeV}$. Then $|\boldsymbol{p}^{(1)}| \simeq \frac{1}{2}m_\pi$, $|\boldsymbol{p}^{(3)}| \simeq 0.67\frac{1}{2}m_\pi$ and from (4.146)

$$\frac{\Gamma(\pi \to \mathrm{e}\nu_3)}{\Gamma(\pi \to \mathrm{e}\nu_1)} = \frac{|U_{\mathrm{e}3}|^2}{|U_{\mathrm{e}1}|^2} 1.65 \times 10^4.$$

Therefore, this mode allows detecting even very small admixtures of heavy neutrinos. Indeed, present limits are of the order of $|U_{\mathrm{e}3}|^2 < 10^{-5}$. In fact, information on possible heavy neutrinos can be obtained both from a direct search for monochromatic charged leptons in the decays (4.147), and from a comparison of measured e/μ branching ratios

$$\frac{\Gamma(\pi \to \mathrm{e}\nu_\mathrm{e})}{\Gamma(\pi \to \mu\nu_\mu)}, \quad \frac{\Gamma(\mathrm{K} \to \mathrm{e}\nu_\mathrm{e})}{\Gamma(\mathrm{K} \to \mu\nu_\mu)}$$

to the expected results (Shrock 1981). The different reactions (4.147) are complementary insofar as they scan different mass regions, owing to their different kinematics.

b) Neutrino masses from three-body decays

From a kinematic point of view decays of elementary systems into three bodies, one of which is a massive neutrino, are more favourable than two-body decays because the neutrino can be produced with arbitrarily small velocity. In this nonrelativistic limit the energy of the neutrino is a function of its mass, and not of the square of the mass. Therefore in the appropriate kinematic situation one expects measurable effects which are *linear* in the mass of the neutrino. Examples of such decays are

$$^3\mathrm{H} \to {}^3\mathrm{He} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}, \tag{4.148a}$$

$$(\mathrm{e}(Z, A))_{\mathrm{K-shell}} \to (Z-1, A) + \gamma + \nu_\mathrm{e}, \tag{4.148b}$$

$$\pi^+ \to \mu^+ + \gamma + \nu_\mu, \tag{4.149a}$$

$$(\mu^{-6}\mathrm{Li})_{\mathrm{s-state}} \to {}^3\mathrm{H} + {}^3\mathrm{H} + \nu_\mu. \tag{4.149b}$$

Similarly, if we suspect $m_{\nu\mu}$ to be much larger than $m_{\nu e}$, and $m_{\nu T}$ to be much larger than $m_{\nu\mu}$, the decays

$$\mu^+ \to \mathrm{e}^+ + \nu_\mathrm{e} + \bar{\nu}_\mu, \tag{4.150a}$$

$$\tau^+ \to \mu^+ + \nu_\mu + \bar{\nu}_\tau, \tag{4.150b}$$

$$\tau^+ \to \mathrm{e}^+ + \nu_\mathrm{e} + \bar{\nu}_\tau, \tag{4.150c}$$

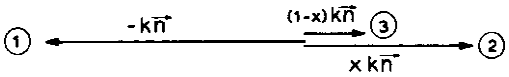

Fig. 4.7 Collinear decay of a particle where decay particle 1 has maximal energy

may also be suitable for detection of these masses. The difference being that in the first group of decays (4.148, 4.149), there are two particles which can be detected, in the second group (4.150), only one particle can be studied.

Here we analyze primarily the first group (4.148, 4.149). For the sake of simplicity we neglect neutrino state mixing, i.e. we take the matrix (4.140) to be (approximately) diagonal. It will be easy, however, to extend the results to the more general situation with nontrivial mixing.

Let us consider the decay of a particle, or atomic system, of mass M into three particles with masses m_1, m_2, m_3, respectively, in a frame of reference where the decaying system is at rest. Suppose that 1 and 2 are particles that can be detected in an experimental arrangement, and that 3 is a neutrino whose mass we wish to determine. We determine first the *maximal* energy of particle 1. At this kinematic point the three momenta of the particles are collinear, as shown in Fig. 4.7, with $\boldsymbol{p}_1 = -k\boldsymbol{n}, \boldsymbol{p}_2 = xk\boldsymbol{n}$, and $\boldsymbol{p}_3 = (1-x)k\boldsymbol{n}$. x is a number between 0 and 1 that one determines as follows. For E_1 to be maximal, $k = k(x)$ must be maximal. From energy conservation we have the condition

$$M = E_1 + E_2 + E_3 = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + x^2k^2} + \sqrt{m_3^2 + (1-x)^2k^2}$$
$$=: F(x, k(x)). \tag{4.151}$$

The maximum of the function $k(x)$ is found from the equation for its derivative

$$\frac{\mathrm{d}k}{\mathrm{d}x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial k} = -kE_1 \frac{xE_3 - (1-x)E_2}{E_2E_3 + x^2E_1E_3 + (1-x)^2E_1E_2} = 0,$$

which yields the condition

$$xE_3 = (1-x)E_2. \tag{4.152a}$$

By squaring this equation, one finds $xm_3 = (1-x)m_2$ and, finally

$$x = \frac{m_2}{m_2 + m_3}. \tag{4.152b}$$

Inserting (4.152a) into (4.151), one has $M - E_1 = E_2/x = (m_2 + m_3)E_2/m_2$, the square of which gives the desired result

$$E_1^{\max} = \frac{1}{2M}\{M^2 + m_1^2 - (m_2 + m_3)^2\}. \tag{4.153a}$$

The energy of particle 2 at this same kinematic point is easily calculated, viz.

$$E_2|_{\text{at } E_1^{\max}} = \frac{m_2}{2M(m_2+m_3)}\{M^2 - m_1^2 + (m_2+m_3)^2\}. \tag{4.153b}$$

As in the last section it is convenient to introduce the mass ratios m_i/M; similarly, it is useful to introduce dimensionless energy variables x_i by dividing each energy E_i by $M/2$, its maximal value in case the other two particles are massless:

$$r_i := m_i/M, \qquad x_i := 2E_i/M. \tag{4.154}$$

Relation (4.151) then reads

$$x_1 + x_2 + x_3 = 2, \tag{4.151'}$$

while (4.153a, b) become

$$x_1^{\max} = 1 + r_1^2 - (r_2+r_3)^2, \tag{4.155a}$$

$$x_2|_{\text{at}x_1^{\max}} = \frac{1}{1+r_3/r_2}\{1 - r_1^2 + (r_2+r_3)^2\}. \tag{4.155b}$$

It is a simple matter to determine the kinematic point where particle 1 has its *minimal* energy $\mathrm{E}_1 = m_1$ or $x_1 = 2r_1$. One finds

$$x_1^{\min} = 2r_1, \tag{4.156a}$$

$$x_2|_{\text{at}\,x_1^{\min}} = 1 - r_1 + \frac{1}{1-r_1}(r_2^2 - r_3^2). \tag{4.156b}$$

Clearly, there is a cyclic symmetry in these equations so that all other extrema can be obtained from them. Figure 4.8 shows the boundary of the kinematically allowed domain of energies in the (x_1, x_2)-plane (for the arbitrarily chosen example $r_1 = \frac{1}{16}, r_2 = \frac{1}{2}, r_3 = \frac{1}{8}$). Equations (4.155, 4.156) give the coordinates of the points B_1 and A_1, respectively. The analogous extrema B_2 and A_2 for the energy variable x_2 are obtained from (4.155) and (4.156), respectively, by the permutation $123 \to 231$.

Of particular interest is the point R at which the neutrino (particle 3) comes to rest. It is in the neighbourhood of this point that we expect effects linear in m_3. Its coordinates $x_3,\ x_1$ are found from (4.156) by the permutation $123 \to 312$, i.e.

$$x_3 = 2r_3,$$

$$x_1 = 1 - r_3 + \frac{1}{1-r_3}(r_1^2 - r_2^2), \tag{4.157a}$$

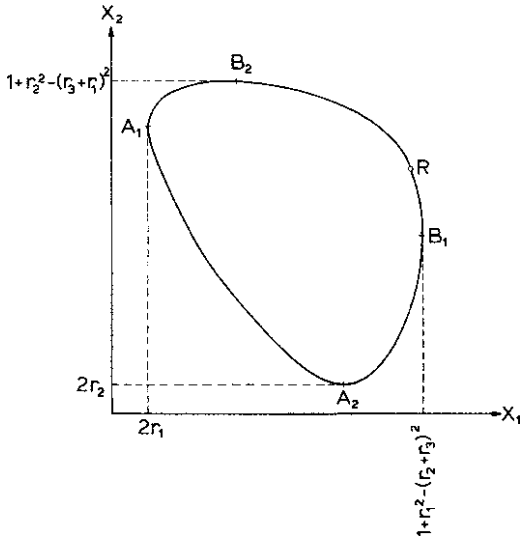

Fig. 4.8 Boundary of the kinematic domain in the plane spanned by the energies of two particles which emanate from a three-body decay. Point R is the projection of the kinematical point where the third particle (not visible in this plane) is produced at rest

while its coordinate x_2 is found by means of (4.151′)

$$x_2 = 1 - r_3 - \frac{1}{1 - r_3}(r_1^2 - r_2^2). \tag{4.157b}$$

Generally speaking, if we compare the shape of Fig. 4.8 to what it would be for a massless neutrino, $m_3 = 0$, we see that the coordinates of the points B_1, R, and B_2 depend on terms which shift *linearly* with m_3. For example, the endpoint energy of particle 1 is shifted as follows:

$$\Delta E_1^{\max} = E_1^{\max}(m_3 = 0) - E_1^{\max}(m_3) = m_3 \frac{m_2}{M}\left(1 + \frac{m_3}{2m_2}\right). \tag{4.158}$$

The larger the ratio m_2/M, the larger the shift. Readers may themselves wish to discuss the specific examples (4.148, 4.149). We quote only one result for the decay (4.149a), $\pi \to \mu\nu_\mu\gamma$. Here $M = m_\pi^+, m_1 = 0, m_2 = m_\mu$. Because the photon is massless the points A_1 and B_2 coincide, while R is rather close to point B_1. The shift of the photon's endpoint energy (4.158) is approximately

$$\Delta E_\gamma^{\max} \simeq m_{\nu_\mu} \frac{m_\mu}{m_\pi} = 0.757 m_{\nu_\mu}, \tag{4.159}$$

a rather large effect indeed.

Unfortunately, a mass determination by simply establishing the exact boundary of the kinematic domain of Fig. 4.8 is very difficult because the differential rates which fill this figure become very small as one approaches the boundary from the inside. In practice, one will only be able to determine partially integrated rates, by

integrating over finite portions of the allowed domain. This raises the question to which extent the differential *rates* are affected by the assumption that the neutrino mass is different from zero. The answer is this: it can be shown that if the CC coupling at the leptonic vertex is exactly V–A, the changes in the differential rates are *quadratic* in m_ν (Missimer et al. 1981). As m_ν^2 appears scaled with M^2, or the square of the energy release, this effect is generally negligible. In other terms, the effect of a nonvanishing neutrino mass, to first approximation, is a purely kinematic one. The differential rate is cut off by the shrinking boundary of the available phase space.

Regarding the possibility of neutrino state mixing, we note that state mixing of several neutrinos n_i, $\mathrm{n}_j, \ldots$ with $m_i < m_j < \ldots$ means superimposing several figures of the type shown in Fig. 4.8. The rates of individual decay channels are added with relative weights $|U_{\mathrm{fi}}|^2$. To first approximation the method will be sensitive only to the lightest, most dominant neutrino component.

In the decays of (4.150), which are of the type

$$f^+ \to f'^+ \nu_{\mathrm{f}'} \bar{\nu}_{\mathrm{f}}, \tag{4.160}$$

the effects due to possible neutrino masses are even more difficult to identify because only one particle in the final state can readily be detected. For simplicity, let us assume that $m_{\nu_{\mathrm{f}}}$ is much larger than $m_{\nu_{\mathrm{f}'}}$ so that, in fact, the latter may be neglected. For the decays (4.150) this is certainly a very good approximation because in the case of muon decay we know already that $m_{\nu_e} \ll m_{\mathrm{e}} \ll m_\mu$, whilst in the case of the τ -decays $m_{\nu_{\mathrm{e}}}$ and m_{ν_μ} are certainly negligible as compared to m_τ. In this approximation we have

$$M \equiv m_{\mathrm{f}},\ m_1 \equiv m_{\mathrm{f}'},\ m_2 \equiv m_{\nu_{\mathrm{f}'}} \simeq 0,\ m_3 \equiv m_{\nu_f} \neq 0.$$

The maximum energy of the charged lepton in the final state is given by (4.153a)

$$E_1^{\max} \equiv W \simeq \frac{1}{2m_{\mathrm{f}}}\{m_{\mathrm{f}}^2 + m_{\mathrm{f}'}^2 - m_{\nu_f}^2\},$$

which should be compared to (4.43). As in our discussion of muon decay with vanishing neutrino masses, let us introduce the dimensionless variable, cf. (4.52),

$$y = E_1/W.$$

The decay spectrum for a polarized initial state, but summing over the polarization of f′, is calculated as in Sect. 4.1.2b. In particular, the integrals (4.44) are evaluated in close analogy to (4.90). One finds for the "V–A" interaction

$$\frac{1}{\Gamma}\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\mathrm{d}(\cos\theta)} = A_{\mathrm{F}}(y)\sqrt{y^2-y_0^2}\left\{3y-2y^2-y_0^2+\frac{6\sigma^2 y}{1-y+3\sigma}\right.$$
$$\left.-\xi\cos\theta\sqrt{y^2-y_0^2}\left[2y-1-\frac{m_{\mathrm{f}'}}{m_{\mathrm{f}}}y_0+\frac{2\sigma(1-y)}{1-y+3\sigma}\right]\right\}, \quad (4.161)$$

where $y_0 = m_{\mathrm{f}'}/W$, as before, and

$$\sigma := m_{\nu_{\mathrm{f}}}^2/2m_{\mathrm{f}}W. \quad (4.162)$$

The function $A_{\mathrm{F}}(y)$ stems from the integration over the phase space of the two neutrinos and is given by

$$A_{\mathrm{F}}(y) = \frac{(1-y)^2(1-y+3\sigma)}{(1-y+\sigma)^3}. \quad (4.163)$$

The differential decay rate (4.161) has two new features as compared to the case $m_2 = m_3 = 0$:

(i) The isotropic part of the spectrum vanishes at the endpoint $y = 1$, due to the phase space factor (4.163).
(ii) The isotropic part of the spectrum has its maximum approximately at the kinematic point (4.157a) where the sensitivity to $m_3 \equiv m_{\nu\mathrm{f}}$ is greatest, i.e. at

$$E_1 \simeq \frac{m_{\mathrm{f}}}{2}\left\{1-\frac{m_{\nu_{\mathrm{f}}}}{m_{\mathrm{f}}}+\frac{m_{\mathrm{f}}'^2/m_{\mathrm{f}}^2}{1-m_{\nu_{\mathrm{f}}}/m_{\mathrm{f}}}\right\}. \quad (4.164)$$

The position of this maximum is indeed a linear function of $m_{\nu_{\mathrm{f}}}$. However, for the cases of practical interest (and taking into account the radiative corrections) this shift is only measurable if $m_{\nu_{\mathrm{f}}}/m_{\mathrm{f}}$ is not smaller than, say, 0.05.[8] With the present limit on $m_{\nu_\mu}^2$, this excludes muon decay as a realistic possibility.

4.3.2 Neutrino Oscillations

In the preceding sections we considered the case of large neutrino masses, i.e. a situation where at least ν_μ and ν_τ have masses of the same order of magnitude as the electron and the muon mass. Although this possibility cannot be excluded on the basis of extant, direct experimental information, it is in conflict with conclusions from cosmology [RPP10], p.556. Indeed, stable massive neutrinos make a contribution to the total energy density of the Universe, viz. $\rho(\nu) = m_{\mathrm{tot}}n_\nu$, where

[8] For details see Missimer et al. (1981).

$$m_{\text{tot}} = 2\sum_f m_{\nu_f} \quad \text{for Dirac neutrinos,}$$

$$m_{\text{tot}} = \sum_f m_{\nu_f} \quad \text{for Majorana neutrinos.}$$

The current upper limit to the matter density would yield the constraint $m_{\text{tot}} < 11\,\text{eV}$. However, the the Cosmic Microwave Background (CMB) and further observations in cosmology combine to an even lower limit of about 0.5 eV. For unstable neutrinos the mass limits depend strongly on their lifetime and, in fact, are then much less tight. The direct search for "large" masses, in laboratory experiments, is of great importance not only as a fundamental question of lepton physics on its own but also as input to astrophysics and cosmology.

In this section we consider the complementary situation of "small" neutrino masses mentioned in the introduction and discuss some experimental possibilities for determining finite masses and mass differences. We base our analysis on the following assumptions:

(i) The weak eigenstates ν_e, ν_μ, ν_τ are nontrivial superpositions of mass eigenstates n_1, n_2, n_3, cf. (4.140).
(ii) One or several of the mass differences $|m_i - m_j|$ are small as compared to the typical resolution in energy of an actual experiment.
(iii) In particular, the mass differences are small also in comparison with the momentum with which the neutrinos are produced in a given decay process.

a) Transition probabilities due to oscillations

For the sake of simplicity we consider first the case of two states. Let ν_{f1} and ν_{f2} be the weak eigenstates, and let

$$|\nu_{f1}\rangle_0 = |n_1\rangle \cos\alpha_1 + |n_2\rangle \sin\alpha_1, \tag{4.165a}$$

$$|\nu_{f2}\rangle_0 = -|n_1\rangle \sin\alpha_1 + |n_2\rangle \cos\alpha_1. \tag{4.165b}$$

be the mixed states (4.140) with $\alpha_2 = \alpha_3 = 0$. We assume further that initially the state $|\nu_{f2}\rangle$ was produced with a given momentum k, e.g. in a two-body decay $\mathbf{P} \to f_2 + \nu_{f2}$ in the system where particle P was at rest. Since n_1 and n_2 are the mass eigenstates the state ν_{f2} has the time evolution

$$|\nu_{f2}\rangle_t = -|n_1\rangle\ e^{-iE_1 t} \sin\alpha_1 + |n_2\rangle\ e^{-iE_2 t}\ \cos\alpha_1, \tag{4.166}$$

where

$$E_i = \sqrt{m_i^2 + \boldsymbol{k}_i^2} \simeq |\boldsymbol{k}| + \frac{1}{2} m_i^2/|\boldsymbol{k}|. \tag{4.167}$$

Note that here we make use of assumption (iii), $m_i^2 \ll \boldsymbol{k}^2$, and, consequently, we neglect the difference in momentum in the different mass eigenchannels. At the

same time it is understood that the momentum resolution in detecting the charged partner f_2 in the two-body decay of particle P is not sufficient to distinguish the two channels. If these conditions are met the state mixture (4.166) leads to observable oscillations between the states ν_{f2} and ν_{f1}.

In order to see this we recall that the only way of detecting neutrinos is by having them induce another weak reaction. Whatever the experimental arrangement, it will always measure either the probability of finding the initial neutrino after a time t,

$$P(\nu_{f2} \to \nu_{f2};\, t), \tag{4.168a}$$

or of finding another neutrino state ν_{f1}, viz.

$$P(\nu_{f2} \to \nu_{f1};\, t). \tag{4.168b}$$

It is easy to calculate these probabilities from the overlap of the state (4.166) with the initial states (4.165), viz.

$$\begin{aligned} P(\nu_{f2} \to \nu_{f2};\, t) &= |_0\langle \nu_{f2}|\nu_{f2}\rangle_t\,|^2 \\ &= 1 - 2\sin^2\alpha_1\cos^2\alpha_1\{1-\cos(E_2-E_1)t\}, \qquad (4.169a) \\ P(\nu_{f2} \to \nu_{f1};\, t) &= |_0\langle \nu_{f1}|\nu_{f2}\rangle_t\,|^2 \\ &= 2\sin^2\alpha_1\cos^2\alpha_1\,\{1-\cos(E_2-E_1)t\}. \qquad (4.169b) \end{aligned}$$

In these formulae $(E_2 - E_1)$ can be replaced by $|E_2 - E_1|$, where

$$|E_2 - E_1| \simeq \frac{|m_2^2 - m_1^2|}{2|\boldsymbol{k}|} \simeq \frac{|m_2^2 - m_1^2|}{2E_\nu}.$$

In practice, the time of flight t is measured by installing the neutrino detectors at a distance L from their source and, if possible, by varying that distance. As υ/c is practically 1, $L \simeq$ t, in natural units. The oscillation pattern in (4.169) is then determined by the quantity

$$\frac{|E_2 - E_1|}{2\pi}\, t = \frac{L}{L_{12}},$$

where the oscillation length L_{12} is defined by

$$L_{12} := \frac{2\pi}{|E_1 - E_2|} \simeq \frac{4\pi E_\nu}{|m_1^2 - m_2^2|}. \tag{4.170}$$

Regarding the units, it is customary to express the length in meters, the neutrino energy E_ν in MeV, the difference of squared masses in $(\mathrm{eV})^2$, so that

$$L_{12}[\mathrm{m}] = \kappa \frac{E_\nu[\mathrm{MeV}]}{|m_1^2 - m_2^2|[\mathrm{eV}]^2} \tag{4.171a}$$

with

$$\kappa = \hbar c(\text{in MeV} \cdot \mathrm{m}) 4\pi \times 10^{12} = 4\pi \cdot 0.19733 \simeq 2.48. \tag{4.171b}$$

With these definitions we have (with $\alpha \equiv \alpha_1$)

$$P(\nu_{\mathrm{f}2} \to \nu_{\mathrm{f}2};\, L) = 1 - \sin^2(2\alpha) \sin^2(\pi L/L_{12}), \tag{4.172a}$$

$$P(\nu_{\mathrm{f}2} \to \nu_{\mathrm{f}2};\, L) = \sin^2(2\alpha) \sin^2(\pi L/L_{12}). \tag{4.172b}$$

In a discussion of these results we have to distinguish three limiting situations:

(i) The greatest sensitivity to the phenomenon of neutrino oscillations is obtained if $L \simeq \frac{1}{2} L_{12}$, i.e. if the experimental conditions are chosen such that $E_\nu/L \simeq |m_1^2 - m_2^2|$. For example, an experiment with $E_\nu = 100\,\mathrm{MeV}$ and $L = 20\,\mathrm{m}$ would be most sensitive to a difference of $|m_1^2 - m_2^2| \simeq 5\,\mathrm{eV}^2$.

(ii) If $L/L_{12} \gg 1$, that is if $E_\nu/L \ll |m_1^2 - m_2^2|$, then the interference term $\sin^2(\pi L/L_{12})$ oscillates very rapidly with L, so that only an average effect of neutrino mixing will be visible,

$$\langle P(\nu_{\mathrm{f}2} \to \nu_{\mathrm{f}2}) \rangle = 1 - \frac{1}{2} \sin^2(2\alpha).$$

This could be the case, for instance, for neutrinos coming from weak processes in the sun, see below.

(iii) Finally, if $L/L_{12} \ll 1$, that is if $E_\nu/L \gg |m_1^2 - m_2^2|$, there is practically no observable effect at all.

The case of three (or even more) mass eigenstates mixed into the weak interaction states is a little more complicated. Instead of going into all details we just mention one case which may be relevant for astrophysics. In the case of three basis states, eq. (4.140), and with a mixing of the form of (3.152), the average probability, for a beam that was initially ν_e, of finding again ν_e after the beam has travelled for a distance $L \gg L_{ij}$, is given by

$$\langle P(\nu_e \to \nu_e) \rangle = 1 - \frac{1}{2} \sin^2(2\alpha_1) - \frac{1}{2} \sin^4 \alpha_1 \sin^2(2\alpha_3). \tag{4.173}$$

The minimum of this expression is reached for $\sin^2 \alpha_1 = \frac{2}{3}$, i.e. $\sin^2(2\alpha_1) = \frac{8}{9}$, and $\sin^2(2\alpha_3) = 1$, in which case $\langle P(\nu_\mathrm{e} \to \nu_\mathrm{e}) \rangle = \frac{1}{3}$. This result may be relevant for the so-called solar neutrino problem to which we turn in the next section.

Finally, we mention that the nondiagonal probabilities $\langle P(\nu_\mathrm{e} \to \nu_\mu) \rangle$ and $\langle P(\nu_\mathrm{e} \to \nu_\tau) \rangle$ depend also on ε, the phase indicating CP violation. Here there is

another though remote possibility of detecting violation of time reversal invariance in the leptonic world.

b) The solar neutrino flux

There is a wealth of experiments on neutrino oscillations with solar, atmospheric and terrestrial neutrino sources, too many to be described here, all of which provide evidence for nonzero neutrino masses and state mixing, cf. [RPP10], review 13.

There is a longstanding problem in solar neutrino physics which may find its solution in the existence of a nontrivial mass matrix of light neutrinos including the electron neutrino. We first describe the problem and then turn to a possible solution.

The sun draws the energy it radiates from the fusion of four hydrogen atoms to helium, ^{4}He, by way of what is called the pp cycle. This cycle starts with the reactions

$$\mathrm{p} + \mathrm{p} \to {}^2\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e} \quad \mathrm{p} + \mathrm{p} + \mathrm{e}^- \to {}^2\mathrm{H} + \nu_\mathrm{e} \quad {}^2\mathrm{H} + \mathrm{p} \to {}^3\mathrm{He} + \gamma. \tag{4.174}$$

It can then terminate by the reaction ^{3}He $+^3$He $\to^4$ He $+$ 2p, or it can produce ^{7}Be via the reaction ^{3}He $+^4$He $\to^7$ Be $+ \gamma$. In the latter case there are two branches leading to two helium-4 nuclei:

(i) Beryllium, through electron capture, converts to lithium which then reacts with another proton to give the two helium nuclei, viz.

$$^7\mathrm{Be} + \mathrm{e}^- \to {}^7\mathrm{Li} + \nu_\mathrm{e}, \quad {}^7\mathrm{Li} + \mathrm{p} \to {}^4\mathrm{He} + {}^4\mathrm{He}; \tag{4.175}$$

(ii) Beryllium reacts with a proton and produces a boron-8 nucleus which undergoes β decay to beryllium-8. The latter is unstable and decays into two helium nuclei, viz.

$$^7\mathrm{Be} + \mathrm{p}^- \to {}^8\mathrm{B} + \gamma, \quad {}^8\mathrm{B} \to {}^8\mathrm{Be} + \mathrm{e}^+ + \nu_\mathrm{e}, \quad {}^8\mathrm{Be} \to {}^4\mathrm{He} + {}^4\mathrm{He}. \tag{4.176}$$

There are four stages in these cycles where electron neutrinos are produced. In the pp reaction (4.174) the first reaction yields ν_e's with an average energy of 0.263 MeV, while the second reaction yields monochromatic ν_e's with energy 1.4 MeV. In the electron capture on ^{7}Be (4.175), the average energy of ν_e is 0.80 MeV, while in the β decay of ^{8}B, (4.176), the average ν_e energy is 7.2 MeV. Thus, there is a resulting neutrino spectrum whose "low- energy" part stems from the pp cycle (4.174) and from electron capture on beryllium (4.175), and whose "high-energy" part stems from the boron decay (4.176).

On the theoretical side, the sun has been studied in great detail and solar models allow one to reliably calculate the neutrino flux from the sun (Bahcall et al. 1992 and earlier references therein). On the experimental side, the neutrino flux from the

sun has been measured directly by means of inverse β decay induced in chlorine and gallium targets,

$$\nu_e + {}^{37}_{17}\mathrm{Cl} \rightarrow {}^{37}_{18}\mathrm{Ar} + e^- \text{ (threshold 0.814 MeV)}, \tag{4.177}$$

$$\nu_e + {}^{71}_{31}\mathrm{Ga} \rightarrow {}^{71}_{32}\mathrm{Ge} + e^- \text{(threshold 0.233 MeV)}. \tag{4.178}$$

The first of these, which has been studied by Davis and collaborators for more than 25 years, is primarily sensitive to the high-energy neutrinos, i.e. those emanating from the boron decay. The second one, which was studied by the GALLEX, GNO and the SAGE groups, is also sensitive to the low-energy part where the flux is highest. Results are traditionally expressed in terms of solar neutrino units (SNU), defined as follows:

$$1 \text{ SNU} := 1 \times 10^{-36} \text{ captures per atom s}^{-1}, \tag{4.179}$$

Typical results are 2 SNU for the reaction (4.177), and 80 SNU for the reaction (4.178). These rates are very small indeed, and large amounts of target material and good screening against other neutrino sources are mandatory. The screening is achieved by installing the experiments deep underground.

A third measurement of the solar neutrino flux makes use of a water Čerenkov detector developed for the Kamioka Nucleon Decay Experiment KAMIOKANDE II which allows one to record the elastic scattering process

$$\nu_e + e \rightarrow e + \nu_e.$$

The threshold for the neutrino energy being 7.5 MeV, for experimental reasons, the experiment is sensitive only to electron neutrinos from the boron-8 decay (4.176).

All these experiments are long-term projects which are being improved by stages unless they were concluded. The reader is advised to consult the latest edition of the review of particle properties for an up-to-date account of the experimental situation (cf. [RPP10]).

When the results are compared to the theoretical predictions, then a sizeable deficit is found in both cases. In the case of the ^{37}Cl experiment the experimental rate is only about 30% of the prediction, in the case of the ^{71}Ga experiment it is about 60% of the prediction. The flux of the ^{8}B neutrinos observed in the KAMIOKANDE II project, too, is smaller than expected: Depending on the calculation one compares with, the measured flux is 50–70% of the expected flux. Although the uncertainties are sizeable, in all three cases the conclusion seems inevitable: The observed flux is significantly *less* than that which is predicted.

There is an interesting, although still unproven, solution to this longstanding puzzle: The calculation of the flux is based on the assumption that neutrinos essentially are massless and do not oscillate. Although the electron neutrinos are produced everywhere over the entire volume of the sun, attenuation of their flux within the solar medium is negligibly small. Therefore, the full intensity calculated under these assumptions should be seen on earth. If, in turn, there is a mechanism

which turns some of the ν_e, on their way out of the sun, into ν_μ then the latter will not be seen in the reactions (4.177, 4.178). This is so because ν_μ cannot induce inverse β decay. A muon neutrino could make the reactions

$$\nu_\mu + {}^{A}_{Z}\mathrm{X} \rightarrow {}^{A}_{Z+1}\mathrm{X}' + \mu^-$$

which are analogous to (4.177, 4.178) provided its energy is larger than m_μ plus the difference in nuclear binding energies. In the case of solar neutrinos, this is obviously not the case.

Suppose ν_e and ν_μ are linear combinations of two mass eigenstates $n_1(0)$ and $n_2(0)$, cf. (4.175),

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\alpha_0 & \sin\alpha_0 \\ -\sin\alpha_0 & \cos\alpha_0 \end{pmatrix} \begin{pmatrix} n_1(0) \\ n_2(0) \end{pmatrix}, \tag{4.180}$$

or, in short, $(\nu_e, \nu_\mu)^T = V(\alpha_0)(n_1(0), n_2(0))^T$. The states ν_e and ν_μ are the weak interaction eigenstates which are produced at vertices involving $\mathrm{W}^\pm$ and Z^0. The states $n_1(0)$ and $n_2(0)$ are the mass eigenstates which propagate in empty space. The squared mass matrix which in the basis of the states $n_i(0)$ is $\overset{\circ}{M}{}^2 = \mathrm{diag}\,(m_1^2, m_2^2)$, when expressed in the basis $(\nu_e, \nu_\mu)^T$, reads

$$\begin{aligned} M^2 &= V(\alpha_0)\,\overset{\circ}{M}{}^2\,V^\dagger(\alpha_0), \quad \text{or} \\ M^2 &= \begin{pmatrix} m_1^2\cos^2\alpha_0 + m_2^2\sin^2\alpha_0 & (m_2^2 - m_1^2)\sin\alpha_0\cos\alpha_0 \\ (m_2^2 - m_1^2)\sin\alpha_0\cos\alpha_0 & m_1^2\sin^2\alpha_0 + m_2^2\cos^2\alpha_0 \end{pmatrix} \\ &= \frac{1}{2}(m_1^2 + m_2^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(m_2^2 - m_1^2)\begin{pmatrix} -\cos 2\alpha_0 & \sin 2\alpha_0 \\ \sin 2\alpha_0 & \cos 2\alpha_0 \end{pmatrix}. \end{aligned}$$

When a neutrino beam travels through matter instead of empty space, the masses will be changed due to weak interactions with the electrons contained in matter (Wolfenstein 1978). While ν_e and ν_μ interact with electrons in exactly the same way via the weak *neutral* current ν_e can also interact via the *charged* weak current, as explained in Sect. 3.6.2 [equation (3.218)]. Thus the diagonal matrix elements of M^2 will be modified according to

$$M^2 \rightarrow M'^2 = \begin{pmatrix} m_1^2\cos^2\alpha_0 + m_2^2\sin^2\alpha_0 + A_N + A_C & (m_2^2 - m_1^2)\sin\alpha_0\cos\alpha_0 \\ (m_2^2 - m_1^2)\sin\alpha_0\cos\alpha_0 & m_1^2\sin^2\alpha_0 + m_2^2\cos^2\alpha_0 + A_N \end{pmatrix},$$

where A_N and A_C are the additional interaction terms due to neutral and charged current interactions, respectively. Note that the term A_C contributes only to the ν_e–ν_e diagonal element. As before this matrix can be written as the sum of a diagonal and a nondiagonal matrix,

$$M'^2 = \frac{1}{2}(\Sigma + A_{\rm C})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A_{\rm C} - \Delta\cos 2\alpha_0 & \Delta \sin 2\alpha_0 \\ \Delta \sin 2\alpha_0 & -A_{\rm C} + \Delta\cos 2\alpha_0 \end{pmatrix}, \quad (4.181)$$

where

$$\Sigma := m_1^2 + m_2^2 + 2A_{\rm N}, \ \Delta := m_2^2 - m_1^2.$$

Obviously, the term $A_{\rm N}$ which is due to the effective neutral current interaction (3.85) is irrelevant because it may be absorbed in a redefinition of the unperturbed masses $m_i'^2 = m_i^2 + A_{\rm N}$.

Before we turn to the discussion of the eigenvalues and eigenstates of this mass matrix let us calculate the quantity $A_{\rm C}$. This term is obtained from the effective CC interaction (3.74) as follows. With the sign as given by (3.74) the interaction hamiltonian density is *minus* that expression, or, after a Fierz reordering according to (4.30), and in momentum space,

$$\mathscr{H}_{\rm int}^{\rm eff} = +\frac{G}{\sqrt{2}}\overline{(u^{(\nu_{\rm e})}}\ \gamma^\mu(\mathbb{1} - \gamma_5)\ u^{(\nu_{\rm e})})(\overline{u^{(\rm e)}}\,\gamma_\mu(\mathbb{1} - \gamma_5)\ u^{(\rm e)}). \quad (4.182)$$

The electron being at rest, to good approximation, one can show that the space components $\mu = i$ of the expression (4.182) vanish. Indeed, using the standard representation (1.78) and the solution (1.90) with $\boldsymbol{p} = 0$, the second factor in (4.182) is equal to $-2m_{\rm e}\chi^{(r)\dagger}\sigma^{(i)}\chi^{(r)}$ which, when summed over the spin orientation r vanishes. For $\mu = 0$ the electron factor gives $2m_{\rm e}$. With $\mu = 0$ and knowing that $\nu_{\rm e}$ is left-handed, the neutrino factor gives $2u_{\rm e}^{(\nu)\dagger}u_{\rm e}^{(\nu)} = 4E_\nu$. Of course, the factors $2m_{\rm e}$ and $2E_\nu$ are remnants of the covariant normalization (1.89). In calculating the interaction Hamiltonian, states must be normalized to 1 and, therefore, these factors are divided out. Thus, the interaction due to charged currents gives rise to a potential energy for $\nu_{\rm e}$ which is given by

$$V = +\frac{G}{\sqrt{2}}\ 2N_{\rm e} = G\sqrt{2}N_{\rm e},$$

where $N_{\rm e}$ is the number of electrons per unit volume. In presence of a potential the energy–momentum relation for an electron neutrino reads

$$k^2 + m^2 = (E - V)^2 \approx E^2 - 2EV,$$

V^2 being negligibly small. This means that the $\nu_{\rm e} - \nu_{\rm e}$ entry of the mass matrix receives the additive contribution

$$A_{\rm C} \approx 2EV = 2\sqrt{2}\frac{G}{2m_{\rm n}}\rho E. \quad (4.183)$$

In this expression ρ is the matter density, $m_{\rm n}$ is the nucleon mass. The number of electrons per nucleon in matter is generally 1/2.

The eigenvalues of M'^2 are easily calculated. They are

$$\lambda^2_{1/2} = \frac{1}{2}\{(\Sigma + A_{\rm C}) \pm \sqrt{(\Delta\cos 2\alpha_0 - A_{\rm C})^2 + \Delta^2 \sin^2 2\alpha_0}\}. \tag{4.184a}$$

In particular, the difference between them is

$$\lambda^2_2 - \lambda^2_1 = \sqrt{(\Delta\cos 2\alpha_0 - A_{\rm C})^2 + \Delta^2 \sin^2 2\alpha_0}. \tag{4.184b}$$

Note that the corresponding mass eigenstates depend on the value of $A_{\rm C}$, hence on the density ρ (for fixed value of the neutrino energy E). Therefore, we write them as $n_i(\rho)$, cf. also (4.180). Let us denote the transformation from the new mass eigenstates $(n_1(\rho), n_2(\rho))^{\rm T}$ to the basis $(\nu_{\rm e}, \nu_\mu)^{\rm T}$ by $V(\alpha_\rho)$, with the same sign convention as in (4.180), i.e. $V^\dagger(\alpha_\rho) M'^2 V(\alpha_\rho) = {\rm diag}(\lambda^2_1, \lambda^2_2)$. Using (4.181) it is not difficult to derive the relation between α_ρ and α_0:

$$\cos 2\alpha_\rho = \frac{\Delta\cos 2\alpha_0 - A_{\rm C}}{\sqrt{(\Delta\cos 2\alpha_0 - A_{\rm C})^2 + \Delta^2 \sin^2 2\alpha_0}}, \tag{4.185a}$$

$$\sin 2\alpha_\rho = \frac{\Delta\sin 2\alpha_0}{\sqrt{(\Delta\cos 2\alpha_0 - A_{\rm C})^2 + \Delta^2 \sin^2 2\alpha_0}}. \tag{4.185b}$$

Clearly, at any time $\nu_{\rm e}$ and ν_μ are given in terms of the mass eigenstates by (4.180), and, conversely, the mass eigenstates are given by the inverse of (4.180)

$$\begin{pmatrix} n_1(\rho) \\ n_2(\rho) \end{pmatrix} = \begin{pmatrix} \cos\alpha_\rho & -\sin\alpha_\rho \\ \sin\alpha_\rho & \cos\alpha_\rho \end{pmatrix} \begin{pmatrix} \nu_{\rm e} \\ \nu_\mu \end{pmatrix}, \tag{4.186}$$

Without restriction of generality we take $\alpha_0 < 45^0$. The difference Δ as well as the interaction term $A_{\rm C}$ are positive. Assume now the parameters to be such that in the solar interior $A_{\rm C}$ is *larger* than $\Delta\cos 2\alpha_0$. Then, as the neutrino moves out of the sun it sees a decreasing density and, hence feels a decreasing potential until it has left the sun where $A_{\rm C}$ becomes equal to zero.

Suppose the neutrino $\nu_{\rm e}$ is created somewhere inside the sun where $A_{\rm C} > \Delta\cos 2\alpha_0$. By (4.186), the mass eigenstate $n_1(\rho)$ is populated with probability $\cos^2\alpha_\rho$, the state $n_2(\rho)$ is populated with probability $\sin^2\alpha_\rho$. In a first, adiabatic, approximation and for the sake of simplicity we could assume the density to vary smoothly as the neutrino moves outwards, such that the mass eigenstates propagate independently. Once they have left the sun they have turned adiabatically into the states $n_1(0)$ and $n_2(0)$, respectively. Using (4.180), the probability for the initial $\nu_{\rm e}$ that was created inside the sun, to remain an electron neutrino is

$$P(\nu_{\rm e}) = \cos^2\alpha_\rho \cos^2\alpha_0 + \sin^2\alpha_\rho \sin^2\alpha_0, \tag{4.187a}$$

while the probability for it to have turned into a muon neutrino is

$$P(\nu_\mu) = \sin^2\alpha_\rho \cos^2\alpha_0 + \cos^2\alpha_\rho \sin^2\alpha_0. \tag{4.187b}$$

Thus, on its way out of the sun the neutrino that was initially an electron neutrino, partially turns into a muon neutrino. This mechanism which explains the reduction of the solar neutrino flux was proposed by S. P. Mikheyev and A. Yu. Smirnov (Mikheyev et al., 1985). In order to illustrate the effect let us consider two examples:

(i) Assume A_C to be just twice the quantity $\Delta \cos 2\alpha_0$. Then, from (4.185) $\alpha_\rho = \pi/2 - \alpha_0$ and, from (4.187),

$$P(\nu_e) = \frac{1}{2}\sin^2 2\alpha_0, \quad P(\nu_\mu) = 1 - \frac{1}{2}\sin^2 2\alpha_0.$$

Thus, if ν_e is very close to the mass eigenstate $n_1(0)$, the mixing angle α is small and the electron neutrino will turn predominantly into a muon neutrino.

(ii) The conversion of the initial electron neutrino is even more enhanced if $A_C \gg \Delta \cos 2\alpha_0$. In this case (4.185a) has the solution $\alpha_\rho \approx \pi/2$, from which

$$P(\nu_e) = \sin^2\alpha_0, \quad P(\nu_\mu) = 1 - \sin^2 2\alpha_0.$$

These results are also well illustrated by Fig. 4.9 which shows the eigenstates (4.184a) as a function of the density ρ and which can be interpreted as follows (Bethe 1986). For *high* values of the density, right-hand side of Fig. 4.9, the state with the higher mass λ_1^2, must be predominantly an electron neutrino, because only that component feels the CC interaction A_C. As it moves outwards, the density decreases (moving from right to left along the abscissa of Fig. 4.9), and the initial state turns slowly into the state $n_2(0)$ whose predominant component is the muon neutrino. At an intermediate point ρ_c where $A_C = \Delta \cos 2\alpha_0$ there is a resonance with $\alpha_\rho = \pi/4$, cf. (4.185). The two energy levels which would cross at this point if Δ were zero, repel each other such that the corresponding eigenstates exchange their roles. This is a well-known phenomenon of quantum mechanics.

In reality, the transition from the interior of the sun to outer space is not adiabatic, at least in the neighbourhood of the density ρ_c where the resonance occurs. There is a finite probability for a nonadiabatic transition from the state $n_1(\rho)$ to the state $n_2(0)$, and, likewise, for a transition from $n_2(\rho)$ to $n_1(0)$, when passing through the resonance point,

$$P_{12} = |\langle n_1(\rho)|n_2(0)\rangle|^2 = |\langle n_2(\rho)|n_1(0)\rangle|^2.$$

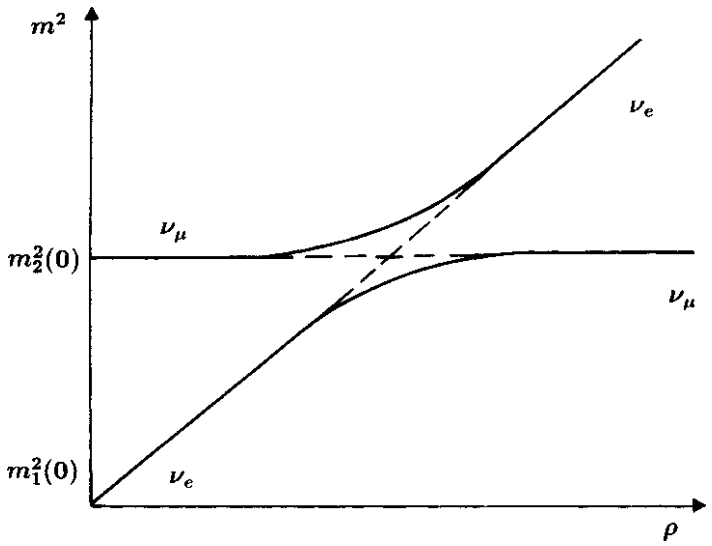

Fig. 4.9 Eigenvalues of the mass matrix (4.181) as a function of the electron density in the sun. For $\rho = 0$ the lower state is the electron neutrino, the higher state the muon neutrino. At high values of the density, in turn, the heavier state is the electron neutrino (because it feels the additional interaction), the lighter is the muon neutrino. If the density is such that the resonance (level crossing) does occur then the initial ν_e, as it moves outwards, turns into a ν_μ

The probability that the ν_e created inside the sun at time t_0 remains a ν_e at a later time t, after is has left the sun, is

$$\begin{aligned} P(\nu_e, t_0, t) = |\, \langle \nu_e(t_0, \rho) | \nu_e(t, 0) \rangle \,|^2 = |\, \langle n_1(t_0, \rho) | n_1(t, 0) \rangle \cos\alpha_\rho \cos\alpha_0 \\ + \langle n_1(t_0, \rho) | n_2(t, 0) \rangle \cos\alpha_\rho \sin\alpha_0 \\ + \langle n_2(t_0, \rho) | n_1(t, 0) \rangle \sin\alpha_\rho \cos\alpha_0 \\ + \langle n_2(t_0, \rho) | n_2(t, 0) \rangle \sin\alpha_\rho \sin\alpha_0 |^2. \end{aligned} \tag{4.188}$$

Making use of the symmetries of the mass Hamiltonian one shows that

$$|\, \langle n_1(t_0, \rho) | n_1(t, 0) \rangle \,|^2 = |\, \langle n_2(t_0, \rho) | n_2(t, 0) \rangle \,|^2 = 1 - P_{12}.$$

Well before ($\rho > \rho_c$) and well after ($\rho = 0$) the crossing of the resonance the mass eigenstates do change adiabatically. Therefore, if one averages the result (4.188) over the initial (production) region and over the final (detection) region, the cross terms in (4.188) oscillate rapidly and, hence, cancel out, so that the average is given by

$$P(\nu_e) = \langle P(\nu_e, t_0, t)\rangle = \cos^2\alpha_\rho \cos^2\alpha_0 + \sin^2\alpha_\rho \sin^2\alpha_0 \\ -P_{12}(\cos^2\alpha_0 - \sin^2\alpha_0)(\cos^2\alpha_\rho - \sin^2\alpha_\rho). \tag{4.189}$$

This result can also be rewritten in terms of $(2\alpha_\rho)$ and $(2\alpha_0)$:

$$P(\nu_e) = \frac{1}{2}\{1 + (1 - 2P_{12})\cos 2\alpha_\rho \cos 2\alpha_0\}. \tag{4.189'}$$

The calculation of P_{12} is a little more complicated[9] and we just quote the result. The radial dependence of the density is assumed to be an exponential

$$\rho(r) = \rho_0 e^{-r/R_S} \quad \text{with} \quad R_s = 0.092\, R_\odot,$$

where $R_\odot$ is the Sun's radius. The transition probability is found to be

$$P_{12} = \exp\{-\pi R_s(1 - \cos 2\alpha_0)\Delta/2E\}. \tag{4.190}$$

Finally, we note that the sign of the quantity A_C, (4.183), is essential for this mechanism to work. If A_C were negative there would be no resonance. For example, replace the ν_e by its antiparticle $\bar{\nu}_e$. Going through the same analysis of $\bar{\nu}_e e$ scattering, the antineutrino factor receives the opposite sign, and therefore A_C is negative and no resonance occurs.

Of course, whether or not this mechanism does indeed explain the solar neutrino puzzle requires a detailed comparison with the experimental information that is available. Without going into a more detailed analysis, we mention that all three results on the solar neutrino flux can be understood with the help of this conversion mechanism if the difference of squared masses is of the order of 10^{-6}–$10^{-5}\,\mathrm{eV}^2$ and $\sin^2\alpha_0 \approx 0.006$. Although very appealing, more tests will be necessary to confirm that this explanation is the correct one.

c) Results from oscillations with three lepton families

Except for one somewhat controversial experiment (the so-called LSND data) all oscillation experiments can be described in terms of three lepton families and a unitary mixing matrix in the neutrino sector of the kind of (3.152). This matrix is customarily written in the following suggestive way

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \mathrm{diag}\{e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1\},$$

where the abbreviations $s_{ik} = \sin\theta_{ik}$ and $c_{jl} = \cos\theta_{jl}$ are used. The angles δ, α_1, and α_2 describe possible CP-violation. With this obvious notation for the mixing angles

[9] P. Pizzochero, *Phys. Rev.* D36 (1987) 2293.

and using the abbreviation $m_{ik}^2 = m_i^2 - m_k^2$ for the differences of squared masses, one finds typically

$$\begin{aligned}
\Delta m_{21}^2 &= (7.59 \pm 0.20) \times 10^{-5}\,\mathrm{eV}^2, \\
\sin^2(2\theta_{12}) &= 0.87 \pm 0.03, \\
\Delta m_{32}^2 &= (2.43 \pm 0.13) \times 10^{-3}\,\mathrm{eV}^2, \\
\sin^2(2\theta_{13}) &< 0.25 \text{ CL} = 90\%.
\end{aligned}$$

Of course, this information is not enough for determining absolute values of any of the masses. Therefore, the results must be discussed under various assumptions on the hierarchy of the lepton families. A good introduction and a guide to the literature is found in [RPP10], Review 13.

4.3.3 Processes Which Change Lepton Family Numbers

In this section we return to the question of lepton family numbers that we discussed briefly in Sect. 3.1.1a. We sketch the various options that are conceivable in breaking the conservation of the family numbers (3.11) and/or of total lepton number (3.12), and we mention some of the most important consequences that one should be able to test in experiments. We then discuss one class of processes which provide an illustrative example of family number violation.

As this field is still open and our ignorance is great, our discussion is neither exhaustive nor conclusive.

a) Internal quantum numbers of neutrinos

A basic problem of lepton physics is the question whether neutrinos carry any additively conserved quantum numbers at all, and if they do, what the nature and dynamical origin of those quantum numbers is. There is very good evidence from astrophysics that neutrinos have no electromagnetic attributes such as electric charge or magnetic moment. So the discussion concentrates upon the lepton and lepton family numbers (3.11, 3.12).

The simple neutrino mixing scheme (4.140) is based on an assumed analogy of the leptons to the quark families. It implies that weak CC interactions do conserve total lepton number exactly but can change individual family numbers such that only their sum (3.12) is conserved. The neutrino states ν_f (weak eigenstates) or n_i (mass eigenstates) belong to the eigenvalue $+1$ of L, (3.12). The weak Lagrangian then possesses a global U(1)-symmetry. The neutrino states n_i, being massive and carrying nonvanishing eigenvalues of this symmetry, are Dirac fermions as defined in Chap. 1. This option can be tested through the mass measurements discussed above, Sects. 4.3.1, 4.3.2. It may also be testable in processes of the type $\mu \to \mathrm{e}\gamma$ provided the basic CC couplings are not strictly "V − A". [Without interference between V − A and V + A currents the rates, though finite, are hopelessly small.]

Another possibility is that neutrinos are massive and are selfconjugate, i.e. Majorana particles (cf. our discussion in Sect. 1.8.4). In this case they cannot carry total lepton number L. This option opens up a number of very interesting consequences some of which can be tested in experiment. The most direct signal would be provided by positive evidence for processes with $\Delta L \neq 0$ such as *neutrinoless double β-decay*. Examples are

$$
\begin{aligned}
&{}_{52}\mathrm{Te} \rightarrow {}_{54}\mathrm{Xe} + 2\mathrm{e}^- (\text{isotopes } A = 128 \text{ and } 130),\\
&{}^{82}_{34}\mathrm{Se} \rightarrow {}^{82}_{36}\mathrm{Kr} + 2\mathrm{e}^-,\\
&{}^{76}_{32}\mathrm{Ge} \rightarrow {}^{76}_{34}\mathrm{Se} + 2\mathrm{e}^-,
\end{aligned}
$$

or, more generally,

$$(Z, A) \rightarrow (Z + 2,\ A) + 2\mathrm{e}^-. \tag{4.191}$$

The decays can be thought of as two-step processes involving the sequence $\mathrm{n} \rightarrow \mathrm{p} + \mathrm{e}^- + \nu_\mathrm{M}$ and $\nu_\mathrm{M} + \mathrm{n} \rightarrow \mathrm{p} + \mathrm{e}^-$ with ν_M being a Majorana neutrino. As yet no positive evidence for any of these reactions was found. Upper limits on neutrinoless double β-decay yield a correlated upper limit on

$$\langle m \rangle = |\sum_i U_{ei} m(\nu_i)| \tag{4.192}$$

where cancellations in the sum may occur. Present limits are of the order of a few eV.

There are immediate consequences, although somewhat indirect, for $\mu \rightarrow \mathrm{e}\gamma$ and related processes. Regarding lepton decays such as μ-decay (4.38), there are consequences in principle but, with present limits on neutrino masses, they are too small to be measurable with any kind of precision.

The central problem in this discussion is the leptonic mass sector about which we know very little. It is the mass matrix which fixes the Dirac and Majorana nature of neutrinos and which determines the mixing matrices in the weak interaction eigenstates. We emphasize again that mixing angles should not be discussed without considering the mass sector. In particular, as should be clear from the discussion in Sect. 1.8.4, the difference between the Majorana and Dirac cases fades away when the masses go to zero.

b) $\mu \rightarrow e\gamma$ *and related processes*

In the framework of a discussion about lepton family numbers the processes

$$\mu^\pm \rightarrow \mathrm{e}^\pm \gamma, \tag{4.193a}$$

$$\mu^\pm \rightarrow \mathrm{e}^\pm \mathrm{e}^\mp \mathrm{e}^\pm \tag{4.193b}$$

$$\mu^-(Z, A) \rightarrow (Z, A)^* \mathrm{e}^-, \tag{4.193c}$$

are of particular interest because they are amenable to experimental investigations of extraordinary sensitivity. The present upper limits for the branching ratios

$$R_{\mu\to e\gamma} := \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to \text{all})}, \tag{4.194a}$$

$$R_{\mu\to 3e} := \frac{\Gamma(\mu \to e\bar{e}e)}{\Gamma(\mu \to \text{all})}, \tag{4.194b}$$

$$R_{\mu^-\to e^-}(Z,A) := \frac{\Gamma(\mu^-(Z,A) \to (Z,A)^* e^-)}{\Gamma(\mu^-(Z,A) \to \text{all})}, \tag{4.194c}$$

are of the order of 10^{-11} to 10^{-12}.

The processes (4.193), if they exist, are processes of second and higher order. As their detailed analysis is technically complicated we restrict the discussion to qualitative considerations here and refer the reader to the literature for the technical details. The first decay mode (4.193a) is a purely electroweak one and, independently of any model, depends on an effective $(\mu e\gamma)$-vertex as shown in Fig. 4.10(a). This vertex is a matrix element of the electromagnetic current $j_\alpha^{\text{e.m.}}$ between interacting lepton states, "dressed" by the weak interactions, viz. (for $\mu^- \to e\gamma$)

$$\begin{aligned}\langle e(p)|j_\alpha^{\text{e.m.}}(0)|\mu(q)\rangle = \frac{1}{(2\pi)^3}\overline{u_e(p)}\Big\{ & G_1(k^2)\left[\gamma_\alpha - \frac{1}{k^2}(m_\mu - m_e)k_\alpha\right] \\ & + G_2(k^2)\left[\gamma_\alpha + \frac{1}{k^2}(m_\mu + m_e)k_\alpha\right]\gamma_5 \\ & + \frac{i}{m_\mu}[F_1(k^2) + F_2(k^2)\gamma_5]\sigma_{\alpha\beta}k^\beta\Big\}\, u_\mu(q). \end{aligned} \tag{4.195}$$

The Lorentz covariants are combined such as to take account of current conservation, $\partial^\alpha j_\alpha^{\text{e.m.}}(x) = 0$. The decomposition (4.195) contains both parity-even and -odd terms, allowing for an arbitrary amount of parity violation from the weak interactions. The current operator j_α is hermitean. If the weak interactions are invariant under time reversal, then one can show that the form factors F_i and G_i, as defined by (4.195), are real. Finally, for photons on the mass shell, $k^2 = 0$, equation (4.195) remains finite only if G_1 and G_2 vanish at least like k^2,

$$G_1(k^2),\ G_2(k^2) \sim k^2 \text{ for } k^2 \to 0. \tag{4.196}$$

It is easy to work out the decay rate for process (4.193a). In the limit $m_e^2 \ll m_\mu^2$ one finds (exercise 4.13)

$$\Gamma(\mu \to e\gamma) = \frac{1}{2}\alpha m_\mu\{|F_1|^2 + |F_2|^2\}. \tag{4.197}$$

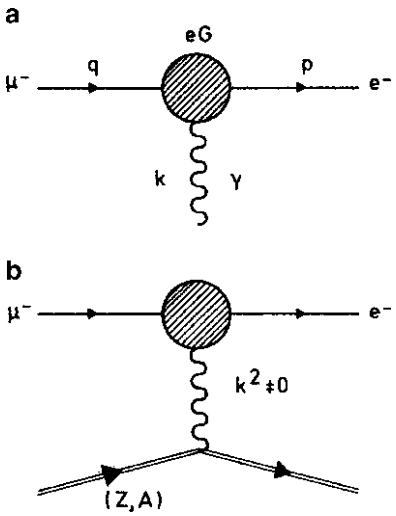

Fig. 4.10 (**a**) Effective $\mu e\gamma$-vertex. (**b**) Photon amplitude in $\mu \to$ e conversion on nuclei

In the processes (4.193b, c) the contribution of Fig. 4.10(b), equation (4.195), with $k^2 \neq 0$ constitutes what is called the *photonic* amplitude. The virtual photon then couples to the e^+e^--pair, or to the nucleus, respectively, via an ordinary electromagnetic vertex. In addition to this there can also be contributions from box diagrams in which two gauge bosons are exchanged between the four external fermion legs, with various possibilities for the intermediate states. The sum of these constitutes what is called the *weak* amplitude. The two amplitudes (weak and photonic) are usually of comparable magnitude.

Specific models which provide nonvanishing rates for the process (4.193) can be classified as follows.

(i) *Lepton mixing models*: These models start from the observation that both the neutral weak eigenstates as well as the charged weak eigenstates may be nontrivial mixtures of a set of mass eigenstates. Furthermore, in case the weak interactions contain left- and right-handed couplings, the left-handed and right-handed fields could be characterized by independent mixing schemes, cf. (4.140, 4.141).

In the standard GSW model, supplemented by the neutral mixing matrix (4.140), but no more, the amplitude for $\mu \to e\gamma$ is found to be proportional to m_i^2/m_W^2, the ratio of a typical squared neutrino mass to the square of the W-mass. For example, in a model with two families one would find typically

$$R_{\mu\to e\gamma} \simeq \frac{75\alpha}{128\pi} \sin^2(2\alpha_1) \frac{|m_1^2 - m_2^2|^2}{m_W^4}, \tag{4.198}$$

which can be at most 10^{-24} and hence remains far below any measurable level. This strong, dynamic suppression comes about in the same way as

the suppression of neutral current interactions with $\Delta S \neq 0$, cf. (3.160) in Sect. 3.4.2, in the quark sector: Let U^-_{ik} be the mixing matrix of the charged, left-handed leptons. The amplitude for $\mu \to e\gamma$ is proportional to $\Sigma_f (U^-_{\mu f})^* (U^-)_{ef}$, where the sum runs over all left-handed doublets $\in$ SU(2). If the sum is complete, i.e. if there is no charged, left-handed lepton in a singlet of SU(2), than this amplitude vanishes due to the unitarity of the matrix U^-.

The situation changes dramatically if the model is extended by allowing for more complicated lepton representations, for a larger Higgs sector and/or for interference terms between left-handed and right-handed CC couplings (such as are assumed in left–right symmetric models). In the latter case, for example, the amplitude for $\mu \to e\gamma$ is proportional to

$$\frac{m_i}{m_\mu} \mathrm{tg}\phi,$$

and also logarithmic terms in m_i / m_w, where m_i is a typical neutrino mass, ϕ is the mixing angle of $\mathrm{W_L}$ and $\mathrm{W_R}$ (see Sect. 4.1.2c). Even with the existing rather low limits on ϕ, the rates for the processes (4.183) could come very close to the present upper limits.

(ii) *Horizontal symmetries*: There is, of course, always the possibility of introducing direct interfamily couplings

$$g' \{ \overline{e(x)} (a\gamma^\alpha + b\gamma^\alpha \gamma_5) \mu(x) X_\alpha(x) + \mathrm{h.c.} \}$$

by means of some new very heavy vector bosons. Assuming coupling constants of the same order of magnitude as in the GSW model, i.e. $g' \sim e$, we would conclude from the present limit for the branching ratio (4.194a)

$$\frac{\alpha}{\pi} \left(\frac{m_\mathrm{W}}{m_\mathrm{X}} \right)^4 \lesssim 10^{-10},$$

or $m_\mathrm{X} \gtrsim 70\, m_\mathrm{W} \simeq 6\mathrm{TeV}$.

These superheavy bosons could be the gauge bosons of some new local gauge theory G_H whose particle multiplets are spread over different lepton and quark families of the standard electroweak gauge theory SU(2) × U(1). While the $\mathrm{W}^\pm$ and Z^0, in the patterns of (3.7) and (3.15), couple only "vertically" within each family, the gauge bosons of G_H would mediate "horizontally" between the families.

In conclusion, the class of processes (4.193), as well as the related rare kaon decays, bears on some fundamental problems of weak interactions: In the first alternative discussed above one is testing the leptonic mass sector, the multiplet structure of the theory, as well as certain aspects of the Lorentz structure of weak CC interactions. In particular, there is a clear relationship to the other topics and problems that we discussed in Sect. 4.1 and Sects. 4.3.1, 4.3.2 above: The Lorentz structure of leptonic CC couplings and the question of neutrino masses. In the

second alternative the rare decays open up a first window onto physics in the region of energies of 10^{12} to 10^{13} eV, a virgin soil still open for our imagination.

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Exercises

4.1 Knowing the behaviour of $\upsilon_\alpha(x)$ and $a_\alpha(x)$ under the discrete symmetries P, C, and T. Sect. 3.2.1 derive the transformation properties of s and p, (4.1), by studying divergences of υ_α and a_α.

4.2 Predict the isotropic spectra of ν_e and of $\bar{\nu}_\mu$ in positive muon decay, for the case of "V – A", equation (4.56). Discuss the relevance of this result for reaction (4.39b) and the experiment that led to the result (4.40).

4.3 Calculate the integrals (4.44). *Hint*: Go to a system where $Q = (Q^0, \mathbf{0})$.

4.4 In the reaction (4.68) introduce the variable $y = E_\mu^{\text{lab}}/E_\nu^{\text{lab}}$, transform the cross section (4.69) to $\mathrm{d}\sigma/\mathrm{d}y$, and integrate from some $y_{\min}$ to 1.

4.5 Calculate the integrals (4.44) for one massive and one massless particle. *Hints*: The integral (4.44a) can be done in a frame where $\boldsymbol{Q} = 0$. The integral (4.44b) can be written as $A(Q^2)(Q^\alpha Q^\beta - B(Q^2)Q^2 g^{\alpha\beta}$. Isolate and calculate $A(Q^2)$ and $B(Q^2)$.

4.6 Using the behaviour of the axial current $a_\alpha(x)$ under T and C, show that F_π, as defined by (3.6) is purely imaginary.

4.7 Compare the result (4.97) for the process $\tau \to \pi \nu_\tau$, divided by the sum of the rates for $\tau \to \mu \nu_\mu \nu_\tau$ and $\tau \to \mathrm{e} \nu_\mathrm{e} \nu_\tau$ to results in the Data Tables [RPP10]. In turn, use the leptonic branching ratios to estimate the lifetime of τ.

4.8 Prove equations (4.105) by making use of the Wigner–Eckart theorem.

4.9 Suppose the pionic axial current were simply

$$a_\alpha^{(i)} = -\frac{1}{\sqrt{2}} f_\pi \partial_\alpha(\phi_\pi^{(i)}(x)),$$

so that the semileptonic interaction were

$$L = -f_\pi \partial_\alpha(\phi_\pi^{(1)}(x) - \mathrm{i}\phi_\pi^{(2)}(x)) \sum_f \overline{\nu_f(x)} \gamma_\alpha(1-\gamma_5) f(x) + \mathrm{h.c.}$$

Introduce the coupling to the photon through minimal substitution and study the process $\pi^+ \to \mathrm{e}^+ \nu_\mathrm{e} \gamma$.

4.10 Apply the result of Sect. 4.3.1b to the special case $\pi \to \mu^+ \nu_\mu \gamma$. Perform the integral over the surface of the allowed region of phase space and compare to $\pi \to \mathrm{e} \nu \gamma$.

4.11 Consider the β^+-decay from $^{14}\mathrm{O}(\mathrm{g.s.}, 0^+)$ to $^{14}\mathrm{N}(0^+)$, whose masses differ by $\Delta E = 2.32\,\mathrm{MeV}$. As these states are members of an isotriplet one can calculate the decay amplitude in the same way as for pion β-decay. Calculate the decay width

$$\Gamma \simeq (2\pi)^7 \int \frac{\mathrm{d}^3 p_\mathrm{e}}{2E_\mathrm{e}} \int \frac{\mathrm{d}^3 p_\nu}{2E_\nu} \delta(E_\mathrm{e} + E_\nu - \Delta E) \sum |T|^2.$$

Show, in particular, that

$$\Gamma \simeq \frac{1}{\pi^3} G^2 \cos^2\theta_1 \int_{m_\mathrm{e}}^{\Delta E} \mathrm{d}E_\mathrm{e} P(E_\mathrm{e}),$$

where $P(E) = Ep(\Delta E - p)^2$ and $p = (E^2 - m_\mathrm{e}^2)^{1/2}$. The integral can be performed analytically giving

$$\Gamma = \frac{1}{\pi^3} G^2 \cos^2\theta_1 m_\mathrm{e}^5 F(\eta),$$

with

$$\eta = \sqrt{(\Delta E/m_e)^2 - 1},$$
$$F(\eta) = -\frac{1}{4}\eta - \frac{1}{12}\eta^3 + \frac{1}{30}\eta^5 + \frac{1}{4}\sqrt{1+\eta^2}\ln(\eta + \sqrt{\eta^2+1}).$$

If an experiment gave the result $\tau F(\eta)\ln 2 = 3075(10)$ s and if G were the same as in muon decay, what would $\cos\theta_1$ be?

4.12 Draw the simplest Feynman diagram for the process $\pi^0 \to e^+e^-$ and estimate $\Gamma(\pi^0 \to e^+e^-)$ knowing that $\tau(\pi^0 \to 2\gamma) \simeq 10^{-16}$ s. Show that the unitarity condition for the decay amplitude yields a lower bound for $\Gamma(\pi^0 \to e^+e^-)$.

4.13 Work out the decay rate (4.197).

Appendix A
Lorentz Invariant Distributions

Theorem. There are precisely two, linearly indendent, Lorentz invariant distributions $\Delta_i(z;\ m)$ which obey the Klein–Gordon equation for mass m. These are

$$\Delta_0(z;m) := -\frac{\mathrm{i}}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2\omega_k}(\mathrm{e}^{-\mathrm{i}kz} - \mathrm{e}^{\mathrm{i}kz}), \tag{A.1}$$

$$\Delta_1(z;m) := \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2\omega_k}(\mathrm{e}^{-\mathrm{i}kz} + \mathrm{e}^{\mathrm{i}kz}), \tag{A.2}$$

with $\omega_k \equiv k^0 = \sqrt{\boldsymbol{k}^2 + m^2}$. These distributions have the following properties:

(i) $$\{\Box + m^2\}\Delta_i(z;m) = 0,\ i = 0, 1, \tag{A.3}$$

(ii) $$\Delta_0(z^0 = 0, z; m) = 0, \tag{A.4a}$$

(iii) $$\frac{\partial}{\partial z^0}\Delta_0(z;m)\bigg|_{z^0=0} = -\delta(\boldsymbol{z}), \tag{A.4b}$$

(iv) $$\Delta_0(-z;m) = -\Delta_0(z;m), \qquad \Delta_1(-z;m) = \Delta_1(z;m), \tag{A.5}$$

(v) $$\Delta_0(z;m) = 0 \text{ for } z^2 < 0. \tag{A.6}$$

Note, however, that $\Delta_1\ (z;\ m)$ does not vanish for spacelike argument, $z^2 < 0$.

Proof. Part (i) is obvious because Δ_i are linear superpositions of exponentials which satisfy the Klein–Gordon equation. Part (ii) is also obvious by noting that, with $z^0 = 0$, the integrand is an *odd* function. Part (iii) gives minus the sum of the integrals $\int \mathrm{d}^3k\ \exp\{\pm ik\cdot z\}$ divided by $2(2\pi)^3$. This is a representation of the three-dimensional delta distribution. Part (iv) is again obvious by noting that the integrand in (A.1) is antisymmetric while that in (A.2) is symmetric when z is replaced by $-z$.

To prove (v) we first note that Δ_0, (A.1), can also be written as

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$$\Delta_0(z;m) = -\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4 q e^{-\mathrm{i}qz}\delta(q^2-m^2)\varepsilon(q^0), \tag{A.7}$$

where $\varepsilon(q^0) = q^0/|q^0|$ is the sign of the time component of the four-vector q. Note that unlike the variable k in (A.1) and (A.2), the four-vector q is unconstrained, i.e. has four independent components. That (A.7) and (A.1) represent the same distribution is seen by using the equality

$$\begin{aligned}\delta(q^2-m^2) &= \delta((q^0)^2-(\boldsymbol{q}^2+m^2)) \\ &= \frac{1}{2q^0}\left\{\delta(q^0-\sqrt{\boldsymbol{q}^2+m^2}) + \delta(q^0+\sqrt{\boldsymbol{q}^0+m^2})\right\}\end{aligned}$$

and by doing the integral over the variable q^0. This gives (A.1) with $\boldsymbol{k} = \boldsymbol{q}$, $k^0 \equiv \omega_k = \sqrt{\boldsymbol{q}^2+m^2}$. Now, for *time-like* z, i.e. for $z^2 > 0$, both z^2 and the sign of z^0, sign z^0 are Lorentz invariants. Thus, for time-like arguments the distribution Δ_0, being itself Lorentz invariant, must be a funtion of z^2 and of sign z^0. This is in accord with the antisymmetry noted in the first equation (A.5). For *space-like* arguments, however, the distinction between positive and negative values of z^0 is not Lorentz invariant. Therefore, for space-like z the distribution Δ_0 is a function of the invariant z^2 only. From this observation one concludes that

$$\Delta(-z;m) = +\Delta(z;m).$$

As this is in contradiction with the antisymmetry (A.5), we conclude that part (v) of the theorem is true. Regarding the distribution Δ_1 we note that this argument does not hold: Indeed, Δ_1 is symmetric under the transformation $z \to -z$, and, hence, does not vanish for space-like z.

Appendix B
S-Matrix, Cross Sections, Decay Probabilities

Write the scattering matrix (somewhat symbolically) as

$$S_{\mathrm{fi}} = \delta_{\mathrm{fi}} + R_{\mathrm{fi}}, \tag{B.1}$$

where f and i are asymptotic free states. δ_{fi} means "no scattering" and R_{fi} is the reaction matrix proper. R_{fi} necessarily contains a δ-distribution expressing conservation of total energy and momentum. Besides this distribution it is convenient to take out a factor $\mathrm{i}(2\pi)^4$ and to define the T-matrix by

$$R_{\mathrm{fi}} = \mathrm{i}(2\pi)^4 \delta(P_{\mathrm{f}} - P_{\mathrm{i}}) T_{\mathrm{fi}}. \tag{B.2}$$

The differential cross section for the reaction

$$\mathrm{a} + \mathrm{b} \to 1 + 2 + \cdots + N$$

(all particles being described asymptotically by plane waves), is given by the general expression

$$\mathrm{d}\sigma_{\mathrm{fi}}(\mathrm{a} + \mathrm{b} \to 1 + \cdots + N) = \frac{(2\pi)^{10}\delta(P_{\mathrm{f}} - P_{\mathrm{i}})}{2E_{\mathrm{a}} 2E_{\mathrm{b}} |\boldsymbol{v}_{\mathrm{ab}}|} |T_{\mathrm{fi}}|^2 \prod_{n=1}^{N} \frac{\mathrm{d}^3 p^{(n)}}{2E_N}. \tag{B.3}$$

In this expression

$$E_{\mathrm{a}} = \sqrt{\boldsymbol{p}^{(\mathrm{a})2} + m_{\mathrm{a}}^2}, \qquad E_{\mathrm{b}} = \sqrt{\boldsymbol{p}^{(\mathrm{b})2} + m_{\mathrm{b}}^2}, \qquad E_{\mathrm{n}} = \sqrt{\boldsymbol{p}^{(\mathrm{n})2} + m_{\mathrm{n}}^2};$$

$$P_{\mathrm{i}} = p^{(\mathrm{a})} + p^{(\mathrm{b})}, \qquad P_{\mathrm{f}} = \sum_{n=1}^{N} p^{(n)},$$

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and $|\boldsymbol{v}_{\mathrm{ab}}|$ is the relative velocity of the incoming particles, $\mathrm{d}^3 p^{(n)}/2E_n$ is the Lorentz invariant volume element in the phase space of particle number n. This term as well as the incoming flux factor in the first denominator is in accord with our covariant normalization $\langle \boldsymbol{p}'|\boldsymbol{p}\rangle = 2E_p \delta(\boldsymbol{p}'-\boldsymbol{p})$. Formula (B.3) holds in all systems of reference where the momenta $\boldsymbol{p}^{(\mathrm{a})}$ and $\boldsymbol{p}^{(\mathrm{b})}$ are collinear (e.g. the laboratory and centre-of-mass systems). It can be extended to any system by replacing the flux factor (so-called Møller factor) with the invariant on the r.h.s. of the following equation:

$$E_{\mathrm{a}} E_{\mathrm{b}} |\boldsymbol{v}_{\mathrm{ab}}| = \sqrt{(p^{(\mathrm{a})} \cdot p^{(\mathrm{b})})^2 - p^{(\mathrm{a})2} p^{(\mathrm{b})2}}. \tag{B.4}$$

(One should verify that this equation does indeed hold if the 3-momenta of particles a and b are collinear.)

The observable cross sections are obtained from (B.3) by integration over those momentum variables in the final state which are not observed. Similarly, depending on whether or not particles a and b have nonvanishing spin and are polarized, the appropriate average over spin projections must be taken. If the spin orientations in the final state are not discriminated, (B.3) must be summed over them.

In a similar fashion the differential decay rate of a particle a, with mass m_{a} and momentum q, into a final state with N particles,

$$\mathrm{a} \to 1 + 2 + \cdots + N,$$

is given by

$$\mathrm{d}\Gamma_{\mathrm{fi}} = (2\pi)^4 \delta(p^{(1)} + p^{(2)} + \cdots + p^{(N)} - q) \frac{(2\pi)^3}{2E_q} |T_{\mathrm{fi}}|^2 \prod_{n=1}^{N} \frac{\mathrm{d}^3 p^{(n)}}{2E_n}. \tag{B.5}$$

Again, depending on what shall be observed, integration over some of the momentum variables and, possibly, sums over spin projections in the final state must be performed. If the decaying particle has nonvanishing spin and if the spin orientation is not known, the formula must be averaged over all spin projections.

From (B.5) one sees that the squared decay amplitude has the dimension $[|T_{\mathrm{fi}}|^2] = E^{2(3-N)}$. Thus, in a two-body decay the dimension is (energy)^2, whereas in a three-body decay T is dimensionless. This can be useful in checking calculations.

Let us consider a few examples:

(i) *Differential cross section for $a + b \to 1 + 2$*. The example of elastic scattering of two massive particles (masses m and M, respectively) is worked out in detail in Sect. 2.4.2. Examples of neutrino reactions are treated in Sects. 3.2.4 and 4.1.2e.

(ii) *Two-body decay*. In the rest system of the decaying particle the two particles in the final state have the momenta

$$p^{(1)} = \{E_1, \boldsymbol{\kappa}\}, \; p^{(2)} = \{E_2, -\boldsymbol{\kappa}\},$$

with $E_1 + E_2 = m_\mathrm{a}$, $E_i = \left(m_i^2 + \kappa^2\right)^{1/2}$, and $\kappa := |\boldsymbol{\kappa}|$. Integrating over the 3-momentum of particle 2 one obtains from (B.5)

$$\mathrm{d}^3\Gamma = \frac{(2\pi)^7}{8m_\mathrm{a}E_1E_2}|T(\mathrm{a} \to 1+2)|^2\delta(E_1 + E_2 - m_\mathrm{a})\mathrm{d}^3p^{(1)},$$

where $\mathrm{d}^3p^{(1)}$ can be expressed in polar coordinates, $\mathrm{d}^3p^{(1)} = \kappa^2\mathrm{d}\kappa\mathrm{d}\Omega$. Noting than $\kappa\mathrm{d}\kappa = E_1\mathrm{d}E_1$ one can convert the integration over κ into an integration over E_1. For this we need the derivative of the argument of the δ-distribution with respect to E_1, viz.

$$\frac{\mathrm{d}}{\mathrm{d}E_1}\{E_1 + E_2 - m_\mathrm{a}\} = 1 + \frac{\mathrm{d}E_2}{\mathrm{d}\kappa}\frac{\mathrm{d}\kappa}{\mathrm{d}E_1} = \frac{E_1 + E_2}{E_2} = \frac{m_a}{E_2},$$

one obtains

$$\mathrm{d}^2\Gamma = \frac{(2\pi)^7\kappa}{8m_\mathrm{a}^2}|T(\mathrm{a} \to 1+2)|^2\mathrm{d}\Omega.$$

The decay probability is independent of the azimuth φ. Integrating over this angle we have

$$\mathrm{d}\Gamma = \frac{(2\pi)^8\kappa}{8m_\mathrm{a}^2}|T(\mathrm{a} \to 1+2)|^2\mathrm{d}(\cos\theta), \tag{B.6}$$

where θ is the opening angle between the spin expectation value of the decaying particle a and the momentum of particle 1. Equation (4.96) for the decay of a polarized τ into a pion and a neutrino provides an example for this case. If the decaying particle is spinless, or if it has spin but is unpolarized, $|T|^2$ is isotropic. Integrating over $\mathrm{d}(\cos\theta)$ one obtains the total decay rate

$$\Gamma = \frac{(2\pi)^8\kappa}{4m_\mathrm{a}^2}|T(\mathrm{a} \to 1+2)|^2. \tag{B.7}$$

(iii) *Three-body decays*. Here we distinguish several situations: If two of the particles in the final state are not observed (cf. the example of $\mu \to \mathrm{e}\nu\bar{\nu}$) one proceeds as described in Sect. 4.1.2a and obtains a differential decay rate

$$\mathrm{d}^2\Gamma/\mathrm{d}E\mathrm{d}(\cos\theta),$$

where θ is the opening angle between the spin of the decaying particle and the momentum of the observed particle in the final state. In other situations one may proceed as follows. Integrate first (B.5) over $\mathrm{d}^3p^{(3)}$ to obtain

$$\mathrm{d}^6\Gamma = \frac{(2\pi)^7}{16m_\mathrm{a}}\frac{\kappa_1\kappa_2}{E_3}|T(\mathrm{a}\to 123)|^2\delta(E_1+E_2+E_3-m_\mathrm{a}) \times \mathrm{d}E_1\,\mathrm{d}E_2\,\mathrm{d}\,\Omega_1\,\mathrm{d}\,\Omega_2,$$

where $E_3 = \left[m_3^2 + (\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)})^2\right]^{1/2}$ and $\kappa_\mathrm{i} := |\boldsymbol{p}^{(i)}|$. Then integrate over $\mathrm{d}\Omega_1$ for particle 1 which is emitted isotropically, take $\boldsymbol{p}^{(1)}$ as the 3-axis and make use of the axial symmetry around this direction, viz.

$$\mathrm{d}^3\Gamma = \frac{(2\pi)^9}{8m_\mathrm{a}}\frac{\kappa_1\kappa_2}{E_3}|T(\mathrm{a}\to 123)|^2\delta(E_1+E_2+E_3-m_\mathrm{a})\mathrm{d}E_1\,\mathrm{d}E_2\mathrm{d}(\cos\theta), \tag{B.8}$$

where now

$$E_3 = (m_3^2+\kappa_1^2+\kappa_2^2+2\kappa_1\kappa_2\cos\theta)^{1/2}.$$

This formula may be transformed to the variables E_1, E_2, E_3 by means of the Jacobian

$$\frac{\partial(E_1, E_2, \cos\theta)}{\partial(E_1, E_2, E_3)} = \frac{E_3}{\kappa_1\kappa_2}$$

and finally to the variables $s := E_2 + E_3$ and $t := E_2 - E_3$, giving

$$\mathrm{d}^2\Gamma = \frac{(2\pi)^9}{16m_\mathrm{a}}|T(\mathrm{a}\to 123)|^2\mathrm{d}E_1\mathrm{d}t, \tag{B.9}$$

from which the total rate is obtained by integration over the kinematic range of E_1 and of t.

Note that T contains a factor $(2\pi)^{-3/2}$ for each external particle. So $|T(\mathrm{a}\to 1+2)|^2$ produces a factor $(2\pi)^{-9}$, $|T(\mathrm{a}\to 123)|^2$ produces a factor $(2\pi)^{-12}$.

Appendix C1
Some Feynman Rules for Quantum Electrodynamics of Spin-1/2 Particles $f^{\pm}$

The rules hold for the matrix R, as defined by (B.1). The T-matrix is obtained upon comparison with the defining equation (B.2).

(i) *Diagrams*. One draws all connected diagrams of the process under consideration, at the order n in the coupling constant that one wishes to calculate. External and internal fermion lines are provided with arrows which point in the direction of the flow of *negative* charge. The momenta of internal lines are chosen such as to follow the arrow. All factors prescribed by the following rules must be written down from *right* to *left* following the direction of the arrows.

(ii) *External lines*. For each external, incoming f^- write a spinor in momentum space $\underline{u_f(p)}$, for each incoming f^+ write $\overline{\upsilon_f(p)}$. Similarly, for an outgoing f^- write $\overline{u_f(p)}$, for an outgoing f^+ write a $\upsilon_f(p)$. For an incoming or outgoing photon write a polarization vector $\varepsilon_\alpha(\boldsymbol{k},\ \lambda)$ with the index α to be contracted with γ^α at the fermion vertex to which it couples. In addition, each external particle obtains a factor $(2\pi)^{-3/2}$.

(iii) *Vertices*. Each vertex (ffγ) has a factor $e\gamma^\alpha$ and a δ-distribution for energy–momentum conservation at that vertex.

(iv) *Internal fermion lines*. An internal fermion line is represented by a propagator

$$\frac{\not{p} + m_f}{p^2 - m_f^2 + i\varepsilon},$$

where the direction of p is chosen in accordance with rule (i).

(v) *Internal photon lines*. An internal photon line with momentum k connects two vertices (ffγ) characterized by the Lorentz indices α and β [cf. rule (iii)] and yields a factor

$$-\frac{g^{\alpha\beta}}{k^2 + i\varepsilon}.$$

(vi) *Integrations*. All internal momenta must be integrated over. In all cases this yields a δ-distribution $\delta(P_i - P_f)$ for conservation of total energy–momentum.

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In orders of e which are higher than the lowest nontrivial order this rule also gives rise to some nontrivial integrations over internal momenta. Such integrals can turn out to be divergent and must then be analyzed in the framework of regularization and renormalization.

(vii) *Factors*. R_{fi} has a factor $(-)^P$ where P is the permutation of the fermions in the final state, as well as a factor $(-)^L$ if L is the number of closed fermion loops. In addition, R_{fi} obtains the following factors:

$$\mathrm{i}^{n+f_{\mathrm{i}}+b_{\mathrm{i}}}(2\pi)^{4(n-f_{\mathrm{i}}-b_{\mathrm{i}})},$$

where n is the order of perturbation theory, f_{i} the number of internal fermion lines, b_{i} the number of internal photon lines.

(viii) *Closed fermion loops*. Closed loops which couple to an *odd* number of photon lines give a vanishing amplitude. This is a consequence of C-invariance of QED.

(ix) *External potentials*. An external potential is an approximation for the interaction with a very heavy particle which therefore can absorb or provide an arbitrary amount of 3-momentum. Therefore, for an external potential one has to write a δ-distribution only for energy conservation, whilst the vertex factor $e\gamma^{\alpha}$ must be replaced by

$$\delta^{\alpha 0}\frac{Ze}{(2\pi)^3}\frac{\upsilon(\boldsymbol{k})}{k^2},$$

where Ze is the total charge that creates the potential, $\upsilon(\boldsymbol{k})$ is the form factor of the corresponding charge distribution,

$$\upsilon(\boldsymbol{k}) = \int \mathrm{d}^3x \mathrm{e}^{-ik\cdot x}\rho(\boldsymbol{x}),$$

with $\int \rho(\boldsymbol{x})\mathrm{d}^3x = 1$.

Appendix C2
Traces

The following formulae are all derived from the equations (1.73–1.75):

$$\mathrm{tr}\,\mathbb{1} = 4, \qquad \mathrm{tr}\gamma^\alpha = \mathrm{tr}\gamma_5 = 0. \tag{C.1}$$

The trace of a product with an *odd* number of factors vanishes. For products with an even number of γ-matrices the following relations are useful:

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\} = 4g^{\alpha\beta}, \tag{C.2}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma_5\} = 0, \tag{C.3}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\tau\} = 4\{g^{\alpha\beta}g^{\sigma\tau} - g^{\alpha\sigma}g^{\beta\tau} + g^{\alpha\tau}g^{\beta\sigma}\}, \tag{C.4}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\tau\gamma_5\} = 4i\varepsilon^{\alpha\beta\sigma\tau}, \tag{C.5}$$

$$\begin{aligned}\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau\} &= g^{\alpha\beta}\mathrm{tr}\{\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau\} - g^{\alpha\mu}\mathrm{tr}\{\gamma^\beta\gamma^\nu\gamma^\sigma\gamma^\tau\} \\ &\quad + g^{\alpha\nu}\mathrm{tr}\{\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\tau\} - g^{\alpha\sigma}\mathrm{tr}\{\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\tau\} \\ &\quad + g^{\alpha\tau}\mathrm{tr}\{\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma\}.\end{aligned} \tag{C.6}$$

In many cases the Lorentz indices of some of the γ-matrices in a product have to be contracted. The following formulae are then useful:

$$\gamma_\alpha\gamma^\alpha = 4, \qquad \gamma_\alpha \not{d}\gamma^\alpha = -2\not{d}, \tag{C.7}$$

$$\gamma_\alpha \not{d}\not{b}\gamma^\alpha = 4ab, \quad \gamma_\alpha \not{d}\not{b}\not{c}\gamma^\alpha = -2\not{c}\not{b}\not{d}, \tag{C.8}$$

$$\gamma_\alpha \not{d}\not{b}\not{c}\not{d}\gamma^\alpha = 2(\not{d}\not{d}\not{b}\not{c} + \not{c}\not{b}\not{d}\not{d}). \tag{C.9}$$

Note that in our conventions $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, $\varepsilon_{0123} = +1$. Note also the relation

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\sigma\tau} = -2\{\delta^\mu_\sigma\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\sigma\}. \tag{C.10}$$

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Appendix D
The Group SU(3)

The group SU(3) is defined as the set of all complex 3×3 matrices which are unitary and have determinant 1,

$$\mathrm{SU}(3) = \{U \in M_3(\mathbb{C}) | U^\dagger U = \mathbb{1},\ \det U = 1\}. \tag{D.1}$$

A matrix U in n complex dimensions depends on $2n^2$ real parameters. The unitarity condition $U^\dagger U = \mathbb{1}$ gives n^2 real constraints (n conditions for the diagonal, $2n(n-1)/2$ conditions for the nondiagonal matrix elements), the determinant gives one more real constraint. Therefore $U \in \mathrm{SU}(n)$ depends on n^2-1 real parameters. In particular, the elements of SU(3) depend on 8 real parameters. This is also seen from their representation (3.102) as exponential series in terms of hermitean 3×3 matrices with vanishing trace,

$$U = \exp\{\mathrm{i}H\} \quad \text{with} \quad \mathrm{tr}H = 0.$$

The condition on the trace of H is seen most easily after diagonalization of H, hence of U, whereby U becomes $U = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\lambda}{}_1, \ldots, \mathrm{e}^{\mathrm{i}\lambda}{}_3\right)$. The constraint $\det U = 1$ yields $\Sigma\lambda_i = 0$, hence $\mathrm{tr}\ H = 0$. Now, any hermitean 3×3 matrix H can be written as a linear combination of 8 linearly independent matrices as follows,

$$H = \sum_{k=1}^{8} \Lambda_k \left(\frac{\lambda_k}{2}\right).$$

The matrices λ_k have a standard form, called Gell-Mann matrices after M. Gell-Mann who constructed them in analogy to the Pauli matrices (1.24), viz.

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$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{D.2}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Apart from a factor 2, the matrices (D.2) are the generators in the three-dimensional representation $\underline{3}$, called the fundamental representation,

$$U(T_k) = \left(\frac{\lambda_k}{2}\right) \quad \text{with normalization } \mathrm{tr}\left\{\left(\frac{\lambda_k}{2}\right)\left(\frac{\lambda_j}{2}\right)\right\} = \frac{1}{2}\delta_{kj}. \tag{D.3}$$

SU(3) has rank 2. This is seen from eqs. (D.2) which show that two generators, T_3 and T_8, are simultaneously diagonal. Furthermore, from the explicit representation (D.2) it is obvious that the generators (T_1, T_2, T_3) form an SU(2) subgroup of SU(3). In the "eightfold way" where baryons and mesons made up of u, d and s quarks are classified according to the flavour group $\mathrm{SU_f}(3)$, this SU(2) subgroup is the (strong) isospin group. In a similar fashion one verifies that the sets

$$\left(U_1 = T_6, U_2 = T_7, U_3 = -\frac{1}{2}T_3 + \frac{\sqrt{3}}{2}T_3\right) \text{ and}$$
$$\left(V_1 = T_4, V_2 = T_5, V_3 = \frac{1}{2}T_3 + \frac{\sqrt{3}}{2}T_8\right)$$

also generate SU(2) subgroups of SU(3). In analogy to the isospin they are called, respectively, U-spin and V-spin.

In SU(3), i.e. when $\mathrm{SU_f}(3)$ is interpreted as the flavour group classifying light mesons and baryons, the electric charge operator is

$$Q_{\mathrm{e.m.}} = T_3 + \frac{1}{2}Y, \qquad \text{with} \quad Y := \frac{2}{\sqrt{3}}T_8, \tag{D.4}$$

Y denoting the operator of hypercharge with respect to strong interactions. One verifies easily that Q commutes with the generators U_i. This means that U-spin connects particles which carry the same electric charge.

Invariant tensors in SU(3) are δ_i^k, ε_{ijk} and ε^{ijk}. They are used in constructing irreducible, unitary representations of SU(3) from the fundamental representation $\underline{3}$, where

$$U(T_3) = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0\right), \qquad U(Y) = \text{diag}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right),$$
$$U(Q) = \text{diag}\left(+\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right),$$

and from its conjugate. The representation conjugate to $\underline{3}$ is the antitriplet $\underline{\bar{3}}$ which cannot be identified with the triplet $\underline{3}$–unlike SU(2) where the doublet and its conjugate are equivalent to each other.

Important Clebsch–Gordan decompositions of SU(3) are

$$\underline{3} \times \underline{\bar{3}} = \underline{1} + \underline{8}, \; \underline{3} \times \underline{3} = \underline{\bar{3}}_{\text{a}} + \underline{6}_{\text{s}}, \; \underline{3} \times \underline{3} \times \underline{3} = \underline{1}_{\text{a}} + \underline{8} + \underline{8} + \underline{10}_{\text{s}}, \tag{D.5}$$

where the subscripts 'a' and 's' indicate that these states are antisymmetric or symmetric, respectively, under exchange of any two of their constituents. In $\text{SU}_{\text{f}}(3)$ the triplet $\underline{3}$ is used to classify quarks (u,d,s), its conjugate $\underline{\bar{3}}$ then describes their antiparticles $(\bar{\text{u}}, \bar{\text{d}}, \bar{\text{s}})$. The singlets $\underline{1}$, the octets $\underline{8}$ and the decuplets $\underline{10}$ serve to classify physical hadrons, mesons made up of quarks and antiquarks and baryons made up of three quarks.

Appendix E
Dirac Equation with Central Fields

The Hamiltonian form (1.82a) of the Dirac equation is well adapted for a discussion of interactions with external fields. For the case of an external, spherically symmetric potential $V(r)$ and for stationary states ($\propto \mathrm{e}^{-\mathrm{i}Et}$) it reads

$$E\Psi(r) = \{-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla + V(r)\mathbb{1} + m\beta\}\Psi(\boldsymbol{r}), \tag{E.1}$$

with α and β as given by (1.81). Using the vector identities

$$\begin{aligned}\nabla &= \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}\cdot\nabla) - \hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\nabla)\\ &= \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}\cdot\nabla) - \frac{\mathrm{i}}{r}\hat{\boldsymbol{r}}\times\boldsymbol{l}\end{aligned}$$

one has

$$\boldsymbol{\alpha}\cdot\nabla = \boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}}\frac{\partial}{\partial r} - \frac{\mathrm{i}}{r}\boldsymbol{\alpha}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{l}) = \gamma_5\boldsymbol{S}\cdot\hat{\boldsymbol{r}}\left\{\frac{\partial}{\partial r} - \frac{1}{r}\boldsymbol{S}\cdot\boldsymbol{l}\right\},$$

where γ_5 is given by (1.78), whilst the matrix $\boldsymbol{S}$ stands for

$$\boldsymbol{S} = \begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}.$$

Finally, upon introduction of Dirac's angular momentum operator

$$K := \beta(\boldsymbol{S}\cdot\boldsymbol{l} + \mathbb{1}) \equiv \begin{pmatrix}K^{(0)} & 0\\ 0 & -K^{(0)}\end{pmatrix} \tag{E.2}$$

with $K^{(0)} = \boldsymbol{\sigma}\cdot\boldsymbol{l} + \mathbb{1}$, equation (E.1) takes the form

$$\left\{-\mathrm{i}\gamma_5\boldsymbol{S}\cdot\hat{\boldsymbol{r}}\left(\frac{\partial}{\partial r} + \frac{1}{r} - \beta\frac{K}{r}\right) + V(r)\mathbb{1} + \beta m\right\}\Psi = E\Psi =: H\Psi. \tag{E.3}$$

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One verifies by explicit calculation that K commutes with H, $[H, K] = 0$, but that H neither commutes with the orbital angular momentum nor with the spin. The operator K contains the entire dependence on angular momenta so that (E.3) lends itself to separation into radial and angular coordinates. To see this, we note first that $K^{(0)}$ can be written as

$$K^{(0)} = \boldsymbol{\sigma} \cdot \boldsymbol{l} + \mathbb{1} = 2\boldsymbol{s} \cdot \boldsymbol{l} + \mathbb{1} = \boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2 + \mathbb{1}.$$

Its eigenfunctions are the coupled states

$$\varphi_{jlm} = \sum_{m_l m_s} \left(l m_l, \frac{1}{2} m_s | j m \right) Y_{l m_l} \chi_{m_s}.$$

Denote the eigenvalues of $K^{(0)}$ by $-\kappa$, i.e.

$$K^{(0)} \varphi_{jlm} = -\kappa \varphi_{jlm} \qquad \text{with} \quad \kappa = -j(j+1) + l(l+1) - \frac{1}{4}.$$

Note also that $(K^{(0)})^2 = \mathbb{1} + \boldsymbol{\sigma} \cdot \boldsymbol{l} + \boldsymbol{l}^2 = \boldsymbol{j}^2 - \boldsymbol{s}^2 + \mathbb{1}$, from which one deduces $\kappa^2 = \left(j + \frac{1}{2}\right)^2$. From these formulae one sees that

$$\begin{aligned} &\text{for } \kappa > 0: \quad l = \kappa, \\ &\text{for } \kappa < 0: \quad l = -\kappa - 1, \\ &\text{in all cases} \quad j = |\kappa| - \frac{1}{2}. \end{aligned} \tag{E.4}$$

Therefore, the eigenfunctions of total angular momentum $\boldsymbol{j}$ can be written in the compact notation $\varphi_{jlm} \equiv \varphi_{\kappa m}$, the modulus of κ giving the value of j, the sign giving the value of $l = j \pm \frac{1}{2}$, according to the rules (E.4). As $K^{(0)} \varphi_{\kappa m} = -\kappa \varphi_{\kappa m}$, the eigenvalues and eigenfunctions of K, (E.2), are

$$K \begin{pmatrix} \varphi_{\kappa m} \\ \varphi_{-\kappa m} \end{pmatrix} = -\kappa \begin{pmatrix} \varphi_{\kappa m} \\ \varphi_{-\kappa m} \end{pmatrix}.$$

For the eigenfunctions Ψ of (E.1) or (E.3) one makes the ansatz

$$\Psi_{\kappa m}(r, \hat{\boldsymbol{r}}) = \begin{pmatrix} g_\kappa(r) \varphi_{\kappa m}(\hat{\boldsymbol{r}}) \\ \mathrm{i} f_\kappa(r) \varphi_{-\kappa m}(\hat{\boldsymbol{r}}) \end{pmatrix}, \tag{E.5}$$

the factor i being introduced for convenience so that the resulting differential equations for the radial functions f and g become real. As a last step one verifies by explicit calculation that $(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) \varphi_{\kappa m} = -\varphi_{-\kappa m}$. With these tools at hand one deduces from (E.3) the following system of differential equations:

$$
\begin{aligned}
f'_\kappa &= \frac{\kappa - 1}{r} f_\kappa - (E - V(r) - m) g_\kappa, \\
g'_\kappa &= -\frac{\kappa + 1}{r} g_\kappa + (E - V(r) + m) f_\kappa.
\end{aligned}
\tag{E.6}
$$

Clearly, this result does not depend on the specific representation (1.82a) of the Dirac equation we started from. For example in the representation (1.74) we would obtain

$$
\Psi_{\kappa m}(\boldsymbol{r}) = \frac{1}{\sqrt{2}} \begin{pmatrix} g_\kappa \varphi_{\kappa m} + \mathrm{i} f_\kappa \varphi_{-\kappa m} \\ g_\kappa \varphi_{\kappa m} - \mathrm{i} f_\kappa \varphi_{-\kappa m} \end{pmatrix}
$$

[cf. (2.107)].

Equation (E.5) shows very clearly that the central field solutions are not eigenfunctions of orbital angular momentum: For example for $\kappa = -1$, the upper component has $l = 0$, the lower has $\bar{l} = 1$. Thus, the relativistic analogue of an s-state has a component proportional to a p-state, cf. the discussing of the M1-transition 2s $\rightarrow$ 1 s in Sect. 4.1.3.

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Exercises: Further Hints and Selected Solutions

Chapter 1: Fermion Fields and their Properties

1.1. The Lagrangian density for a Majorana field is given by (1.108). One takes the partial derivatives with respect to, say, ϕ_B^* and with respect to $\partial_\mu \phi_B^*$,

$$\frac{\partial \mathscr{L}}{\partial \phi_B^*} = \frac{\mathrm{i}}{2}(\hat{\sigma}^\mu \partial_\mu)^{Bb} \phi_b + m\varepsilon^{BC}\phi_C^* \tag{1}$$

$$\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \phi_B^*\right)} = -\frac{\mathrm{i}}{2}(\hat{\sigma}^\mu)^{Bb}\phi_b. \tag{2}$$

The Euler–Lagrange equation which reads (1) – ∂_μ(2) = 0 yields

$$\mathrm{i}(\hat{\sigma}^\mu \partial_\mu)^{Bb}\phi_b + m\varepsilon^{BC}\phi_C^* = 0.$$

This is (1.107b). Similarly, (1.107a) is obtained by taking the derivatives with respect to ϕ_b and to $\partial_\mu \phi_b$.

1.3. With $J_2 = \sigma^{(2)}/2$ and noting that all even powers of $\sigma^{(i)}$ are equal to the unit matrix, while all odd powers are equal to $\sigma^{(i)}$,

$$(\sigma^{(i)})^{2n} = (\sigma^{(i)2})^n = \mathbb{1}, \quad (\sigma^{(i)})^{2n+1} = \sigma^{(i)},$$

one finds

$$\mathrm{e}^{\mathrm{i}\theta \mathrm{J}_2} = \mathrm{e}^{\mathrm{i}\theta/2\sigma^{(2)}} = \mathbb{1}\cos\frac{\theta}{2} + i\sigma^{(2)}\sin\frac{\theta}{2}.$$

With $\mu = (1/2, -1/2)$ counting the rows, $m = (1/2, -1/2)$ counting the columns, one has

$$(\mathrm{e}^{\mathrm{i}\pi J_2})_{\mu m} = \mathrm{i}(\sigma^{(2)})_{\mu m} - (-)^{1/2-\mu}\delta_{\mu,-m} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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Let U denote the transformation that effects the transition to contragedience. Regarding rotations, this simply means that U must be such that

$$UD(R)U^{-1} = (D(R)^{-1})^{\mathrm{T}} = D^*(R), \tag{3}$$

for any rotation matrix $D(R)$. When expressed in terms of the generators, this gives the condition

$$U\mathrm{e}^{\mathrm{i}\alpha_k J_k}U^{-1} = \mathrm{e}^{\mathrm{i}\alpha_k (U J_k U^{-1})} = \mathrm{e}^{-\mathrm{i}\alpha_k J_k^*}, \tag{4}$$

and, hence

$$UJ_kU^{-1} = -J_k^*, \quad \text{or} \quad UJ_k + J_k^*U = 0. \tag{5}$$

The phase convention which is standard in the theory of angular momentum (and which goes back to Condon and Shortley) yields J_1 real and positive, J_2 pure imaginary, and, of course, J_3 real and diagonal. Therefore the condition (5) is met if we choose

$$U = D(0, \pi, 0) = \mathrm{e}^{\mathrm{i}\pi J_2},$$

because a rotation by π about the 2-axis leaves invariant J_2 but transforms J_1 and J_3 into $-J_1$ and $-J_3$, respectively. Applying U twice leads back to the original representation.

1.4. For vanishing mass $m = 0$, and omitting the spinor indices, (1.69b) reads

$$\mathrm{i}\hat{\sigma}^\mu \partial_\mu (\mathrm{e}^{\pm \mathrm{i}px}\tilde{\phi}(p)) = 0.$$

The wave function in momentum space obeys the equation

$$(\sigma^0 p^0 - \boldsymbol{\sigma}\cdot\boldsymbol{p})\tilde{\phi}(p) = 0,$$

with $p^0 = |\boldsymbol{p}|$. Obviously, its solutions describe *positive* helicity, $\boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}| = +1$. In a similar way (1.69a) with $m = 0$ yields plane wave solutions with *negative* helicity.

1.6. With $\mathscr{L}$ as given in (1.163) we calculate the derivatives

$$\frac{\partial \mathscr{L}}{\partial \phi_A^*} = \frac{\mathrm{i}}{2}(\hat{\sigma}^\mu \partial_\mu)^{Ab}\phi_b - m_D^*\chi^A + m_1^*\varepsilon^{AB}\phi_B^*$$

$$\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_A^*\right)} = -\frac{\mathrm{i}}{2}(\hat{\sigma}^\mu)^{Ab}\phi_b.$$

Note that according to the rules (1.56) $\varepsilon^{AB}\phi_B^* = -\varepsilon^{ab}(\phi_b)^* = +\phi^{*A}$. Therefore, the Euler–Lagrange equation reads

$$\mathrm{i}(\hat{\sigma}^\mu \partial_\mu)^{Ab}\phi_b = m_D^*\chi^A - m_1^*\phi^{*A}.$$

Taking derivatives with respect to χ^* and to $\partial_\mu\chi^*$ one obtains in the same way

$$\mathrm{i}(\sigma^{\mu}\partial_{\mu})_{aB}\chi^{B} = m_D\phi_a + m_2^*\chi_a^*.$$

1.7. The behaviour of the two-spinors ϕ_a and χ^A under P, C, and T is given explicitly in Sect. 1.5. The transformation behaviour of the Lagrangian density (1.163) is studied in the same way as in (3.50), (3.55), and (3.56), and we recommend that the reader goes through these first. It then follows that (1.163) transforms according to the pattern

$$\begin{aligned} P &: \{m_D, m_1, m_2\} \to \{m_D^*, m_2, m_1\}, \\ C &: \{m_D, m_1, m_2\} \to \{m_D^*, m_2^*, m_1^*\}, \\ T &: \{m_D, m_1, m_2\} \to \{m_D, m_1^*, m_2^*\}. \end{aligned}$$

Thus, if one combines the three discrete operations, $\mathscr{L}$ is indeed found to be invariant

$$TCP : \{m_D, m_1, m_2\} \to \{m_D, m_1, m_2\}. \tag{6}$$

1.8. The left-hand side is the direct product of the matrices (1.26) and is easily calculated to be

$$\mathbb{1} \otimes \mathbb{1} - \sum_i \sigma^{(i)} \otimes \sigma^{(i)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The right-hand side is also a direct product but with indices c and B interchanged. As such it reads

$$-2\varepsilon_{ac}\varepsilon_{BD} = 2\begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 0 & \boxed{0} & \boxed{1} \\ \boxed{0} & \boxed{0} & -1 & 0 \\ 0 & -1 & \boxed{0} & \boxed{0} \\ \boxed{1} & \boxed{0} & 0 & 0 \end{pmatrix}.$$

Interchanging c and B means interchanging the second half of the first row with the first half of the second row, as well as interchanging the second half of the third row with the first half of the fourth row. This gives indeed the result (7).

1.9. First verify that the trace of ρ is real, viz

$$(tr\rho)^* = \mathrm{tr}\rho^{\dagger} = \mathrm{tr}(\gamma_0^2\rho^{\dagger}) = \mathrm{tr}(\gamma_0\rho^{\dagger}\gamma_0) = \mathrm{tr}\rho.$$

Now, let A be an observable expressed as an operator in the space of Dirac fields Ψ. One verifies that its expectation value in a pure state $|n\rangle$, say $\langle A\rangle_n = \langle n|\bar{\Psi}A\Psi|n\rangle$,

is real only if $\gamma^0\, A^\dagger\, \gamma^0 = A$. Thus, for a pure or a mixed state described by the density matrix ρ one finds

$$(\mathrm{tr}(\rho A))^* = \mathrm{tr}(A^\dagger \rho^\dagger) = \mathrm{tr}(\gamma^0 A^\dagger \gamma^0 \gamma^0 \rho^\dagger \gamma^0) = \mathrm{tr}(A\rho).$$

Here, we have made use of the relation (1.154b) but we did not need ρ to be hermitean.

1.10. First we work out the explicit form of ρ. Using the standard representation (1.78) one finds

$$\rho = \frac{1}{2}\begin{pmatrix} (E+m)(\mathbb{1}+\boldsymbol{\varsigma}\cdot\boldsymbol{\sigma}) & -(\boldsymbol{p}\cdot\boldsymbol{\sigma}) - (\boldsymbol{p}\cdot\boldsymbol{\varsigma}) + \mathrm{i}(\boldsymbol{p}\times\boldsymbol{\varsigma})\cdot\boldsymbol{\sigma} \\ (\boldsymbol{p}\cdot\boldsymbol{\sigma}) - (\boldsymbol{p}\cdot\boldsymbol{\varsigma}) + \mathrm{i}(\boldsymbol{p}\times\boldsymbol{\varsigma})\cdot\boldsymbol{\sigma} & -(E-m)(\mathbb{1}-\zeta\cdot\boldsymbol{\sigma}) - \frac{2}{E+m}(\boldsymbol{p}\cdot\boldsymbol{\varsigma})(\boldsymbol{p}\cdot\boldsymbol{\sigma}) \end{pmatrix}. \tag{8}$$

Then choose the 3-axis such that $\boldsymbol{p} = p\hat{e}_3$. The answer is then read off from the explicit representation (8).

1.12. The matrix $A \in SL(2,\mathrm{C})$ that represents the boost with rapidity parameter λ along the direction $\hat{w}$ is given by

$$\begin{aligned} A = \mathrm{e}^{\lambda/2\boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}}} &= \mathbb{1} + \left(\frac{\lambda}{2}\right)^{2n} (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}})^{2n} + \left(\frac{\lambda}{2}\right)^{2n+1} (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}})^{2n+1} \\ &= \mathbb{1}\cosh(\lambda/2) + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}}\sinh(\lambda/2). \end{aligned} \tag{9}$$

(Cf. also the solution to exercise 1.3 above). With $\sigma^{(1)T} = \sigma^{(1)}$, $\sigma^{(2)T} = -\sigma^{(2)}$, and $\sigma^{(3)T} = \sigma^{(3)}$, it is easy to find the transpose of the expression (9) as well as its inverse. The latter is

$$(A^{-1})^{\mathrm{T}} = \mathbb{1}\cosh(\lambda/2) - (\sigma^{(1)}\hat{\boldsymbol{w}}_1 - \sigma^{(2)}\hat{\boldsymbol{w}}_2 + \sigma^{(3)}\hat{\boldsymbol{w}}_3)\sinh(\lambda/2). \tag{10}$$

(Verify that (10) is indeed the inverse of A^{T}.) From (5), on the other hand, we have

$$\begin{aligned} UAU^{-1} &= \exp\left(\frac{\lambda}{2}\hat{\boldsymbol{w}}\cdot U\boldsymbol{\sigma}U^{-1}\right) = \exp\left(-\frac{\lambda}{2}\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}^*\right) \\ &= \exp\left(-\frac{\lambda}{2}(\hat{w}_1\sigma^{(1)} - \hat{w}_2\sigma^{(2)} + \hat{w}_3\sigma^{(3)}\right). \end{aligned}$$

The right-hand side, when expanded as in (9), is identical with (10).

1.13. The momenta conjugate to ϕ_a and to ϕ_B^* are, respectively,

$$\begin{aligned} \pi^a &= \frac{\partial\mathscr{L}}{\partial(\partial_0\phi_a)} = \frac{\mathrm{i}}{2}\phi_B^*(\hat{\sigma}^0)^{Ba}, \\ \pi^{*B} &= \frac{\partial\mathscr{L}}{\partial(\partial_0\phi_B^*)} = -\frac{\mathrm{i}}{2}(\hat{\sigma}^0)^{Ba}\phi_a. \end{aligned}$$

The Hamiltonian density is

$$\mathscr{H} = \frac{\mathrm{i}}{2}\phi_B^*(\hat{\sigma}^0)^{Ba}\partial_0\phi_a - \frac{\mathrm{i}}{2}\partial_0\phi_B^*(\hat{\sigma}^0)^{Ba}\phi_a - \mathscr{L},$$

with $\mathscr{L}$ as given in (1.108). Making use of the equations of motion (1.107) and taking the integral over 3-space, at constant time, one obtains

$$H = \int_{x^0=\text{const.}} \mathrm{d}^3x\ \mathscr{H} = \frac{\mathrm{i}}{2}\int_{x^0=\text{const.}} \mathrm{d}^3x\phi_B^* \overset{\leftrightarrow}{\partial}_0 \delta^{Ba}\phi_a.$$

It is not difficult to derive from the equations of motions that a plane wave solution must have the form

$$\phi_a(x) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}px}\varphi_1(p) + \mathrm{e}^{\mathrm{i}px}\psi_1(p) \\ \mathrm{e}^{+\mathrm{i}px}\varphi_2(p) + \mathrm{e}^{-\mathrm{i}px}\psi_2(p) \end{pmatrix}$$

and that the functions φ_i and ψ_i satisfy the equations

$$\begin{aligned}
(p_0 + p_3)\varphi_1(p) + (p_1 - \mathrm{i}p_2)\psi_2(p) &= m\varphi_2^*(p) \\
(p_0 - p_3)\varphi_2(p) + (p_1 + \mathrm{i}p_2)\psi_1(p) &= m\varphi_1^*(p) \\
(p_0 + p_3)\psi_1(p) + (p_1 - \mathrm{i}p_2)\varphi_2(p) &= -m\psi_2^*(p) \\
(p_0 + p_3)\psi_2(p) + (p_1 + \mathrm{i}p_2)\varphi_1(p) &= -m\psi_1^*(p).
\end{aligned}$$

The latter are seen to be invariant under the replacement

$$\varphi_1(p^0,\boldsymbol{p}) \to \varphi_2^*(p^0,-\boldsymbol{p}), \quad \psi_2(p^0,\boldsymbol{p}) \to -\psi_1^*(p^0,-\boldsymbol{p}).$$

Thus, $|\varphi_1(p^0,\boldsymbol{p})|^2 = |\varphi_1(p^0,\ -\boldsymbol{p})|^2$, and $|\psi_1(p^0,\ \boldsymbol{p})|^2 = |\psi_2(p^0,\ -\boldsymbol{p})|^2$. Insert now the Fourier decomposition

$$\phi_a(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3p}{2E_p}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}px}\varphi_1(p) + \mathrm{e}^{\mathrm{i}px}\psi_1(p) \\ \mathrm{e}^{+\mathrm{i}px}\varphi_2(p) + \mathrm{e}^{-\mathrm{i}px}\psi_2(p) \end{pmatrix}$$

of the field ϕ and its hermitean conjugate into the expression for $\mathscr{H}$ and perform the integral over d^3x, giving a δ distribution in the momentum variables. The result is

$$H = \int \frac{\mathrm{d}^3p}{2E_p}E_p[|\varphi_1(p)|^2 - |\varphi_2(p)|^2 - |\psi_1(p)|^2 + |\psi_2(p)|^2].$$

Indeed, H vanishes because the integrand is odd.

Chapter 2: Electromagnetic Processes and Interactions

2.1. Reflection with respect to a plane through the origin and perpendicular to the 3-direction inverts the direction of the incident momentum but leaves the spin orientation unchanged. A rotation by 180° about, say, the 2-axis, brings back the incident momentum to its initial configuration but interchanges the solutions u_+ and u_-. If the interaction is invariant the scattering amplitudes for the two spin orientations must be the same.

2.2. Repeat the calculation described in Sect. 2.4.2., inserting the matrix element

$$\langle p'|j_\beta(0)|p\rangle = \frac{Z}{(2\pi)^3}F(q^2),$$

instead of (2.46′), and use $m \approx 0$.

2.3. For an infinitesimally small translation four-vector ε with $|\varepsilon_\mu| \ll 1$, (2.39) gives

$$F'(x') \approx F(x) + \varepsilon_\mu \partial^\mu F(x) = F(x) + \mathrm{i}\varepsilon_\mu [P^\mu, F(x)].$$

On the other hand, using $[P^\mu, P^\nu] = 0$, and expanding the operator $U(\varepsilon) \approx \mathbb{1} + \mathrm{i}\varepsilon_\mu P^\mu$, we find

$$U(\varepsilon)F(x)U^{-1}(\varepsilon) \approx F(x) + \mathrm{i}\varepsilon_\mu [P^\mu, F(x)].$$

Now choose ε to be $\varepsilon_\mu = a_\mu/n$ for some large n and take the limit

$$\lim_{n\to\infty}\left(\mathbb{1} + \mathrm{i}\frac{a_\mu}{n}P^\mu\right)^n = \mathrm{e}^{\mathrm{i}a_\mu P^\mu}.$$

2.4. With $\boldsymbol{x} := \boldsymbol{r}_n - \boldsymbol{r}_e$ the integral over d^3q can be calculated using spherical coordinates in $\boldsymbol{q}$-space, i.e. $\mathrm{d}^3q = q^2\mathrm{d}q\mathrm{d}(\cos\theta_q)\mathrm{d}\phi_q$. The integrand being even in the modulus q of q we extend the integral over q to the interval $(-\infty, +\infty)$. With $z := \cos\theta_q$ and noting that the integral over the azimuth ϕ_q gives a factor 2π, we find

$$\int \mathrm{d}^3q \frac{\mathrm{e}^{\mathrm{i}q\cdot x}}{q^2} = 2\pi\int_0^\infty \mathrm{d}q \int_{-1}^{+1} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}qxz} = \frac{\pi}{\mathrm{i}x}\int_{-\infty}^{+\infty}\mathrm{d}q\left(\frac{\mathrm{e}^{\mathrm{i}qx}}{q} - \frac{\mathrm{e}^{-\mathrm{i}qx}}{q}\right) = \frac{2\pi^2}{x}.$$

In the second step we extended the integral over q to the interval $(-\infty, \infty)$, in the last step we made use of Cauchy's theorem.

2.7. Due to time dilation the effective lifetime that is recorded in the laboratory frame is $\tau_{\mathrm{lab}} = \gamma\tau$, with $\gamma = E/m$. The average length over which a particle of energy E can be transported is then estimated from τ_{lab} and its velocity $\upsilon = p/(m\gamma)$ in the laboratory frame.

2.8. For negative κ we have $\ell = -\kappa - 1$ and (2.105) read, dropping the index κ,

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{-\ell - 2}{r} f - (E - V - m)g \tag{11}$$

$$\frac{\mathrm{d}g}{\mathrm{d}r} = \frac{\ell}{r} g + (E - V + m)\, f. \tag{12}$$

By (12) f can be expressed in terms of g and of $\mathrm{d}g/\mathrm{d}r$. The derivative $\mathrm{d}f/\mathrm{d}r$ can also be expressed in terms g and $\mathrm{d}g/\mathrm{d}r$, by means of (11) and the previous result. We then take the derivative of (12) with respect to r to obtain a second- order differential equation for the function g alone. One obtains

$$\frac{\mathrm{d}^2 g}{\mathrm{d}r^2} + \left(\frac{2}{r} + \frac{\mathrm{d}V/\mathrm{d}r}{E - V + m} \right) \frac{\mathrm{d}g}{\mathrm{d}r} + \left((E - V)^2 - m^2 - \frac{\ell(\ell + 1)}{r^2} - \frac{\ell}{r} \frac{\mathrm{d}V/\mathrm{d}r}{(E - V + m)} \right) g = 0.$$

Let ε be the energy E from which the rest energy is subtracted, $E = m + \varepsilon$. In the limit of ε and V being small as compared to the rest energy m,

$$(E - V)^2 - m^2 \approx 2m(\varepsilon - V).$$

In this limit the second order differential equation for g reduces to

$$\frac{\mathrm{d}^2 g}{\mathrm{d}r^2} + \frac{2}{r} \frac{\mathrm{d}g}{\mathrm{d}r} - \left(\frac{\ell(\ell + 1)}{r^2} + 2m(V - \varepsilon) \right) g = 0. \tag{13}$$

The analogous analysis for $\kappa' = -\kappa - 1$ gives the same value of ℓ and leads to the same differential equation. The result (13) is identical with the nonrelativistic radial equation for the wave function $R_{\ell n}(r) = y_{n\ell}(r)/r$.

2.9. Making use of the identity

$$\boldsymbol{\ell} \cdot \boldsymbol{s} = \frac{1}{2}(\boldsymbol{j}^2 - \boldsymbol{\ell}^2 - \boldsymbol{s}^2)$$

the angular matrix element for the states $\ell = 1$, $j = 3/2$ or $j = 1/2$, is easily calculated,

$$\langle \boldsymbol{\ell} \cdot \boldsymbol{s} \rangle = \frac{1}{2} \left(j(j + 1) - \ell(\ell + 1) - \frac{3}{4} \right).$$

The former has $\kappa = -2$, the latter has $\kappa = 1$. In the nonrelativistic limit they have the same radial function (2.145) which, for a circular orbit reads

$$y_{n,n-1} = \frac{2^n}{n^{n+1} \sqrt{(2n-1)!}\, a_B^{n+1/2}} r^n \mathrm{e}^{-r/na_B}.$$

The expectation value of $1/r^3$ is calculated in an elementary way,

$$\left\langle 1/r^3 \right\rangle_{n,n-1} = \frac{2}{n^4(2n-1)(n-1)} \frac{1}{a_B^3}.$$

Thus, the fine structure splitting, when calculated in first order perturbation theory, is found to be

$$\Delta E = \frac{m(Z\alpha)^4}{2n^4(n-1)}. \tag{14}$$

On the other hand, taking the difference of (2.164) for $\kappa = n-1$ and for $\kappa = -n$ one finds the same expression (14). Thus, up to terms of higher order in $Z\alpha$ the relativistic result agrees with the perturbative estimate.

2.10. The strategy for solving this exercise is to use partial integration in such a way that the radial part of the Laplacian

$$\Delta(V_1 - V_2) = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}(V_1 - V_2)}{\mathrm{d}r} \right)$$

appears in the integrand. By Poisson's equation this is then replaced by the difference of the corresponding charge densities. Accordingly, we transform the integral

$$\begin{aligned} I :&= \int_0^\alpha \mathrm{d}r (ar^2 + br^3 + cr^4)(V_1 - V_2) \\ &= \int_0^\alpha r^2 \mathrm{d}r \left(\frac{a}{2} r^2 + \frac{b}{12} r^3 + \frac{c}{20} r^4 \right) \Delta(V_1 - V_2). \end{aligned}$$

Making use of Poisson's equation $\Delta(V_1 - V_2) = -4\pi Ze(\rho_1 - \rho_2)$, the difference of the potentials is replaced by the difference of the charge distributions. The formula for I now contains the moments $\Delta\langle r^n \rangle = 4\pi \int r^2 \mathrm{d}r\, r^n$. This yields the desired formula for ΔE,

$$\Delta E = e \int \mathrm{d}^3x |\psi|^2 (V_1 - V_2) \approx 4\pi \mathrm{e} I.$$

The coefficients a, b, and c for the $2p$ state and for the $2s$ state are obtained by expanding the squared radial functions (2.145) in power of r, viz.

$$\begin{aligned} a^{(2p)} = b^{(2p)} = 0, \quad c^{(2p)} &= \frac{1}{12a_B^5}, \\ a^{(2s)} = \frac{1}{2a_B^3}, \quad b^{(2s)} &= -\frac{1}{a_B^4}, \quad c^{(2s)} = \frac{7}{8a_B^5}. \end{aligned}$$

2.11. Using the equations in momentum space $\not{p}u(p) = m\,u(p)$ and $\overline{u(p)}\not{p} = m\,\overline{u(p)}$ we calculate

$$\begin{aligned}\overline{u(p')}\,i\sigma_{\alpha\beta}(p-p')^{\beta}u(p) &= -\frac{1}{2}\overline{u(p')}\,[\gamma_\alpha\not{p} - \gamma_\alpha\not{p}' - \not{p}'\gamma_\alpha + \not{p}'\gamma_\alpha]u(p)\\ &= -\overline{u(p')}\,\left[\gamma_\alpha\not{p} - p_\alpha + \not{p}'\gamma_\alpha - p'_\alpha\right]u(p)\\ &= -2m\,\overline{u(p')}\,\gamma_\alpha u(p) + \left(p_\alpha + p'_\alpha\right)\overline{u(p')}\,u(p).\end{aligned}$$

In the second step we have used $\gamma_\alpha\not{p}' = -\not{p}'\gamma_\alpha + 2p'_\alpha$ and $\not{p}\gamma_\alpha = -\gamma_\alpha\not{p} + 2p_\alpha$. Analogous relations hold with $u(p)$ replaced by $v(p)$ and $\overline{u(p')}$ by $\overline{v(p')}$. In view of weak interactions (Chaps. 3 and 4) it is instructive to derive analogous identities which hold when $\sigma_{\alpha\beta}$ is multiplied by γ_5.

Chapter 3: Weak Interactions and the Standard Model of Strong and Electroweak Interactions

3.1. Applying parity or charge conjugation to one of the indicated states obviously reproduces the same state. The eigenvalues are determined as follows. The parity of a quark–antiquark state contains a factor $(-)^{\ell}$ from the angular part $Y_{\ell m}$ of the orbital wave function, ℓ being the relative angular momentum, and a factor -1 from the relative intrinsic parity of quark and antiquark, thus $P = (-)^{\ell+1}$.

Charge conjugation interchanges quark and antiquark, hence the spins must be recoupled, giving a factor $(-)^{1/2+1/2-S}$. Regarding the orbital wave function, we note that interchanging q and $\bar{q}$ means relabeling coordinates in the relative coordinate $\boldsymbol{r}_1 - \boldsymbol{r}_2$. While the radial part does not change, the angular part changes by a factor $(-)^{\ell}$. Finally, charge conjugation gives an extra minus sign when applied to a doublet (with respect to an internal symmetry such as strong isospin); cf. Sect. 1.10. Therefore, $C = (-)^{\ell+S}$. If the ω meson is purely nonstrange, i.e., if $\omega = (u\bar{u} - d\bar{d})/\sqrt{2}$, then ϕ is a purely strange state, i.e., $\phi = s\bar{s}$. This is so because ϕ is an isoscalar and must be orthogonal to ω.

From the results above the P and C eigenvalues for low-lying mesons are as follows:

$$\begin{aligned}&\pi_0,\ \eta\,(\ell = 0, S = 0): && P = -1 && C = +1\\ &\rho_0, \omega, \phi\,(\ell = 0, S = 1): && P = -1 && C = -1\\ &a_2\,(\ell = 1, S = 1): && P = +1 && C = +1\end{aligned}$$

3.4. The generators fall into two sets, say T_k for SU(P) and S_j for SU(Q), which commute for all k and j, $[T_k, S_j] = 0$. Therefore, expressions such as (3.112) for the gauge potential and (3.121) for the field strength tensor will read explicitly

$$A_\alpha(x) = \mathrm{i}\left[e_P \sum B_\alpha^{(k)}(x) T_k + e_Q \sum C_\alpha^{(j)}(x) S_j\right],$$
$$F_{\alpha\beta}(x) = \mathrm{i}\left[e_P \sum G_{\alpha\beta}^{(k)}(x) T_k + e_Q \sum H_{\alpha\beta}^{(j)}(x) S_j\right].$$

In fact, a more precise way of writing the generators would be $\mathrm{T}_k \otimes \mathbb{1}$ and $\mathbb{1} \otimes S_j$, where the first factor acts on the internal space spanned by representations of $\mathrm{SU}(P)$, the second acts on the internal space pertaining to $\mathrm{SU}(Q)$. The construction of the gauge theory for the group $\mathrm{SU}(P) \times \mathrm{SU}(Q)$, along the lines of Sect. (3.3), goes through for any choice of the coupling constants e_P and e_Q. For example, one verifies that the Lagrangian for the pure gauge fields is just the sum of the corresponding Lagrangians for the $\mathrm{SU}(P)$ fields B_α and for the $\mathrm{SU}(Q)$ fields C_α,

$$-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{4} G_{\alpha\beta} G^{\alpha\beta} - \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta}.$$

Here we have used $\mathrm{tr}\,\{T_k \otimes \mathbb{1}\} = \mathrm{tr}\, T_k = 0$ and likewise for S_j.

3.5. The equation

$$-(\partial_\alpha g^{-1}(x)) g(x) = A_\alpha(x)$$

is interpreted as a differential equation for the gauge transformation $g(x)$, the inhomogeneity being $A_\alpha(x)$. Rewriting this equation

$$\partial_\alpha g^{-1}(x) = -A_\alpha(x) g^{-1}(x), \tag{15}$$

we obtain the integrability condition

$$(\partial_\beta \partial_\alpha - \partial_\alpha \partial_\beta) g^{-1}(x) = 0.$$

Inserting (15) we find

$$\begin{aligned}
&\partial_\alpha (A_\beta(x) g^{-1}(x)) - \partial_\beta (A_\alpha(x) g^{-1}(x)) \\
&\quad = (\partial_\alpha A_\beta - \partial_\beta A_\alpha) g^{-1} + A_\beta \partial_\alpha g^{-1} - A_\alpha \partial_\beta g^{-1} \\
&\quad = (\partial_\alpha A_\beta - \partial_\beta A_\alpha) g^{-1} + (-A_\beta A_\alpha + A_\alpha A_\beta) g^{-1} \\
&\quad = F_{\alpha\beta}(x) g^{-1}(x) = 0.
\end{aligned}$$

In the second step we have used (15) to replace $\partial_\alpha g^{-1}$, in the last step we have inserted the definition of the field strength tensor. As $g^{-1}(x)$ is not identically zero we conclude: $A_\alpha(x)$ is gauge equivalent to 0 if and only if $F_{\alpha\beta}(x)$ vanishes identically. Thus, $F_{\alpha\beta}$ is a measure which tells us to which extent the potential A_α cannot be gauged to zero.

3.6. With $G = \mathrm{SO}(3)$ the structure constants are $C_{ijk} = \varepsilon_{ijk}$. For simplicity consider a set of real scalar fields

$$\Phi(x) = \{\phi_1(x), \phi_2(x), \ldots, \phi_M(x)\}$$

forming an M-dimensional representation of G. The invariant scalar product is

$$(\Phi, \Phi) = \sum_{k=1}^{M} \phi_k(x)\phi_k(x).$$

A *globally* invariant theory involving these fields could have the form

$$\mathscr{L}_0 = \frac{1}{2}(\partial_\alpha \Phi, \partial^\alpha \Phi) - m^2(\Phi, \Phi) + \lambda(\Phi, \Phi)^2.$$

This is turned into a *locally* gauge invariant theory by introducing gauge fields $A_\alpha(x) = \mathrm{ie}\sum_{i-1}^{3} T_i A_\alpha^{(i)}(x)$ in the adjoint representation of SO(3) and the covariant derivative $D(A)$ which acts on the scalar multiplet. Symbolically, we obtain

$$\mathscr{L} = -\frac{c}{4}(F^{\alpha\beta}, F_{\alpha\beta}) + \frac{1}{2}(D_\alpha \Phi, D^\alpha \Phi) - m^2(\Phi, \Phi) + \lambda(\Phi, \Phi)^2, \tag{16}$$

with $c = 1/(e^2\kappa)$ and $\mathrm{tr}(T_i T_k) = \kappa\delta_{ik}$. The second, third, and fourth terms are obvious. We work out the first one explicitly, viz.

$$\begin{aligned} -\mathscr{L}_A &:= \frac{c}{4}(F^{\alpha\beta}, F_{\alpha\beta}) \\ &= \frac{1}{4}\sum_{i=1}^{3} f_{\alpha\beta}^{(i)} f^{(i)\alpha\beta} - \frac{\mathrm{e}}{2}\sum_{ijk} \varepsilon_{ijk} f_{\alpha\beta}^{(i)} A^{(j)\alpha} A^{(k)\beta} \\ &\quad + \frac{\mathrm{e}^4}{4}\sum_{ikpq}\sum_{i} \varepsilon_{ijk}\varepsilon_{ipq} A_\alpha^{(j)} A_\beta^{(k)} A^{(p)\alpha} A^{(q)\beta} \qquad (17) \\ &= \frac{1}{4}\boldsymbol{f}_{\alpha\beta}\cdot\boldsymbol{f}^{\alpha\beta} - \frac{\mathrm{e}}{2}\boldsymbol{f}_{\alpha\beta}\cdot(\boldsymbol{A}^\alpha \times \boldsymbol{A}^\beta) \\ &\quad + \frac{\mathrm{e}^2}{2}\{(\boldsymbol{A}_\alpha\cdot\boldsymbol{A}^\alpha)(\boldsymbol{A}_\beta\cdot\boldsymbol{A}^\beta) - (\boldsymbol{A}_\alpha\cdot\boldsymbol{A}^\beta)(\boldsymbol{A}_\beta\cdot\boldsymbol{A}^\alpha)\}. \end{aligned}$$

Here the boldface notation stands for $\boldsymbol{A}_\alpha = \left(A_\alpha^{(i)}, i = 1, 2, 3\right)$ and, likewise, for $\boldsymbol{f}_{\alpha\beta} = \left(f_{\alpha\beta}^{(i)} = \partial_\alpha A_\beta^{(i)} - \partial_\beta A_\alpha^{(i)}, i = 1, 2, 3\right)$. (Remember that the gauge fields and the field strengths belong to the adjoint representation of $G = \mathrm{SO}(3)$. Thus, they form a triplet and, therefore, are isomorphic to a vector in real, three-dimensional space $\mathbb{R}^3$.) We have made use of the identity

$$\sum_i \varepsilon_{ijk}\varepsilon_{ipq} = 2(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}),$$

and have introduced the well-known scalar and cross products of $\mathbb{R}^3$. This theory is perhaps the simplest, nontrivial example of a gauge theory. In the explicit form (16) and (17) it is easy to interpret in terms of interactions between the three gauge bosons and with the scalar fields. In the example studied here, G is both a spectrum symmetry and the structure group from which the gauge group is constructed. Without the gauge principle, i.e., without the geometric framework on which it rests, it would be difficult to guess the specific form (17) of the Lagrangian.

3.7. A priori the mixing matrix between any two generations is a unitary matrix $U \in U(2)$ and, hence, depends on four parameters: A common phase φ in $U = \mathrm{e}^{i\varphi} U_S$ and three angles which parametrize the factor with determinant $+1$, $U_S \in \mathrm{SU}(2)$. Clearly, the former is irrelevant and may be omitted from the start. In analogy to $D^{(1/2)}$, cf. Sect. 1.8.3.a, the latter must have the form, for the example of mixing between the first and the third generation,

$$
\begin{aligned}
U_S^{(1,3)} &= \begin{pmatrix} \cos\alpha_{13}\mathrm{e}^{\mathrm{i}(\beta_{13}+\gamma_{13})} & 0 & \sin\alpha_{13}\mathrm{e}^{\mathrm{i}(\beta_{13}-\gamma_{13})} \\ 0 & 1 & 0 \\ -\sin\alpha_{13}\mathrm{e}^{-\mathrm{i}(\beta_{13}-\gamma_{13})} & 0 & \cos\alpha_{13}\mathrm{e}^{-\mathrm{i}(\beta_{13}+\gamma_{13})} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\beta_{13}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\beta_{13}} \end{pmatrix} \begin{pmatrix} \cos\alpha_{13} & 0 & \sin\alpha_{13} \\ 0 & 1 & 0 \\ -\sin\alpha_{13} & 0 & \cos\alpha_{13} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\gamma_{13}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\gamma_{13}} \end{pmatrix}.
\end{aligned}
$$

What remains to be done is to multiply the three mixing matrices, $U_S^{(1,2)} U_S^{(2,3)} U_S^{(1,3)}$, and to realize that both the initial, unmixed, states, say $b^{(m)} \equiv (d, s, b)$, and the mixed states $d^{(n)}$ can each be multiplied by an unobservable phase, such that the mixing matrix

$$
\mathrm{diag}(\mathrm{e}^{i\phi_1}, \mathrm{e}^{i\phi_2}, \mathrm{e}^{i\phi_3}) U_S^{(1,2)} U_S^{(2,3)} U_S^{(1,3)} \mathrm{diag}(\mathrm{e}^{i\psi_1}, \mathrm{e}^{i\psi_2}, \mathrm{e}^{i\psi_3})
$$

is physically equivalent to $U_S^{(1,2)} U_S^{(2,3)} U_S^{(1,3)}$. This freedom in the choice of ϕ_i and ψ_i reduces the number of observable phases to four.

3.8. In this exercise, which we do not write out here, it is important to realize that the coupling of the charged W-bosons to the photon is fixed through the generalized kinetic term $(F_{\alpha\beta},\ F^{\alpha\beta})$, so that, in particular, the anomalous magnetic moment of the W can be identified. If the coupling had been constructed from the minimal coupling principle (1.200) of electrodynamics, the W would have obtained no anomalous magnetic moment.

3.9. The neutral partner of the Higgs doublet has $y = -2t_3$. The right-hand side of (3.173) then becomes

$$
-\mathrm{i}\left(g A_\alpha^{(3)} - g' A_\alpha^{(0)}\right)\ t_3\, \phi^0.
$$

When this is inserted into (3.172) we obtain

$$\frac{1}{2}(\phi^0, \phi^0)\, t_3^2 \left(g A_\alpha^{(3)} - g' A_\alpha^{(0)}\right) \left(g A^{(3)\alpha} - g' A^{(0)\alpha}\right) \equiv \frac{1}{2} M_{ik}^2 \, A_\alpha^{(i)} \, A^{(k)\alpha}, \; i, k = 0, 3.$$

Thus the mass matrix for the neutral gauge bosons is given by

$$M^2 = (\phi^0, \phi^0)\, t_3^2 \begin{pmatrix} g'^2 & -gg' \\ -gg' & g^2 \end{pmatrix}. \tag{18}$$

Diagonalization of this matrix gives $m_\gamma^2 = 0$ and $m_Z^2 = (\phi^0, \phi^0) t_3^2 (g^2 + g'^2)$. The result for m_w^2 is as given in Sect. 3.4.3c.

3.10. Return to (3.222) and write the second factor in terms of helicity projection operators, viz.

$$\gamma^\alpha - \lambda \gamma^\alpha \gamma_5 = \frac{1+\lambda}{2} \gamma^\alpha (\mathbb{1} - \gamma_5) + \frac{1-\lambda}{2} \gamma^\alpha (\mathbb{1} + \gamma_5).$$

The longitudinal polarization of the outgoing neutrino is easily obtained from this;

$$P_l = \frac{(1-\lambda) - (1+\lambda)}{(1-\lambda) + (1+\lambda)} = -\lambda.$$

3.12. The calculation, which is a little lengthy, may be done along the lines of our calculation with $m_\tau = 0$, as given in Sect 3.6.3. Alternatively, this might be a good example to try out an algebraic program such as REDUCE.

3.13. Inserting $c_A^{(e)} = 0$ and $c_V^{(F \equiv \mu)} = 0$ into (3.242) gives a vanishing asymmetry. Although the electromagnetic and the neutral weak (NC) couplings have negative relative parity, this is not seen in the asymmetry. This interference could only be seen by measuring a spin-momentum correlation.

Chapter 4: Beyond the Minimal Standard Model

4.2. A straightforward way of solving this exercise is to repeat the calculation of the isotropic spectrum along the lines of Sect. 4.1.2. and for the coupling $(\upsilon_\alpha - a_\alpha)(\upsilon^\alpha - a^\alpha)$, by singling out the electron neutrino and the muonic antineutrino, respectively, and integrating over the other two leptons in the final state. A more elegant way is the following: First, realize that the parameter ρ determines the intercept of the unpolarized spectrum at $x = 1$, cf. Fig. 4.2. Recall that e^+ and $\bar{\nu}_\mu$ are right-handed, while ν_e is left-handed.

Suppose it is the electron neutrino ν_e that is measured. The point $x = 1$ corresponds to the kinematic situation where ν_e moves in a given direction with maximal energy, while e^+ and $\bar{\nu}_\mu$ are parallel and move in the opposite direction. The three helicities adding up to 3/2 in the final state, this configuration is

forbidden by angular momentum conservation. Hence, the parameter ρ must be zero, $\rho(\nu_e) = 0$.

Suppose now that one measures the $\bar{\nu}_\mu$. At $x = 1$ this neutrino moves in a given direction, and carries positive helicity. The helicities of e^+ and ν_e who move in the opposite direction, now add up to zero. Therefore, there is no obstacle for this configuration to occur. In fact, up to charge conjugation, this is exactly the situation drawn in Fig. 3.2. Regarding the weak couplings the electric charge of the leptons participating in a four-fermion process does not matter. The "V–A" interaction is invariant under Fierz reordering (possibly up to an overall sign) so that for the interaction term that is relevant here, we have

$$(\bar{\mu}\gamma_\alpha\,(\mathbb{1}-\gamma_5)\,\nu_\mu)\;(\bar{\nu}_e\,\gamma^\alpha\,(\mathbb{1}-\gamma_5)e) = -(\bar{\mu}\gamma_\alpha(\mathbb{1}-\gamma_5)e)\;(\bar{\nu}_e\,\gamma^\alpha(\mathbb{1}-\gamma_5)\nu_\mu).$$

Thus, the calculation of the $\bar{\nu}_\mu$ spectrum is exactly the same as the one of the positron. As we know that ρ was equal to 3/4 there, we conclude $\rho(\bar{\nu}_\mu) = 3/4$.

If the reaction (4.39b) did indeed occur, one would like to identify the electronic antineutrino $\bar{\nu}_e$ through inverse β-decay. Obviously, interpreting data for $\bar{\nu}_e$ in terms of the branching ratio (4.40) rests on the knowledge of the isotropic decay spectrum.

4.3. Let

$$I := \int \frac{\mathrm{d}^3k_1}{2E_1}\;\int \frac{\mathrm{d}^3k_2}{2E_2}\;(k_1\cdot k_2)\;\delta\,(Q-k_1-k_2). \tag{19}$$

As $k_1+k_2 = Q, k_1^2 = k_2^2 = 0$ we have $(k_1\cdot k_2) = Q^2/2$. The integral over $\boldsymbol{k}_2$ takes care of the spatial delta distribution. In a frame where $\boldsymbol{Q} = (0,0,0)$ and introducing polar coordinates for $\boldsymbol{k}_1$, i.e. $\mathrm{d}^3k_1 = E_1^2\mathrm{d}E_1\mathrm{d}\Omega$ we find

$$I = \frac{Q^2}{2}\pi\;\int_0^\infty \mathrm{d}E_1\,\delta(Q^0-2E_1) = \frac{\pi}{4}\,Q^2.$$

The integral (4.44b) is best decomposed into covariants which are then isolated by taking various contractions as follows. The momentum Q being the only variable, the integral must be a linear combination of the covariants $g^{\alpha\beta}$ and $Q^\alpha Q^\beta$ with Lorentz invariant coefficients. We set

$$\begin{aligned} I^{\alpha\beta} :&= \int \frac{\mathrm{d}^3k_1}{2E_1}\;\int \frac{\mathrm{d}^3k_2}{2E_2}\;\left\{k_1^\alpha\,k_2^\beta - (k_1\cdot k_2)g^{\alpha\beta} + k_2^\alpha\,k_1^\beta\right\}\;\delta\,(Q-k_1-k_2)\\ &\equiv AQ^\alpha Q^\beta + Bg^{\alpha\beta}\,Q^2, \end{aligned} \tag{20}$$

and calculate the invariant integrals

$$g_{\alpha\beta}I^{\alpha\beta} = (A+4B)Q^2, \quad \text{and} \quad Q_\alpha Q_\beta I^{\alpha\beta} = (A+B)(Q^2)^2.$$

From the definition of the tensor integral $I^{\alpha\beta}$ we find its contraction with the metric tensor $g_{\alpha\beta}I^{\alpha\beta} = -2I$, with I as calculated above. Inserting $Q = k_1 + k_2$

in the integrand of $Q_\alpha I^{\alpha\beta} Q_\beta$, the second combination is zero. Thus $B = -A$, $(A + 4B)Q^2 = -\pi Q^2/2$, and, from these,

$$A = \frac{\pi}{6}, \quad B = -\frac{\pi}{6}. \tag{21}$$

4.4. Let k and k' denote the initial and final neutrino four-momenta, respectively, p the electron momentum, q the muon momentum. Neglecting terms quadratic in the electron mass the invariant kinematic variables s and t are expressed in terms of the laboratory energies as follows,

$$\begin{aligned} s &= (k + p)^2 \approx 2m_e E_\nu^{\text{lab}}, \\ t &= (k - q)^2 = (k' - p)^2 \approx -2m_e E_\nu^{\prime\text{lab}} \approx -2m_e \left(E_\nu^{\text{lab}} - E_\mu^{\text{lab}}\right) \approx -s(1 - y). \end{aligned}$$

With $\mathrm{d}t/\mathrm{d}y = s$ and the fact that the cross section (4.69) is a quadratic function in the variable y, the integration is elementary. Note that it is the integral over y from some value $y_{\min}$ (that depends on the experimental arrangement), to its maximum $y = 1$ which is determined in the quoted experiments.

4.5. The calculation is similar to the one of exercise 4.3, except that now one of the neutrinos is massive, say, $k_2^2 = \lambda^2$. We first calculate the integral

$$I_0 := \int \frac{\mathrm{d}^3 k_1}{2E_1} \int \frac{\mathrm{d}^3 k_2}{2E_2} \, \delta(Q - k_1 - k_2), \tag{22}$$

say, in a frame where the spatial part of Q vanishes. The integral over $\mathrm{d}^3 k_2$ eliminates the spatial δ-distribution. In a frame where $\boldsymbol{Q} = (0, 0, 0)$, this leaves E_2 a function of E_1, so that we obtain

$$I_0 = \pi \int_0^\infty \mathrm{d}E_1 \, E_1 \, \frac{1}{E_2(E_1)} \, \delta(E_1 + E_2(E_1) - Q^0),$$

with $E_2(E_1) = \sqrt{\lambda^2 + E_1^2}$. The integral over E_1 is calculated using the well-known replacement rule for the delta distribution

$$\delta(g)(E_1)) \;\to\; \frac{1}{\left|g'\left(E_1^{(i)}\right)\right|} \, \delta(E_1^{(i)}),$$

where $E_1^{(i)}$ is a simple zero of the function $E_2(E_1)$. This gives

$$I_0 = \frac{\pi}{2} \, \frac{Q^2 - \lambda^2}{(Q^0)^2} \;\to\; I_0 = \frac{\pi}{2} \, \frac{Q^2 - \lambda^2}{Q^2}.$$

In the last step we returned to an arbitrary frame (where $\boldsymbol{Q}$ is possibly non-zero), using our knowledge that the integral I_0 must be a Lorentz scalar. As the neutrino number 2 is massive, the scalar product of the neutrino four-momenta becomes $(k_1 \cdot k_2) = (Q^2 - \lambda^2)/2$ and the integral (19) is given by

$$I = \frac{\pi}{4} \frac{(Q^2 - \lambda^2)}{Q^2}.$$

The tensor integral (20) is calculated as in exercise 4.3., making use of the result above. One finds

$$Q_\alpha I^{\alpha\beta} Q_\beta = (Q^2)^2 \ (A + B) = \lambda^2 I, \ g_{\alpha\beta} I^{\alpha\beta} = Q^2(A + 4B) = -2I,$$

and, from these relations,

$$A = \frac{\pi}{6} \frac{(Q^2 - \lambda^2)^2 \ (Q^2 + 2\lambda^2)}{(Q^2)^3}, \quad B = -\frac{\pi}{12} \frac{(Q^2 - \lambda^2)^2 \ (2Q^2 + \lambda^2)}{(Q^2)^3}. \tag{23}$$

Clearly, for $\lambda = 0$ the result (23) goes over into the result (21) of exercise 4.3.

4.6. It is simplest to use a Cartesian basis in isospin space and to drop all real factors such as $(2\pi)^{3/2}$ so that the ansatz reads

$$\left\langle 0 \left| A_\alpha^{(i)}(0) \right| \pi_j(q) \right\rangle = F \delta_{ij} q_\alpha. \tag{24}$$

Inserting first $T^{-1}T$ and then $P^{-1}P$ one obtains successively

$$\begin{aligned}
\left\langle 0 \left| A_\alpha^{(i)}(0) \right| \pi_j(q) \right\rangle &= \left\langle 0 \left| T^{-1} \ \left(T A_\alpha^{(i)}(0) \, T^{-1} \right) \ T \right| \pi_j(q) \right\rangle \\
&= (-)^{1+\delta_{\alpha 0}} \left\langle 0 \left| A_\alpha^{(i)}(0) \right| \pi_j(q_0 - \boldsymbol{q}) \right\rangle {}^* \eta_T \\
&= (-)^{1+\delta_{\alpha 0}} \left\langle 0 \left| P \ A_\alpha^{(i)}(0) \, P^{-1} \right) \ P \right| \pi_j(q_0 - \boldsymbol{q}) \right\rangle {}^* \eta_T \\
&= -\left\langle 0 \left| A_\alpha^{(i)\dagger}(0) \right| \pi_j(q_0, \ \boldsymbol{q}) \right\rangle \ {}^* \eta_T \eta_P \\
&= -F^* \delta_{ij} q_\alpha \eta_T \eta_P
\end{aligned}$$

Here η_P is the intrinsic parity of the pion, η_T is an analogous phase which appears when T is applied to a one-pion state. The pion has negative intrinsic parity. It is even with respect to charge conjugation, i.e. $\eta_C = 1$. With $\eta_P \eta_P \eta_C = 1$ we conclude that the product $\eta_P \eta_T$ is $+1$. Thus, F, as defined in (24) is pure imaginary, or, equivalently, f_π is pure real.

4.9. It is not difficult to check that this model for the pionic axial current yields the terms represented by the diagrams of Fig. 4.5 and, hence, that the structure terms vanish.

4.10. In the electronic decay mode $\pi \to e\nu\gamma$ the structure terms are found to be comparable in magnitude to the contributions from internal bremsstrahlung. The latter, however, contain the dynamic suppression factor $\propto m_e^2$ that we found in comparing $\pi \to e\nu$ to $\pi \to \mu\nu$. In the muonic decay mode $\pi \to \mu\nu\gamma$, the bremsstrahlung does not show this strong suppression while the structure terms, up to kinematic differences due to the muon mass, are the same as in the electronic mode. From this simple observation one draws two conclusions:

(i) as the contributions due to internal bremssstrahlung are dominant in the muonic mode, the branching ratio

$$\frac{\Gamma(\pi^+ \to e^+ \nu_e \gamma)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu \gamma)}$$

must be practically the same as the branching ratio

$$\frac{\Gamma(\pi^+ \to e^+ \nu_e)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)}$$

(ii) the structure terms are very small, hence not easily detectable, in the muonic decay mode. The integral over the available phase space for $\pi \to \mu\nu\gamma$ is easily calculated using the formulae of Sect. 4.3.1 a for three-body decays. It is found markedly smaller than the surface of the triangle of Fig. 4.6 for $\pi \to e\nu\gamma$.

4.11. The calculation is analogous to the calculation of pion β-decay; cf. Sect. 4.2.2.

4.12. This problem is a little lengthier. We give only a few hints and a reference where further details will be found. A Feynman graph where the π^0 first converts to a single virtual photon which then decays via pair decay to the $e^+ e^-$ final state gives no contribution because of conservation of C. Indeed, the neutral pion has $C(\pi^0) = +1$ while the photon has $C(\gamma) = -1$. Therefore, the diagram of lowest order is the triangular graph with π^0 decaying into a pair of virtual photons, and with the electron–positron pair being created via (virtual) Compton effect. It is not difficult to write down the analytic expression that corresponds to this diagram. It will contain the amplitude (4.126a) for $\pi^0 \to \gamma\gamma$, with the two photons off-shell, the electron and the positron as external legs, and the electron propagator joining the two vertices where the photons are annihilated. A lower limit on the branching ratio is obtained by means of the unitarity relation,

$$\mathrm{i}(T^\dagger - T) = (2\pi)^4 T^\dagger T,$$

taken between the initial state $|\pi_0\rangle$ and the final state $\langle e^+ e^-|$. It is important to note that the product $T^\dagger T$, after insertion of a complete set of intermediate states, contains only *on-shell* amplitudes. The dominant contribution stems from the two-photon intermediate state,

$$\mathrm{i}(\langle e^+e^-|\pi_0\rangle^* - \langle \pi_0|e^+e^-\rangle) = (2\pi)^4 \langle e^+e^-|\gamma\gamma\rangle^* \langle \gamma\gamma|\pi_0\rangle + \cdots,$$

with the two photons on their mass shell. This will be a lower limit because the unitarity relation yields only the imaginary part of the $\pi_0 \to e^+e^-$ amplitude. If this is worked out one finds the so-called *unitarity* limit

$$\Gamma(\pi^0 \to e^+e^-)|_{\text{unitarity}} \approx \frac{1}{2}(\alpha m_e/m_\pi)^2 \Gamma(\pi^0 \to \gamma + \gamma).$$

This gives a lower limit on the branching ratio of 4.75×10^{-8} (cf., e.g., L.G. Landsberg, *Phys. Rep*. **128** (1985) 301). Compare this to the present experimental value for the branching ratio: $(7.5 \pm 2.0) \times 10^{-8}$ (Deshpande et al., *Phys. Rev. Lett*. **71** (1993) 27, McFarland et al., ibid. p.31.)

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